NONLINEAR DYNAMIC ANALYSIS OF A CABLE UNDER FIRST AND SECOND ORDER PARAMETRIC EXCITATIONS

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Abstract. It is well known that small periodic vibrations of a cable support through its axial direction produce large spectacular oscillations of the cable. This may occur when the frequency of the anchorage motion is close to the first natural frequency or twice the fundamental frequency of the cable. In this paper, a nonlinear dynamic study of a cable under first and second order parametric excitations is presented. The cable model takes into account sag as well as quadratic and cubic nonlinear couplings between in-plane and out-of-plane motions. As a numerical example, a single-d.o.f. planar model of a horizontal cable is used to study the effect of frequency and amplitude of excitation as well as the natural damping of the cable on its transient and steady state responses with a particular focus on the time needed to trigger first and second order parametric resonance.

Keywords: small sagged cables, nonlinear vibration, parametric excitation, general support movement.

1. Introduction

Cables have been widely used in many engineering applications such as guyed towers, suspended bridges, cable stayed bridges, stadium roofs, etc. These structures are very flexible and sensitive to traffic, wind and seismic excitations. Thus, stay cables are prone to vibrate locally both in-plane and out-of-plane. Furthermore, cable vibrations can be accompanied by support motion. The coupling between the local (cable) and the global modes (structure) may give rise to parametric excitations (Sun *et al.* 2003). In fact, when a tuning process converges to certain conditions, even small periodic vibrations of the cable support through its axial direction produce large spectacular oscillations of the cable. This may occur when the frequency of the anchorage motion is close to the first natural frequency or twice the fundamental frequency of the cable (Lilien, Pinto da Costa 1994).

Cable dynamics bibliography is very rich. Classical linear modelling was addressed by Irvine (1992) in a book, Triantafyllou (1984, 1987, 1991) and Starossek (1994) in some review articles. As for nonlinear dynamics, modeling and analysis were highlighted in some review articles by Rega (2004a, b) and Ibrahim (2004). A small part of the extremely rich literature is cited here. Linear free and forced oscillations of elastic cables with a small sag-to-chord-length ratio based on the parabolic equilibrium approximation were first developed by Irvine (1992) and Irvine and Caughey (1974). Based on a single-degree-of-freedom model for the in-plane vibrations of the cable, Hagedorn and Schäfer (1980) extended the linear theory by considering the effect of quadratic and cubic non-linearities on eigen frequencies. These non-linear terms come up due to the stretching of the cable associated with the large-amplitude vibration (Takahashi, Konishi 1987). The existence of the quadratic and cubic non-linear terms makes the in-plane cable motion couples with the out-of-plane cable motion and induces modal interaction (Perkins 1992). To determine the nonlinear dynamic response of a sag cable, the Galerkin method is often used to convert non-linear partial differential equations of motion to non-linear ordinary differential equations with respect to time functions only. Then, either the perturbation method (Benedettini *et al.* 1995; Chen, Xu 2009), or the harmonic balance method (Takahashi 1991) is applied to find the solution for the time functions. Nayfeh *et al.* (1995) investigated the nonlinear differential equations that govern the in-plane motion of a taut string under parametric excitation. They predicted the steady state response using the method of multiple scales. Ben Kahla (1995) presented a numerical model, of a single cable with large oscillations, based on a step by step integration method coupled with an iterative procedure. Juozapaitis and Norkus (2004) derived the expressions of maximum vertical and horizontal displacements of a asymmetrically loaded cable. Wu *et al.* (2005) did a modification in the expressions for the in-plane natural frequencies of an inclined cable as derived before by Irvine (1992). Ren *et al.* (2005) proposed empirical formulas to estimate cable tension based on the cable fundamental

frequency only and validated them by comparing the results with those reported in the literature and with the experimental results carried out on a stay cable mock-up. Wu *et al.* (2003) examined local parametric vibrations in the stay cables of an existing cable-stayed bridge under sinusoidal excitations, traffic loadings and earthquake. Srinil and Rega (2007) studied the effects of kinematic condensation on internally resonant forced vibrations of shallow horizontal cables. Rega *et al.* (2008) carried experimental studies on free and parametrically-forced vibrations of sagged inclined cables and discussed them against theoretical and numerical analyses. Hu and Frank Pai (2010) conducted an experimental study of the nonlinear dynamic characteristics of taut cables using an extraordinary 3D motion analysis system. The planar and nonplanar/whirling vibration, hardening effect, and modal coupling vibration were successfully recorded and the hysteretic phenomenon of response was also observed. Wang and Zhao (2009a, b) applied the shooting method and the continuation technique to investigate the large amplitude motion and non-planar motion characteristic of an inclined cable subjected to support motion. Recently, Nielsen and Sichani (2011) analyzed the stochastic response and chaotic behaviour of a shallow cable by two comparable stochastic models of the chord elongation for the sub- and the superharmonic resonances.

Wang and Rega (2010) developed 3D nonlinear equations of motion of a suspended cable with moving mass and studied the effects of the inertia force, mass, sag and velocity of the moving mass on the transient dynamics of the cable. Takahashi and Chen (2006) studied the effect of cable loosening on the nonlinear parametric vibrations of inclined cables with small sag using a new technique that takes into account flexural rigidity and damping. An investigation on accurate finite element modelling of large-diameter sagged cables taking into account flexural rigidity and sag extensibility is carried out by Ni *et al.* (2002). Lacarbonara and Pacitti (2008) presented a geometrically exact formulation of cables suffering axis stretching and flexural curvature based on nonlinearly visco-elastic constitutive laws for the tension and bending moment with an additional constitutive nonlinearity accounting for the no-compression condition. Treyssède (2010) studied the effect of thermal change on the frequencies of a cable taking into account flexural rigidity and sag. Sousa *et al.* (2011) analyzed the influence of bending and shear stiffness and rotational inertia in the natural frequencies of overhead transmission line conductors and compared the results with a vibrating string where only geometrical stiffness is considered. As the flexural rigidity has an effect on the natural frequencies in the case of suspended cables with large diameters, it can be neglected in the case of stay cables with smaller diameter. In this paper, we present a cable model which takes into account sag as well as quadratic and cubic nonlinear couplings between in-plane and out-of-plane motions and movable support. The motivation of the use of a model accounting for general support movement is that it can simply be coupled with a finite element model of a cable stayed bridge which will be used to study the instabilities of stay cables under parametric excitation

and active control of vibrations. The objective of the present study is to analyze the nonlinear dynamic behaviour of a cable under first and second order (i.e., principal and fundamental) parametric excitations with a comparative construction of frequency, force and damping-response diagrams for the two cases of parametric excitation with a particular focus on the time needed to trigger the two resonances.

