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ACCURACY ESTIMATES IN FREE VIBRATION ANALYSIS

R.Baušys

1. Introduction

The construction of a posteriori error estimates to control numerical simulation procedure is very attractive subject for the researchers in the field of the finite element method. By now a considerable success has been achieved mainly on problems of linear elliptic type, such as linear elastostatics and stationary heat conduction problems, see e.g. Babuška et al. [1] and Oden [2]. Recently, an objective methodology for assessing the reliability of a posteriori error estimators has been developed by Babuška et al. [3]. For free vibration problems, however, theory and computer implementation for error estimation are far from completed and need to be further exploited.

When standard Galerkin finite element approximation is used, a priori error estimation is available for the generalised eigenvalue problem [4,5]. However, from the computational view-point applications of a priori error estimates, based upon knowledge of the general properties of solutions for the model equations and the approximation properties of the discretization methods, are in practice very limited as they provide only a qualitative assessment of the error and the asymptotic rate of convergence when the number of degrees of freedom in the approximation tends to infinity. A priori estimates provide indications of the error based upon upper bounds for Sobolev norms of the solution. However, they usually do not provide much information about the actual error in the discrete approximation. Instead, more precise information to evaluate the actual discretization error of the eigenfrequencies can be gained only by a posteriori error estimates which utilise the finite element solution itself.

New methods to improve accuracy of the eigenfrequencies of the discretized engineering structure and to give error bounds have recently appeared. Friberg et al. [6] propose an error estimate and an adaptive procedure for eigenpairs computation within a framework of the hierarchical finite element method. This approach represents an iterative procedure for the selection of hierarchical refinements based on the activation of positive maximum indicators which approximate the relative change in an eigenfrequency. Avrashi and Cook [7] present an approach for the error estimation for C^0 eigenproblems by smoothing gradient (stress) and primary variable (displacement) fields. The improved eigenfrequency for the error estimate is obtained from the Rayleigh quotient with the modified field of the primary variables and its gradients by use of some users defined parameters.

The Superconvergent Patch Recovery (SPR) technique, originally proposed by Zienkiewicz and Zhu [8,9] has been applied to static and dynamic wave-propagation problems in order to improve the solution of stresses and to estimate the spatial discretization errors [10-11]. In this paper we focus our attention on the application of the SPR technique to free vibration problems for improving eigenmodes and eigenfrequencies giving not only an improved solution but also an error estimation of the eigenfrequencies. The proposed SPRD technique [12] is based on a higher order accuracy displacement field fitted to superconvergence values of the original finite element solution in a least square sense over local element patches. In order to maintain locality of the least squares fit we use a reduced element patch with the size 2h in the present patch recovery technique for displacements. From the higher order accuracy field both improved kinetic energy and strain energy can be calculated and thus an improved eigenfrequency can be obtained. A separate patch recovery must be made for each eigenmode and eigenfrequency.

Numerical experiments show for the eigenfrequencies an improved accuracy and an improved convergence rate of the error of the improved eigenfrequencies compared to the FE solution.

2. Model problem and basic equations

In free vibration analysis, the equations of motion for linear elastic continuum, in small deformation theory, can be expressed by

$$\rho\ddot{u} - \tilde{\nabla}^T D \tilde{\nabla} u = 0 \,, \quad in\ \Omega \tag{1}$$

with boundary conditions

$$u(x) = u_b \quad , \quad on\ \Gamma_u \tag{2}$$

$$\tilde{\nabla}_n^T D \tilde{\nabla} u(x) = \sigma_b \quad , \quad on\ \Gamma_\sigma \tag{3}$$

where $\tilde{\nabla}$ is the strain operator, $\tilde{\nabla}_n$ is the boundary operator, $\boldsymbol{D}$ is the constitutive elasticity matrix and ρ is the mass density. The Ω is the spatial domain with the boundary

$\Gamma = \Gamma_u \cup \Gamma_\sigma$ $(\Gamma_u \cap \Gamma_\sigma = \varnothing)$, where Γ_u is the boundary with essential boundary conditions and Γ_σ is the boundary with natural boundary conditions.