The paper is organized as follow: in section 2, a non-linear model of a cable with small sag and movable support is presented. In section 3, a numerical example of a single-d.o.f. planar model of a horizontal cable is used to study the effect of frequency and amplitude of excitation as well as the natural damping of the cable on its transient and steady state responses with a particular focus on the time needed to trigger first and second order parametric resonance. The paper ends with some conclusions.

2. Nonlinear modelling of an inclined cable with small sag

The non-linear model of the inclined cable takes into account the sag effect, coupling between the in-plane and out-of-plane motions and also the displacements of the anchorage points. The model of the cable is written in a local coordinate system as indicated in Fig. 1; the local x axis is taken along the string line and y axis in the horizontal plane, while the z axis is in the gravity plane and perpendicular to the chord line. The cable displacements can be separated into three parts: the static, the quasi-static and the dynamic contributions. A brief description of the nonlinear model of the cable is presented hereafter (for more details see Bossens 2001).
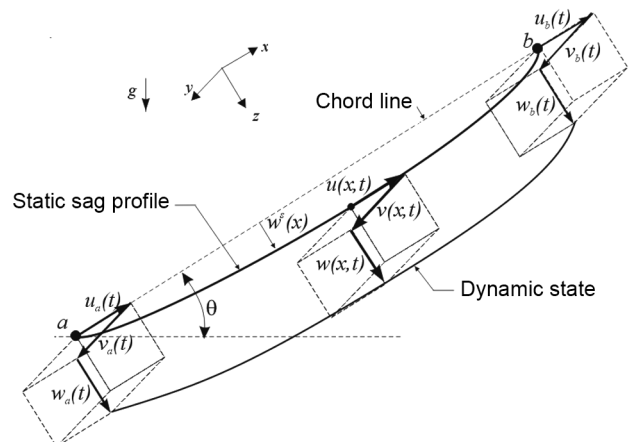

Fig. 1. 3D-model of an inclined cable with support motions

2.1. Static configuration

The static configuration of a cable is a function of its weight and pre-stress when the anchorage points are fixed in the initial position. Assuming small sag and a constant stress along the cable span, the static profile of an inclined cable can be approximated by a parabola. The static sag of a given point of coordinate x along the cable chord is given by the following relation:

$$w^s(x) = \frac{\gamma l^2}{2\sigma^s}\left[\frac{x}{l} - \left(\frac{x}{l}\right)^2\right], \qquad (1)$$

where: σ^s is the static stress; l the length of the cable's chord; γ is the component of weight density along z axis which is equal to $\rho g \cos\theta$. With ρ the cable mass density, g the gravity and θ the angle of the chord line with respect to the horizontal.

2.2. Quasi static motion

The quasi-static motion due to the support movements of anchorage points a and b can be written as:

$$u^q(x,t) = u_a - \frac{w_b - w_a}{l} w^s(x) + (u_b - u_a)\frac{x}{l} ; \quad (2)$$

$$v^q(x,t) = v_a + (v_b - v_a)\frac{x}{l} ; \quad (3)$$

$$w^q(x,t) = w_a + (w_b - w_a)\frac{x}{l} - \frac{E_q}{\sigma^s}\frac{u_b - u_a}{l} w^s(x), \quad (4)$$

where: E_q is the effective modulus of elasticity (see Appendix); u_a, u_b, v_a, v_b, w_a and w_b are the movements imposed to anchorage points.

2.3. Dynamic motion

The dynamic motion of the cable subjected to the excitation of its supports formulated in a discrete manner is obtained by employing the Ritz method. This method is based on approaching the solution of variational problem by a finite sum of shape functions. The "separation of variables" method is used to write the shape functions with two well separated factors (geometric and temporal).

For small sag, the axial dynamic component can be neglected and:

$$u^d(x,t) \approx 0 ; \quad (5)$$

$$v^d(x,t) = \sum_n y_n(t)\, \varphi_n(x) = \sum_n y_n(t) \sin\frac{n\pi x}{l} ; \quad (6)$$

$$w^d(x,t) = \sum_n z_n(t)\, \psi_n(x) = \sum_n z_n(t) \sin\frac{n\pi x}{l} . \quad (7)$$

The global displacement of the cable is obtained from the superposition of the static displacement, quasi-static and dynamic motion as:

$$u(x,t) = u^q(x,t) ; \quad (8)$$

$$v(x,t) = v^q(x,t) + v^d(x,t) ; \quad (9)$$

$$w(x,t) = w^s(x) + w^q(x,t) + w^d(x,t) . \quad (10)$$

The total displacement of the cable will be completely defined by the following coordinates:

$$u_a, u_b, v_a, v_b, w_a, w_b, \dots y_i \dots, \dots z_i .$$

The kinetic energy of the cable is given by:

$$K_c = \frac{1}{2}\rho A \int_0^l [\dot{u}^2 + \dot{v}^2 + \dot{w}^2]\ dx . \quad (11)$$

The potential energy of the cable contains the contribution of the elastic elongation of the cable and the gravitational potential:

$$U_c = \frac{1}{2} EA \int_0^l \varepsilon(x,t)^2\ dx -$$

$$\rho g A \int_0^l [(w^s + w^q + w^d)\cos\theta - u^q \sin\theta]\ dx , \quad (12)$$

where E is the modulus of elasticity and ε is the axial strain in the cable which is supposed to be the only contribution to the strain energy.

The axial strain is evaluated using Green's tensor of strain:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) . \quad (13)$$

The expressions of the Kinetic and potential energies are given in the Appendix.

2.4. Equation of horizontal modes of the cable

Substituting the expressions of the kinetic and potential energies in the Lagrange equations, we obtain the differential equations governing the generalized coordinates y_n of the nth cable mode:

$$\frac{d}{dt}\left(\frac{\partial K_c}{\partial \dot{y}_n}\right) + EAl\,\overline{\varepsilon}\,\frac{\partial(\overline{\varepsilon})}{\partial y_n} = F_{y_n} , \quad (14)$$

where F_{y_n} is the modal component, associated to y_n, of the external forces applied to the cable.

Thus, by adding a damping term, the equations of motion become:

$$\frac{1}{2}ml\left\{\ddot{y}_n + 2\xi_{yn}\omega_n \dot{y}_n + \frac{n^2\pi^2}{m\ l^2}(T_0 + T_q + T_d)y_n\right\} =$$

$$F_{y_n} - \frac{m\ l}{n\pi}(\ddot{v}_a + (-1)^{n+1}\ddot{v}_b), \quad (15)$$

where ξ_{y_n}, ω_{y_n} and F_{y_n} are respectively the modal damping, the frequency and the modal component of the external forces applied to the cable, associated to the generalized coordinates y_n of the nth cable mode. The cable mass per unit length is m, $\ddot{v}_a$ and $\ddot{v}_b$ are respectively the transverse acceleration of the anchorage points a and b with respect to the y axis, T_0 is the static tension in the cable at its equilibrium and $T_q = T_q^{(1)} + T_q^{(2)}$ is the tension increment induced by the support movement with:

$$T_q^{(1)} = E_q A \frac{u_b - u_a}{l} ; \quad (16)$$

$$T_q^{(2)} = EA \quad (1+\frac{E_q}{\sigma^s}\frac{\lambda^2}{12+\lambda^2}) \quad \frac{(u_b-u_a)^2}{2l^2} ;$$

$$+EA\frac{(v_b-v_a)^2}{2l^2} + EA(1+\frac{\lambda^2}{12}\frac{\sigma^s}{E})\frac{(w_b-w_a)^2}{2l^2}, \quad (17)$$

where: A is the cable cross section area and λ^2 is the Irvine's parameter (Irvine 1992) (see Appendix), $T_d = T_d^{(1)} + T_d^{(2)}$ is the tension increment induced by the dynamic motion of the cable, which is responsible for the quadratic and cubic nonlinear couplings between in-plane and out-of-plane motion:

$$T_d^{(1)} = \frac{EA^2\gamma}{T_0}\sum_n[\frac{z_n}{n\pi}(1+(-1)^{n+1})] ; \quad (18)$$

$$T_d^{(2)} = \frac{EA}{2}\sum_n(y_n^2\frac{n^2\pi^2}{2l^2}) + \frac{EA}{2}\sum_n(z_n^2\frac{n^2\pi^2}{2l^2}) -$$

$$\frac{E_q}{(\sigma^s)^2}\frac{EA}{l}\frac{\gamma}{l}\frac{u_b-u_a}{l}\sum_n[\frac{z_n}{n\pi}(1+(-1)^{n+1})] . \quad (19)$$

2.5. Equation of transverse modes of the cable

The equations of motion describing the transverse modes of vibration can be developed using the same strategy as for determining the equation governing the horizontal modes of vibration of an inclined cable with small sag:

$$\frac{d}{dt}(\frac{\partial K_c}{\partial \dot{z}_n}) + EAl\,\overline{\varepsilon}\,\frac{\partial(\overline{\varepsilon})}{\partial z_n} + \frac{\partial(U_{c,g})}{\partial z_n} = F_{z_n} . \quad (20)$$

By assuming that $\frac{T_q^2}{T_0^2} \ll 1$ and $\frac{T_d}{T_0} \ll 1$ and limiting T_q to the first order, one can write:

$$\frac{1}{2}ml\left\{\ddot{z}_n + 2\xi_{z_n}\omega_n\dot{z}_n + \frac{n^2\pi^2}{m\,l^2}(T_0+T_q+T_d)z_n\right\} = F_{z_n}$$

$$-\frac{ml}{n\pi}(\ddot{w}_a + (-1)^{n+1}\ddot{w}_b) + \frac{ml^2 E_q\gamma}{(\sigma^s)^2}\frac{(1+(-1)^{n+1})}{(n\pi)^3}(\ddot{u}_b - \ddot{u}_a) -$$

$$\frac{\gamma}{T_0}\frac{A}{n\pi}\frac{l}{(1+(-1)^{n+1})}T_d , \quad (21)$$

where: ξ_{z_n}, ω_{z_n} and F_{z_n} are respectively the modal damping, the frequency and the modal component of the external forces applied to the cable, associated with the generalized coordinates z_n of the nth cable mode. The in-plane acceleration of the anchorage points a and b with respect to the z axis are respectively $\ddot{w}_a$ and $\ddot{w}_b$, the longitudinal acceleration of the anchorage points a and b with respect to the x axis are respectively $\ddot{u}_a$ and $\ddot{u}_b$.

This model had been implemented into a MATLAB/SIMULINK program.

3. Numerical example

In this section, the dynamic response of a single-d.o.f. planar model of a horizontal cable (*ab*) with small sag (nearly taut cable), fixed at its right end and attached to a roller support at its left end so that only horizontal motion is allowed at this node, had been studied under a first and a second order (i.e., principal and fundamental) parametric excitation. The kth order parametric excitation is obtained for a frequency of excitation $\omega_k = 2\omega_0/k$ ($k=1,2,3,...$). The geometrical and physical properties of the stainless steel cable are given in Table 1. The cable is located before the first crossover of natural frequencies. In fact, the cable had been excited through its axial direction (u_a) at its left end with a frequency close to its fundamental frequency then close twice that of its first natural frequency (see Fig. 2). The effect of frequency and amplitude of excitation as well as the natural damping of the cable on the steady state response had been investigated.

Table 1. Cable parameters

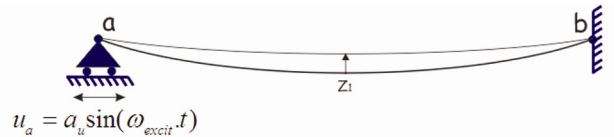| Parameters	Values
Length l	1.745 m
Diameter φ	1×10^{-3} m
Modulus of elasticity E	105×10^9 Pa
Density	6251 kg/m^3
Lumped mass every 10 cm along the cable	10 g
Initial static tension T_0	100 N
Sag d/l	0.195%
Irvine's parameter	0.2
Modal damping ξ_0	0.3%