By finite element approximation

$$u(x,t) \approx u^h(x,t) = N(x)\tilde{u}(t) \qquad (4)$$

where $N(x)$ contains the basis functions of polynomial order p, and applying standard Galerkin procedure, weak form of equation (1) leads to a linear system of the differential equations of the semidiscrete form

$$[M]\ddot{\tilde{u}} + [K]\tilde{u} = 0 \qquad (5)$$

Assuming a harmonic behaviour

$$\tilde{u}(t) = ue^{i\omega t} \qquad (6)$$

equation (5) leads to

$$([K] - \lambda^h [M])u^h = 0 \, , \quad \lambda^h = (\omega^h)^2 \qquad (7)$$

where $[K]$ is the stiffness matrix and $[M]$ is the consistent mass matrix of the structure. Equation (7) is of the form of a generalised eigenvalue problem, with eigenvalues λ_i^h being equal to the squares of the eigenfrequencies ω_i^h of the appropriate mode u_i^h of vibration.

Determination of the eigenfrequency ω_i^h and the associated eigenmode u_i^h for each mode i of interest can be done by any convenient solution procedure of the generalised eigenvalue problem.

3. Eigenfrequency error estimation

For the elliptic eigenvalue problem of order $2m$, an a priori error estimation of the eigenfrequencies for standard Galerkin finite element approximation is given as [4,5]

$$\omega_i \leq \omega_i^h \leq \omega_i + C_1 h^{2(p+1-m)} \omega_i^{(p+1)/m} \qquad (8)$$

and the L_2 - estimate for the eigenmodes

$$\left\| u_i^h - u_i \right\|_0 \leq C_2 h^\sigma \omega_i^{(p+1)/m} \tag{9}$$

where $\sigma = min(p+1, 2(p+1-m))$, C_1 and C_2 are positive constants independent of h and ω_i, h is the maximum element size, p is the degree of complete polynomial appearing in the element basis functions and $2m$ is the order of the differential operator $\tilde{\nabla}$.

The appearance of powers of the eigenfrequencies on the right hand sides of eq.(8) and eq.(9) suggests that higher eigenfrequencies of finite element solution give not reasonable approximations to the corresponding exact eigenfrquencies. From eq. (8) we can observe that finite element approximations are upper bounds for eigenfrequencies $\omega_i^h \geq \omega_i$ only if the standard Galerkin rules are employed (a consistent mass matrix is used).

The a priori error estimate gives us a qualitative assessment and the asymptotic rate of convergence which we can anticipate by finite element solutions but it is of limited use to evaluate an actual numerical error in the discrete approximation.

Since the exact solutions are generally unknown, the exact error in the eigenfrequencies can only be estimated a posteriori by utilising the information which can be extracted from the original finite element approximation. The essence of the error estimate of the post-processing type is to replace the exact solution by an improved one ω_i^*

$$\Delta\varpi_i^h = \omega_i^h - \omega_i^* \tag{10}$$

In order to assess numerically the quality of the performance of the eigenfrequency error estimator we introduce an effectivity index giving the ratio of the estimated errors in the eigenfrequency to the exact ones as

$$\theta_i = \frac{\Delta\varpi_i^h}{\Delta\omega_i^h} \tag{11}$$

where $\Delta\omega_i^h$ is the discretization error in appropriate eigenfrequency of the finite element solution which can be expressed as

$$\Delta\omega_i^h = \omega_i^h - \omega_i \tag{12}$$

and ω_i is an exact eigenfrequency of the structure. A reliable error estimate is said to be asymptotic exact if $\theta_i \to 0$, when $h \to 0$. Hence, the effectiveness of the eigenfrequency error estimator depends mainly on the superior accuracy and convergence properties of the postprocessed eigenfrequency ω_i^*. Alternatively, this condition can be expressed as

$$\left(1 - \frac{\Delta\omega_i^*}{\Delta\omega_i^h}\right) \le \theta_i \le \left(1 + \frac{\Delta\omega_i^*}{\Delta\omega_i^h}\right) \tag{13}$$

where $\Delta\omega_i^*$ is the error in the postprocessed eigenfrequency which can be expressed as

$$\Delta\omega_i^* = \omega_i^* - \omega_i \tag{14}$$

Assuming that the true error $\Delta\omega_i$ has $C_1 h^{2(p+1-m)}\omega_i^{(p+1)/m}$ rate of the convergence and the error of the postprocessed solution $\Delta\omega_i^*$ has $C_* h^{2(p+1-m+\alpha)}\omega_i^{(p+1)/m}$ rate of the convergence for some superconvergent solution with $\alpha \ge 0$, we readily obtain

$$1 - \frac{C_* h^{2(p+1-m+\alpha)}\omega_i^{(p+1)/m}}{C_1 h^{2(p+1-m)}\omega_i^{(p+1)/m}} \le \theta_i \le 1 + \frac{C_* h^{2(p+1-m+\alpha)}\omega_i^{(p+1)/m}}{C_1 h^{2(p+1-m)}\omega_i^{(p+1)/m}} \tag{15}$$

Replacing the constant C_*/C_1 by some constant C we may simplify eq.(15) as

$$1 - C h^{2\alpha} \le \theta_i \le 1 + C h^{2\alpha} \tag{16}$$

From eq.(16) follows that the postprocessed eigenfrequencies should exhibit superior accuracy and convergence properties than the eigenfrequencies of the original finite element approximation.