Fig. 2. Parametric excitation of a cable

3.1. First order parametric excitation

3.1.1. Effect of the frequency of excitation

The tension of the cable is tuned to 100 N which corresponds to a first natural frequency of the cable $\omega_0 = 60.2$ rad/s and excited harmonically ($u_a = a_u\sin(\omega_{excit}t)$) through its axial direction by a frequency $\omega_{excit} = 2\omega_0$ and with an amplitude of excitation $a_u = 0.1$ mm. The time evolution of the mid span vertical displacement (Z1) is plotted in Fig. 3a. This figure shows large spectacular vertical oscillations of the cable at mid span under first order parametric excitation. First the response of the cable is dominated by small oscillations with a period corresponding to a frequency of $2\omega_0$ which characterises the anti-symmetric mode of vibration of the cable. Next, a progressive change of the cable vibration frequency is established and converges to ω_0 (Fig. 3b) the first natural frequency of the cable. Large oscillations of the cable characterising the symmetric mode of vibration are obtained.
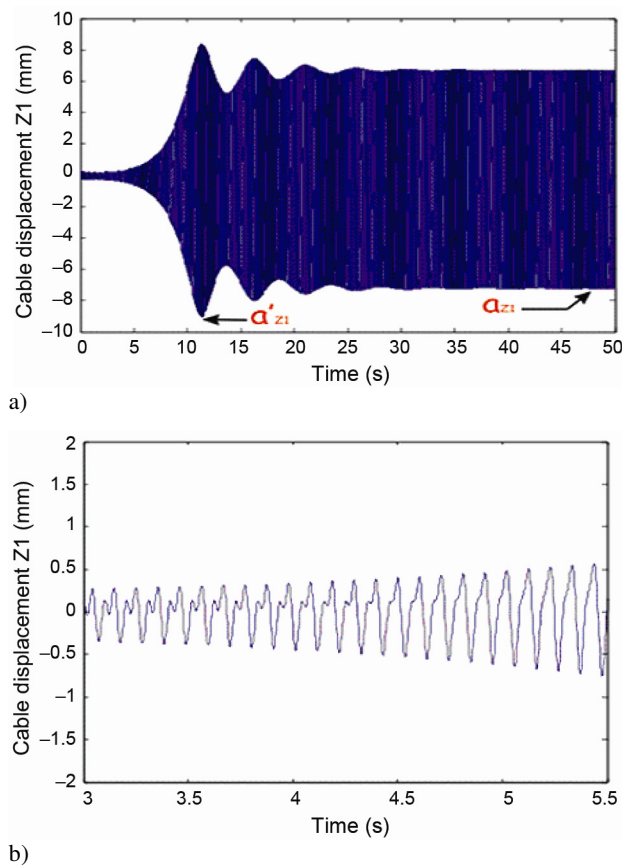
Fig. 3. In-plane cable vibration at mid span under first order parametric excitation as a function of time (a); zoom between 3 s and 5.5 s (b)
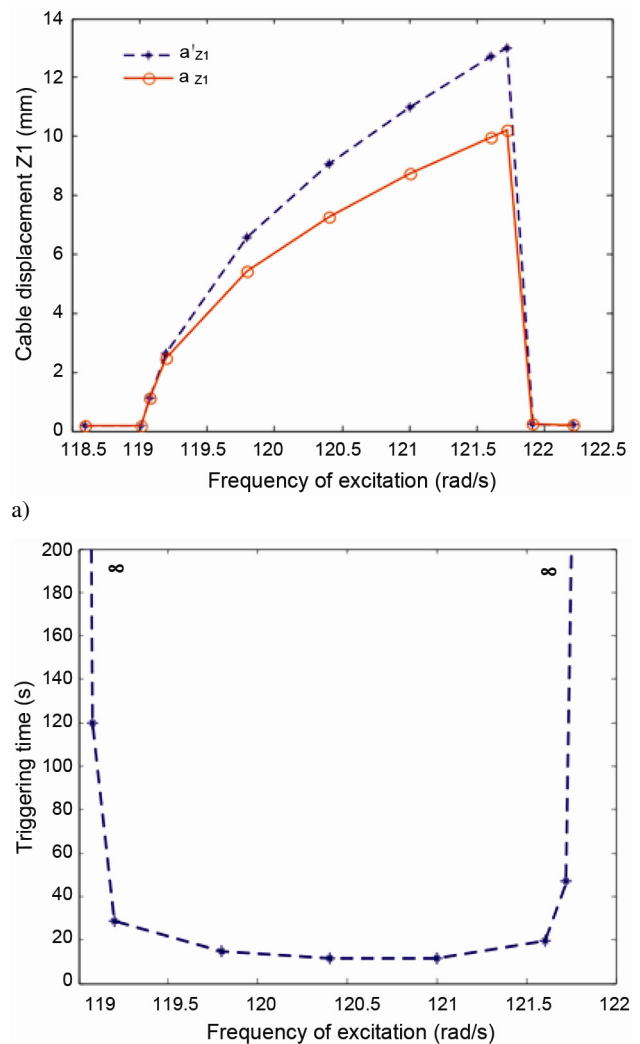

Fig. 4. Maximum mid span cable displacement (a); necessary time to trigger first order parametric resonance as a function of the frequency of excitation (b)

Now, let's change the frequency of excitation near the initial frequency of excitation ($2\omega_0$). The maximum amplitude of both transient (a'_{Z1}) and steady state (a_{Z1}) in-plane response of the cable at mid span is plotted as a function of the frequency of excitation. Fig. 4a shows that the maximum in-plane displacement Z1 depends on a minimum (ω_{min}) and maximum (ω_{max}) threshold frequency of excitation (equal to 119.1 and 121.7 rad/s in this case) to be triggered. The in-plane displacement Z1 increases as the frequency of excitation travels from ω_{min} to ω_{max} then jumps down. Otherwise, out of this interval the first order parametric resonance does not occur. The transient response disappears when the frequency is close to ω_{min} and the difference between the maximum amplitude of both transient a'_{Z1} and steady state a_{Z1} responses decreases when the frequency of excitation increases from ω_{min} to ω_{max}. The shape of the frequency-response curve in Fig. 4a is qualitatively similar with the analogous curve obtained by Rega and Benedettini (1989) for the single d.o.f model of suspended cable using multiple scale method.

The necessary time to trigger parametric resonance is also plotted as a function of the frequency of excitation (Fig. 4b). Parametric resonance triggers faster when the frequency of excitation is close to $2\omega_0$ but triggers slower when frequency is near ω_{min} and ω_{max}.

3.1.2. Effect of the amplitude of excitation

Let's fix the frequency of the excitation at $\omega_{excit} = 2\omega_0$ and progressively change the amplitude of excitation. The transient and steady state response of the cable are plotted as a function of the amplitude of excitation (a_u). Fig. 5a shows that the parametric resonance needs threshold amplitude of excitation to be triggered, as proved by Rega and Benedettini (1989), and that the maximum in-plane displacement a_{Z1} of the cable increases by increasing the amplitude of excitation. This means that in cable stayed bridges, first order parametric excitation could be prevented by damping the deck or the pylon and keeping their amplitudes of vibration under the threshold value needed to trigger parametric resonance. The difference between the maximum amplitude of both transient and steady state responses increases by increasing the amplitude of excitation.

Fig. 5b shows that the time needed to trigger a parametric resonance decreases by increasing the amplitude of excitation and goes to infinity when the amplitude of excitation is under the threshold value.
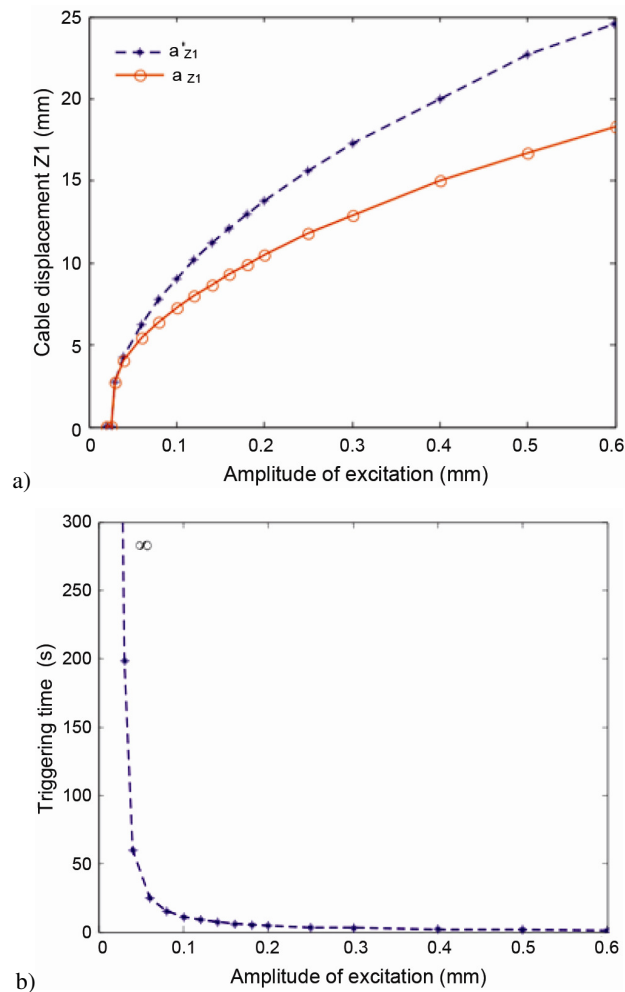
a)

b)