4. Element patch recovery technique

The original finite element solution of the eigenfrequency can be expressed using Rayleigh quotient as follows

$$(\omega_i^h)^2 = \frac{(u_i^h)^T K u_i^h}{(u_i^h)^T M u_i^h} \tag{17}$$

A new improved eigenfrequency will be of the form

$$(\omega_i^*)^2 = \frac{\sum\limits_K \int\limits_{\Omega_e} (\tilde{\nabla} u_i^*)^T D \tilde{\nabla} u_i^* \, dx}{\sum\limits_K \int\limits_{\Omega_e} (u_i^*)^T \rho u_i^* \, dx} \tag{18}$$

where K is summed over the total number of elements, u_i^* is a displacement field over the element which has a higher order of accuracy. The recovered displacement field of the eigenmode u_i^* will be determined by the SPRD technique, described below.

The SPR technique is based on the fact that for finite element solutions there exist certain points in each element at which the prime variables (displacements) and the derivatives (stresses) have superior accuracy to that found globally. These points are called the superconvergent points of the finite element solution.

In the original SPR technique [8,9] stresses are accurately recovered by fitting polynomials over local nodal patch to the stresses at superconvergent points of the finite element solution. In one dimensional problems, the superconvergent points were somewhat easy and always existed irrespective of the mesh uniformity. The situation is not that simple for the higher dimensional problems. Mackinnon and Carey [13] dealt with Taylor series analysis in higher dimensions to arrive at the conclusion: the midside points of linear elements are superconvergent points for the derivatives in tangential directions. For the quadratic elements Andreev and Lazarov [14] have shown that Gauss points on the element edges are the superconvergent points for the derivatives in tangential directions. So, original SPR technique do not use a full superconvergent information at the sampling points for stresses and do not provide the local projection for the displacement field that is necessary for the improvement of the appropriate eigenfrequency in free vibration analysis. From the expression of the Rayleigh quotient it is clear that to improve eigenfrequency it is necessary to determine a new displacement field of higher accuracy of the corresponding eigenmode. So our attention will be focused on a SPRD technique for the prime variables of the finite element approximation. It has been known that the nodal points of the finite element approximation are found to be the exceptional points at which the prime variables (displacements) have higher order accuracy with respect to the global accuracy [15].

The main idea of the proposed recovery procedure consists of a least squares fit of a local polynomial to displacement values at higher accuracy points. The recovered displacement field of the corresponding eigenmode $\boldsymbol{u}_i^*$ over an element $\tau \in T_h$ is constructed as

$$\boldsymbol{u}_i^*(\boldsymbol{x}) = \sum_r \boldsymbol{N}_r^*(\boldsymbol{x})(\boldsymbol{u}_r^*)_i + \sum_s \boldsymbol{N}_s^*(\boldsymbol{x})(\boldsymbol{u}_s^*)_i \tag{19}$$

where *r* is used to denote finite element τ nodes, s denotes additional nodes of the element of the recovered displacement field, $\boldsymbol{N}_r^*(\boldsymbol{x})$ and $\boldsymbol{N}_s^*(\boldsymbol{x})$ are local basis functions of the order $p+1$ associated with the original element nodes and the additional ones, respectively.