Fig. 5. Maximum mid span cable displacement (a); necessary time to trigger first order parametric resonance; as a function of the amplitude of excitation (b)

3.1.3. Effect of the natural damping of the cable

Next, the amplitude and the frequency of excitation are fixed respectively to $a_u = 0.1$ mm and $\omega_{excit} = 2\omega_0$. The natural damping ($\xi_0 = 0.3\%$) of the cable is changed from small to high values. For small values of natural damping, a beating phenomenon is obtained as could be seen in Fig. 6a and Fig. 6d. A progressive change of the frequency of the cable from $2\omega_0$ to ω_0, then from ω_0 to $2\omega_0$ is observed after the first beating as shown in Fig. 6c. The beating disappears as the damping increases (see Fig. 7). The maximum amplitude of both transient and steady state in-plane response of the cable at mid span are plotted as a function of the damping. Fig. 8a shows that the amplitude of both transient and steady state responses, and also the difference between them, decreases by increasing the damping in the cable. For high damping, the transient response disappears and the response goes directly to a steady state. The parametric resonance also disappears when the damping achieve a critical value of damping ($\xi = 1.19\%$ in this case). Note that this critical value depends also on the amplitude of excitation. The time needed to trigger parametric resonance is also plotted as a function of the damping (Fig. 8b). Here we can see that parametric resonance triggers faster when the damping is small. And for high values of damping it needs a very long time to occur which means its disappearance.
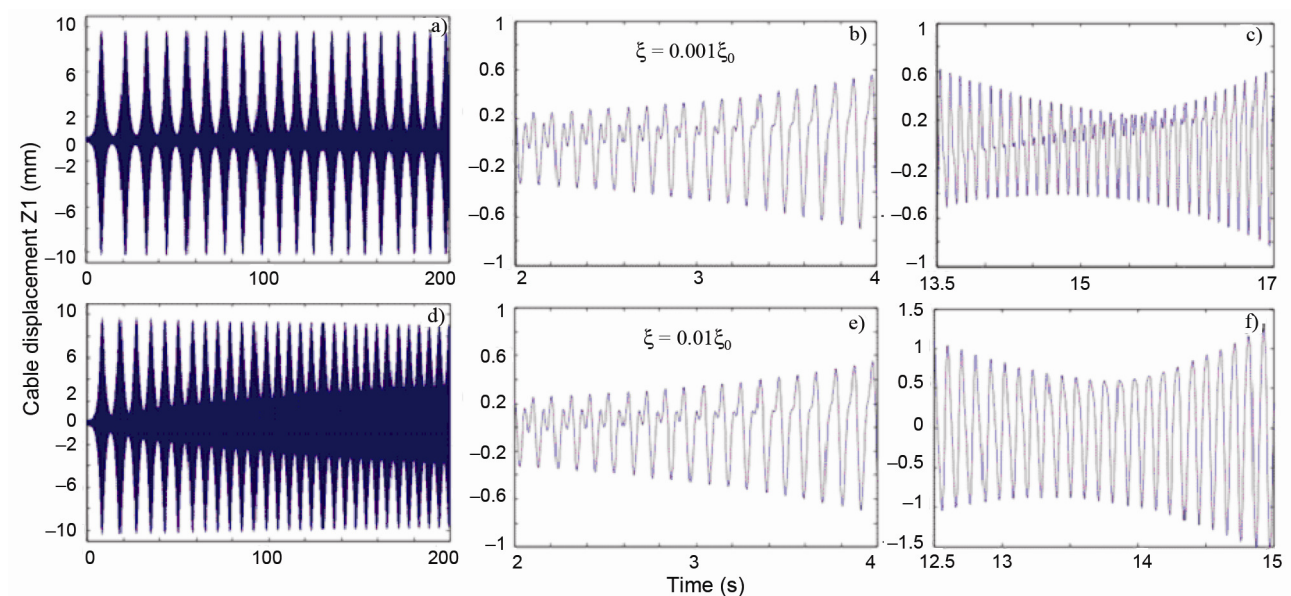