The nodal values of the original finite element displacements are assumed fixed $(\boldsymbol{u}_r^*)_i \equiv (\boldsymbol{u}_r^h)_i$ and recovered displacement values $(\boldsymbol{u}_s^*)_i$ at the additional nodes are obtained by solving least squares problem in the element patch Ω_τ:

Find $\boldsymbol{u}_i^* \in \mathcal{P}_{p+1}$ such that

$$J_{\Omega_\tau}(\boldsymbol{u}_i^*) = \min_{(\boldsymbol{u}^{f*})_i \in \mathcal{P}_{p+1}} J_{\Omega_\tau}(\boldsymbol{u}^{f*})_i \tag{20}$$

where

$$J_{\Omega_\tau}(\boldsymbol{u}^{f*})_i = \sum_{j=1}^{ns} w_j^2 \boldsymbol{R}_u^T(\boldsymbol{x}_j)\boldsymbol{R}_u(\boldsymbol{x}_j) \tag{21}$$

where the residual $\boldsymbol{R}_u(\boldsymbol{x}_j)$ is defined by expression as

$$\boldsymbol{R}_u = (\boldsymbol{u}^{f*})_i - (\boldsymbol{u}_r^h)_i \tag{22}$$

and

$$(\boldsymbol{u}^{f*})_i = [Q(\boldsymbol{x})]\boldsymbol{b} \tag{23}$$

Here $\boldsymbol{x}_j$ is the location of j-th sampling point in the element patch Ω_τ, w_j is the weight assigned to the j-th sampling point and ns is the total number of the sampling (nodal)points in the element patch Ω_τ and $[Q(x)]$ contains the appropriate polynomial terms of $p+1$ order, b are unknown coefficients.

The unknown parameters b are determined by the solution of the weighted least squares problems which gives us the system of the linear equations

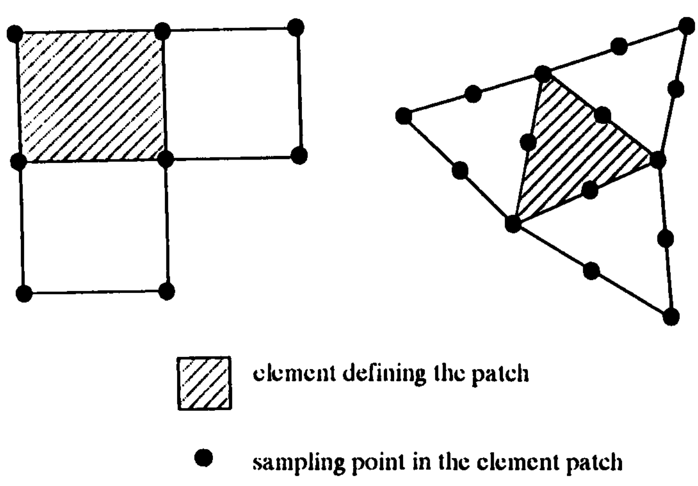
$$\left(\sum_j w_j^2 [Q(x)]^T [Q(x)]\right) b = \sum_j w_j^2 [Q(x)]^T u_j^h \tag{24}$$

where w is a weighting function.

For each element $\tau \in T_h$ we denote be $E(\tau)$ the set of its edges. So for each element τ the patch Ω_τ which consists of the part of elements surrounding the master element is denoted by

$$\Omega_\tau = \bigcup_{\tau \in E^*(\tau')} \tau' \tag{25}$$

For the triangular elements, set $E^*(\tau')$ coincides with $E(\tau)$, and for quadrilateral elements, set $E^*(\tau')$ consists of the adjacent edges connected to one of the nodes of the element ($E^*(\tau') \subset E(\tau)$) as shown in Figure 1.



a) Element patch for linear quadrilaterals b) Element patch for quadratic triangles

Figure 1. Possible element patches for linear quadrilateral and quadratic triangular elements

When the recovered displacement field u_i^* of the appropriate eigenmode is determined over all elements $\tau \in T_h$, we obtain an eigenmode of improved accuracy and an improved eigenfrequency can be found.

5. Numerical example

We consider in-plane vibrations of a rectangular flat plate of uniform thickness shown in Figure 2.

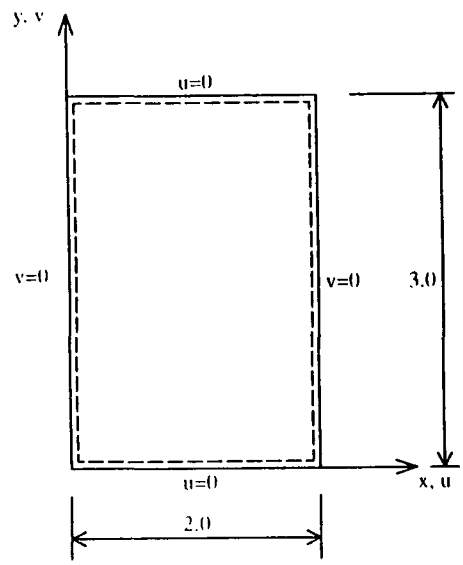


Figure 2. Geometry and boundary conditions of the flat rectangular plate

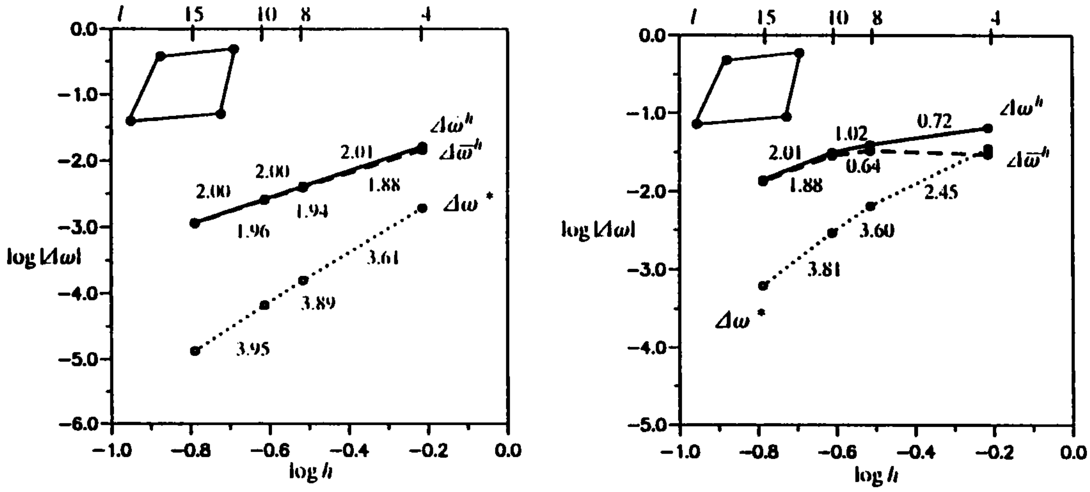
A plane stress condition and isotropic material with Poisson's ratio set to be 0.3 are assumed. We use nondimensionalized frequencies $\omega\sqrt{\frac{E}{\rho}}$, where E is Young's modulus and ρ is mass density, to study the quality of the proposed eigenfrequency error estimator. A sequence of four regular meshes with 4x4, 8x8, 10x10 and 15x15 are used for the numerical experiments for both quadrilateral and triangular elements.

For this problem an analytical solution is available [16]. The nondimensionalized frequencies of an analytical; solution can be expressed by

$$\omega^2 = \frac{\pi^2}{4(1-\upsilon^2)}\left(\frac{m^2}{c^2} + \frac{n^2}{d^2}\right)[(3-\upsilon)\pm(1+\upsilon)] \qquad (26)$$

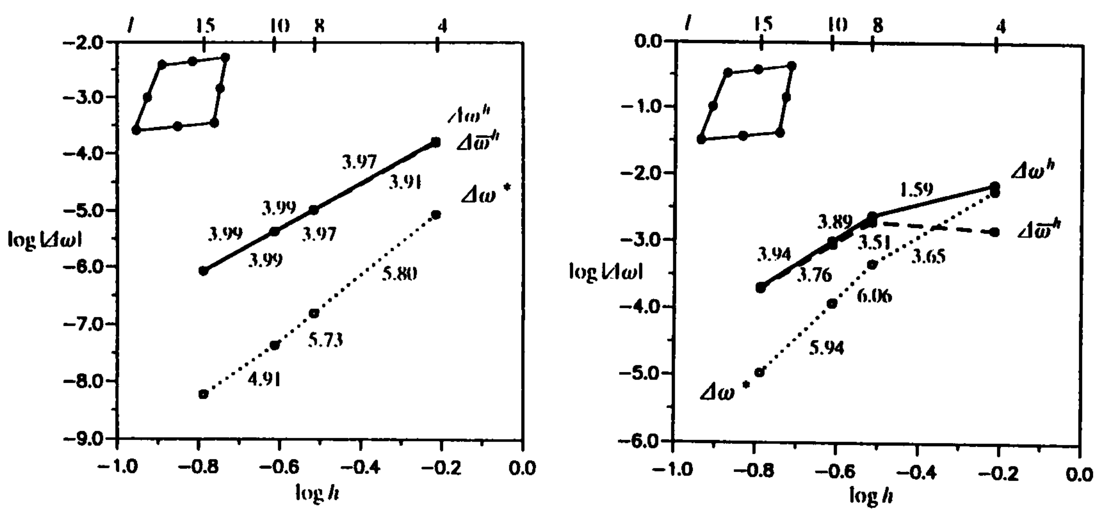
where $c=2.0$ and $d=3.0$, m and n are integer numbers.