Fig. 6. Evolution in time of the cable mid span displacement under first order parametric excitation: (a), (b) and (c) – when $\xi = 0.001\xi_0$; (d), (e) and (f) – when $\xi = 0.01\xi_0$; (b) and (e) zoom between 2 s and 4 s; (c) zoom between 13.5 s and 17 s; (f) zoom between 12.5 s and 15 s
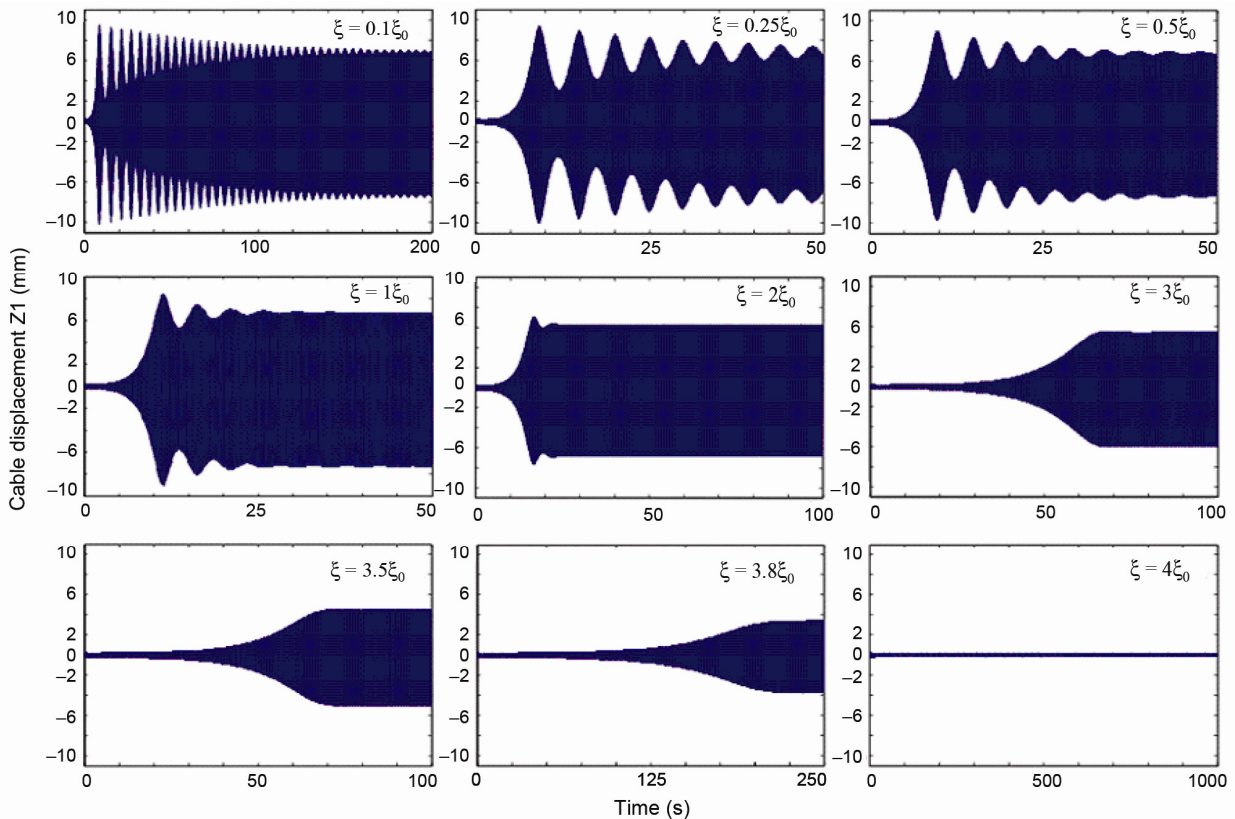
Fig. 7. Evolution in time of the in-plane cable mid span displacement under first order parametric excitation for different values of natural damping
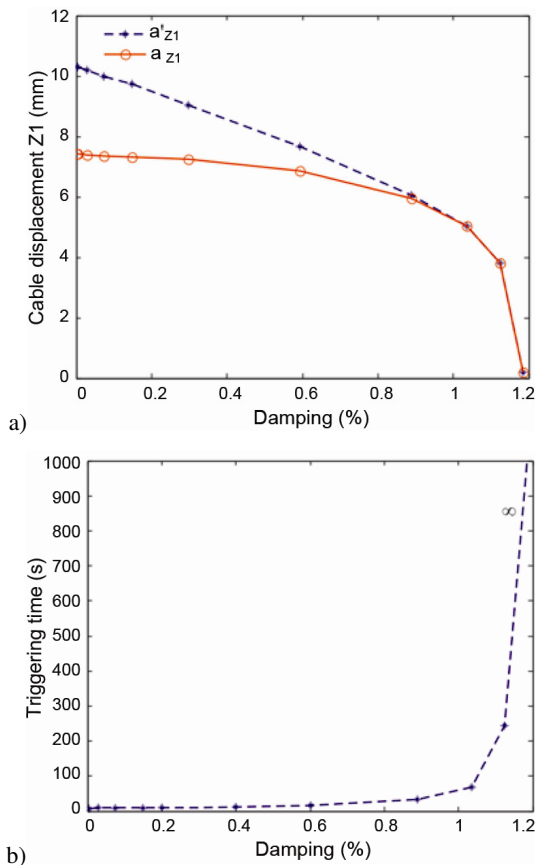

Fig. 8. Maximum mid span cable displacement (a); necessary time to trigger first order parametric resonance, as a function of the natural damping (b)

3.2. Second order parametric excitation

The same numerical experiences described in section 3.1 are reproduced for $k = 2$, which corresponds to a frequency of excitation equal to the fundamental frequency of the cable. Investigating the dynamic response of the cable under this particular frequency of excitation, allows studying the second order parametric excitation. The time evolution of the mid span vertical displacement ($Z1$) is plotted for this case in Fig. 9a. Large vertical oscillations of the cable at mid span are also observed but without any change of the cable vibration frequency (Fig. 9b).

By changing the frequency of excitation near the initial frequency of excitation (ω_0) and plotting in Fig. 10a the maximum amplitude of both transient (a'_{Z1}) and steady state (a_{Z1}) in-plane response of the cable at mid span as a function of the frequency of excitation, one can see that the shape of the frequency-response curve is similar to that of a single-d.o.f. model under primary external resonance (Berlioz, Lamarque 2005). The parametric resonance occurs in a frequency band larger than for the case of first order parametric excitation. This means that the risk of producing a second order parametric resonance is higher than for a first order parametric excitation.

The transient in-plane response appears for all frequencies near the first cable frequency whereas for the case of first order parametric excitation the transient response disappears for frequencies close to ω_{min}.
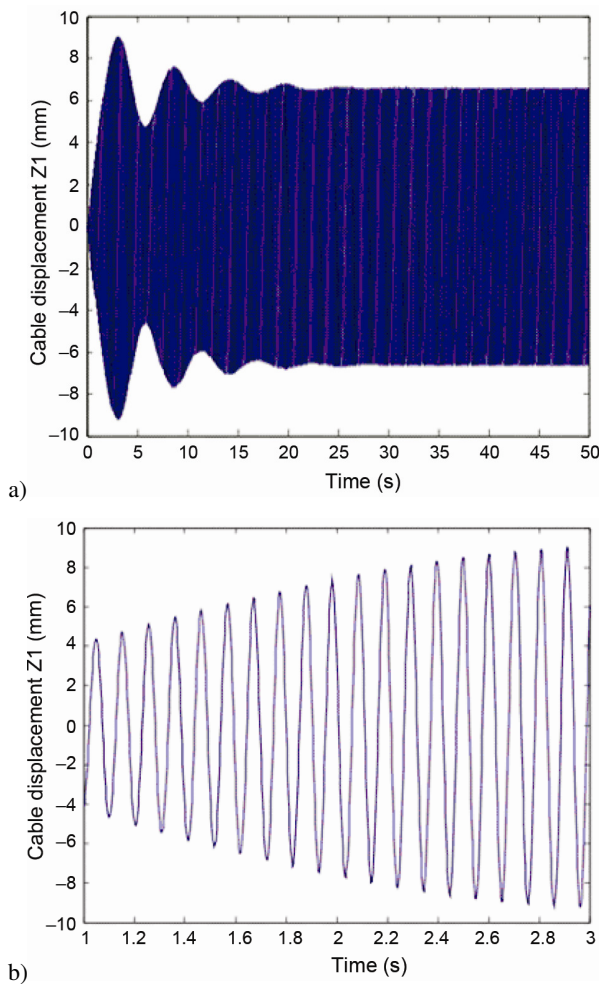
a)

b)

Fig. 9. In-plane cable vibration at mid span under second order parametric excitation as a function of time (a); zoom between 1 s and 3 s (b)

a)

b)

Fig. 10. Maximum mid span cable displacement (a); necessary time to trigger second order parametric resonance as a function of the frequency of excitation (b)

The necessary time to trigger parametric resonance is also plotted as a function of the frequency of excitation (Fig. 10b). Second order parametric resonance triggers slower when the frequency of excitation is very close to ω_0 but triggers faster when frequency changes around ω_0. The second order parametric resonance triggers faster than the first parametric excitation.

Next, the effect of the amplitude of excitation on the second order parametric resonance is studied. The maximum mid span cable displacements are plotted as a function of the amplitudes of excitation and compared to those obtained from first order parametric excitation (Fig. 11a). This figure reveals that the second order parametric excitation doesn't need threshold amplitude to trigger and that the maximum response increases by increasing the amplitude of excitation. The second order parametric excitation produces a maximum in-plane displacement Z1 larger than the one under first order excitation up to an amplitude of excitation equal to 0.08 mm. Beyond this value, the first order parametric excitation produces the largest oscillations of the cable at mid span. Fig. 11b shows that the time needed to trigger a second order parametric resonance is faster than the first order parametric resonance.
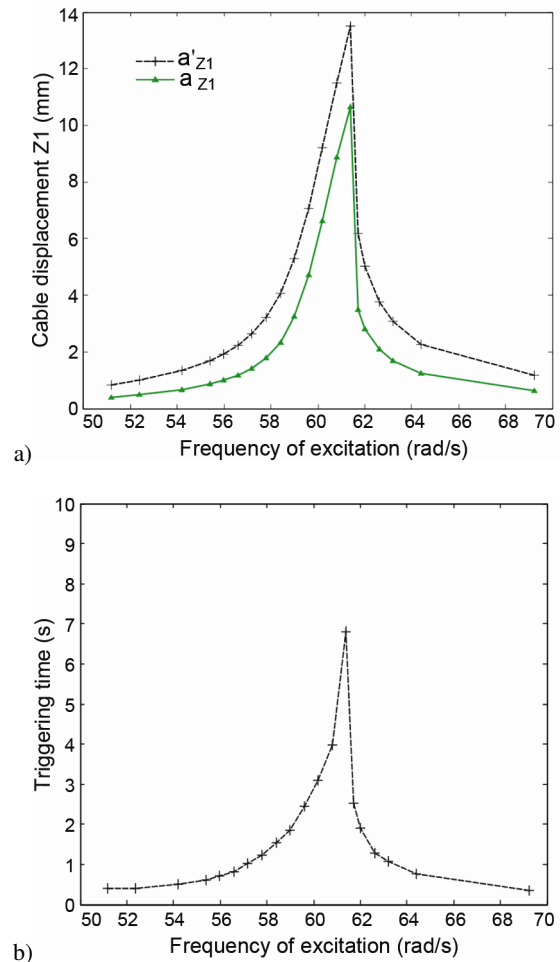
Finally, the effect of natural damping on the second order parametric resonance is investigated. The amplitude and the frequency of excitation are fixed respectively to $a_u = 0.1$ mm and $\omega_{excit} = \omega_0$. The natural damping ($\xi_0 = 0.3\%$) of the cable is changed from small to high values. The maximum amplitude of both transient and steady state in-plane response of the cable at mid span are plotted as a function of the damping. Fig. 12a shows that the cable vibration induced by second order parametric excitation needs higher values of damping ratio to be reduced, unlike the first order parametric excitation. The damping has no significant effect on the time needed to trigger a second order parametric resonance as shown in Fig. 12b.

4. Conclusions

A nonlinear model of a cable with small sag and support movement is used to investigate the nonlinear oscillations of a single-d.o.f. planar cable under first and second order parametric excitations. Maximum amplitudes of the in-plane transient and steady state cable responses at its mid span are examined as a function of amplitude and frequency of excitation as well as natural damping of the cable. It had been shown that the existence of parametric
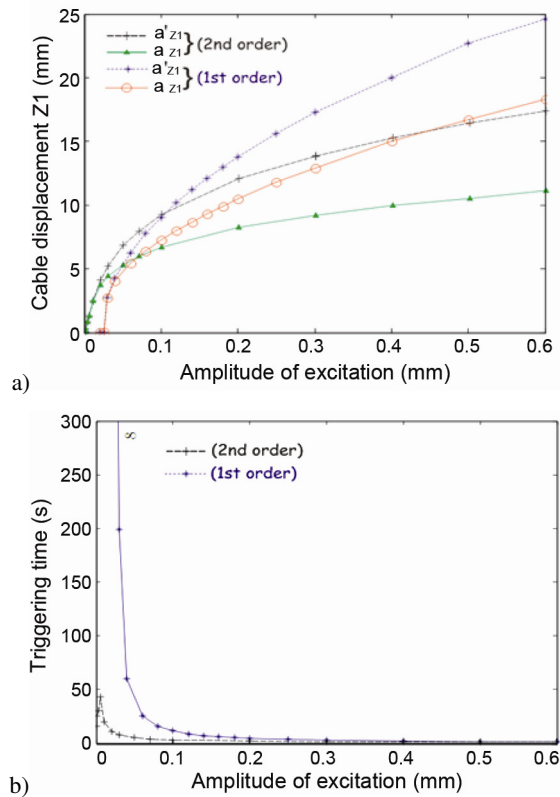
Fig. 11. Maximum mid span cable displacements (a); necessary time to trigger parametric resonance; as a function of the amplitude of excitation (b)
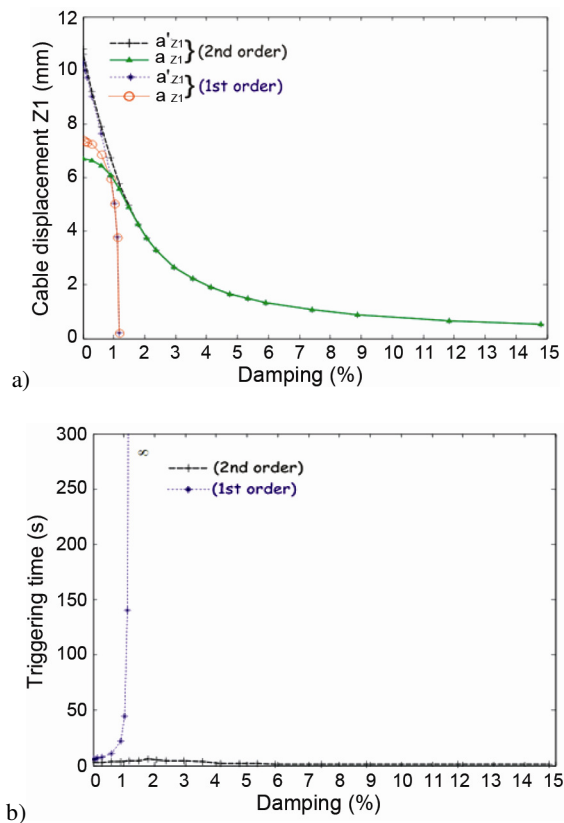

Fig. 12. Maximum mid span cable displacements (a); necessary time to trigger parametric resonance, as a function of the natural damping (b)

excitation depends on these three parameters. The first order parametric resonance triggers faster as the amplitude of excitation increases, the natural damping decreases and the frequency of excitation is exactly twice that of the fundamental frequency of the cable. The second order parametric resonance is more difficult to damp and triggers faster than the first order one for any amplitude of excitation.

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Appendix

The Irvine's parameter (Irvine 1992) involving cable elasticity and geometry is defined by:

$$\lambda^2 = \left(\frac{\rho g\, l \cos\theta}{\sigma^s}\right)^2 \frac{E}{\sigma^s}, \tag{A.1}$$

where E is the cable modulus of elasticity.

The effective modulus of elasticity is defined as:

$$E_q = \frac{1}{1 + \dfrac{\lambda^2}{12}} E. \tag{A.2}$$

The kinetic energy of the cable is:

$$K_c = \frac{1}{2}\rho Al \left\{ \left(\frac{1}{1+\frac{\lambda^2}{12}}\right)^2 \left(\frac{1}{3} + \frac{\lambda^2}{18} + \frac{\lambda^4}{432} + \frac{\lambda^2}{120}\frac{E}{\sigma^s}\right)(\dot{u}_b - \dot{u}_a)^2 + \right.$$

$$\dot{u}_b\dot{u}_a + \frac{1}{3}(\dot{v}_b - \dot{v}_a)^2 + \dot{v}_b\dot{v}_a + \frac{1}{3}(\dot{w}_b - \dot{w}_a)^2 + \dot{w}_b\dot{w}_a -$$

$$\frac{1}{12}\frac{1}{1+\frac{\lambda^2}{12}}\sqrt{\frac{\sigma^s}{E}}\left[1 + \frac{\lambda^2}{12} + \frac{E}{\sigma^s}\right](\dot{w}_b - \dot{w}_a)(\dot{u}_b - \dot{u}_a) +$$

$$\frac{\lambda^2}{120}\frac{E}{\sigma^s}(\dot{w}_b - \dot{w}_a)^2 - \frac{1}{6}\frac{1}{1+\frac{\lambda^2}{12}}\sqrt{\frac{\sigma^s}{E}}(\dot{u}_b - \dot{u}_a)\ \dot{w}_a -$$

$$\frac{\lambda}{6}\sqrt{\frac{\sigma^s}{E}}(\dot{w}_b - \dot{w}_a)\ \ddot{u}_a + \sum_n [\frac{2}{n\pi}(\dot{v}_a + (-1)^{n+1}\dot{v}_b)\dot{y}_n -$$

$$2\frac{\gamma}{\sigma^s}\frac{l}{n\pi}\frac{(1+(-1)^{n+1})}{(n\pi)^2}\frac{1}{\sigma^s}\frac{E_q}{\sigma^s}(\dot{u}_b - \dot{u}_a)\dot{z}_n +$$

$$\frac{1}{2}\dot{y}_n^2 + \frac{1}{2}\dot{z}_n^2 + \frac{2}{n\pi}(\dot{w}_a + (-1)^{n+1}\dot{w}_b)\dot{z}_n]\}. \quad (A.3)$$

In our case, the strain energy can be written as:

$$\varepsilon(x,t) = \varepsilon_{xx}(x,t) = \frac{\partial u^q}{\partial x} + \frac{1}{2}[(\frac{\partial u^q}{\partial x})^2 +$$

$$(\frac{\partial(v^q + v^d)}{\partial x})^2 + (\frac{\partial(w^s + w^q + w^d)}{\partial x})^2]. \quad (A.4)$$

Since we assume that the strain (tension) is constant along the cable span, the dynamical strain is a function of time only; averaging (x,t) over the cable span, we get:

$$\overline{\varepsilon}(t) = \frac{1}{l}\int_0^l \varepsilon(x,t)\,dx\ ; \quad (A.5)$$

$$\overline{\varepsilon}(t) = \frac{T_0}{EA} + \frac{E_q}{E}\frac{u_b - u_a}{l} + \frac{\gamma}{T_0}\frac{A}{n\pi}\sum_n[\frac{z_n(t)}{n\pi}(1+(-1)^{n+1})] +$$

$$\frac{1}{2}\sum_n(y_n(t)^2\frac{n^2\pi^2}{2l^2}) + \frac{1}{2}\sum_n(z_n(t)^2\frac{n^2\pi^2}{2l^2}) +$$

$$\frac{E_q A^2 \gamma}{T_0^2}\frac{u_b - u_a}{l}\sum_n\frac{z_n(t)}{n\pi}\frac{((-1)^n - 1)}{n\pi} +$$

$$\frac{(u_b - u_a)^2}{2l^2}[1 + \frac{E_q}{\sigma^s}\frac{\lambda^2}{12+\lambda^2}] + \frac{(v_b - v_a)^2}{2l^2} +$$

$$\frac{(w_b - w_a)^2}{2l^2}[1 + \frac{\lambda^2}{12}\frac{\sigma^s}{E}]. \quad (A.6)$$

So the strain energy due to elastic elongation of the cable is:

$$U_{c,d} = \frac{EA}{2}\int_0^l \varepsilon(x,t)^2\,dx = \frac{EAl}{2}\overline{\varepsilon}(t)^2. \quad (A.7)$$

The gravitational potential energy is:

$$U_{c,g} = \rho A\,g\,l\int_0^l\left[-(w^s + w^q + w^d)\cos\theta + u^q\sin\theta\right]dx =$$

$$-\rho\ gA\cos\theta\ \frac{\gamma}{12}\frac{l^3}{\sigma^s} + \frac{1}{2}\rho A\ g\ l\sin\theta\ (u_b + u_a) +$$

$$\frac{1}{12}\frac{\rho\ A^3 gE_q\ \gamma\ l^2\ \cos\theta}{T_0^2}\ (u_b - u_a) -$$

$$\frac{1}{2}\rho A\ g\ l\cos\theta\ (w_b + w_a) -$$

$$\frac{1}{12}\frac{\rho\ A^2 g\ \gamma\ l^2\sin\theta}{T_0}\ (w_b - w_a)$$

$$-\rho A\ g\ l\cos\theta\ \sum_n\frac{(1+(-1)^{n+1})}{n\pi}\ z_n. \quad (A.8)$$

The Total potential energy is:

$$U = U_{c,g} + U_{c,d}. \quad (A.9)$$

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