The numerical results of the convergence of the 1-st and the 6-th eigenfrequencies for all quadrilateral and triangular elements are plotted in Figures 3-6.



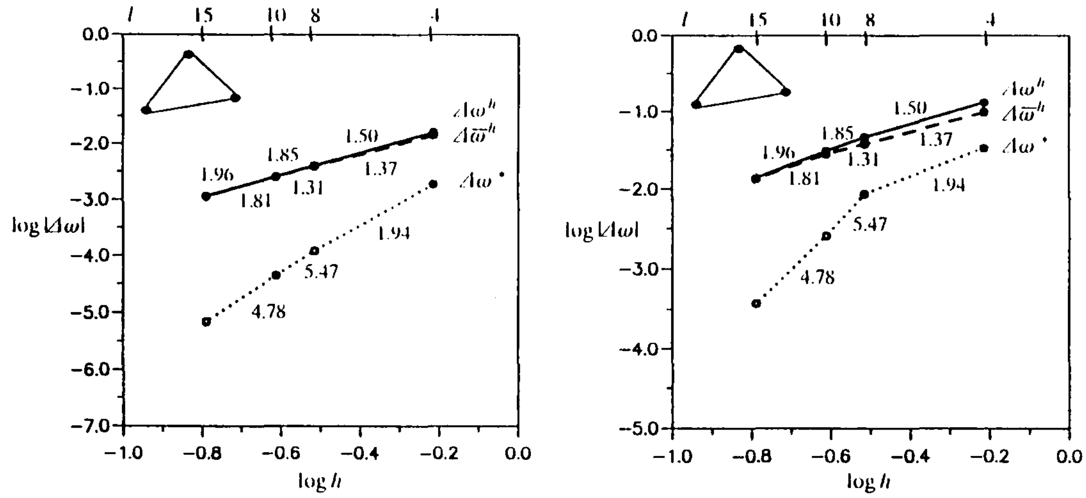
a) 1-st eigenfrequency b) 6-th eigenfrequency

Figure 3. Rate of convergence of the eigenfrequencies using linear quadrilateral elements



a) 1-st eigenfrequency b) 6-th eigenfrequency

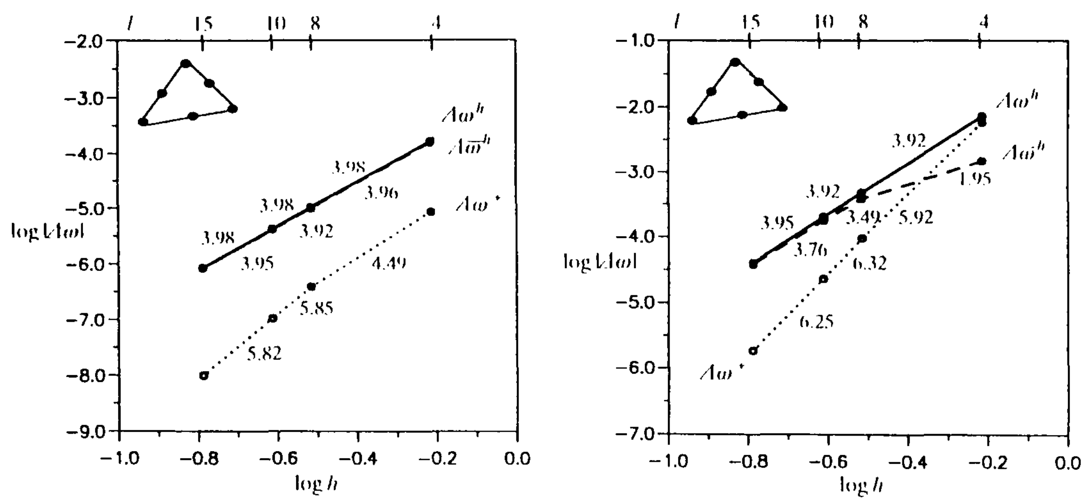
Figure 4. Rate of convergence of the eigenfrequencies using quadratic quadrilateral elements



a) 1-st eigenfrequency

b) 6-th eigenfrequency

Figure 5. Rate of convergence of the eigenfrequencies using linear triangular elements



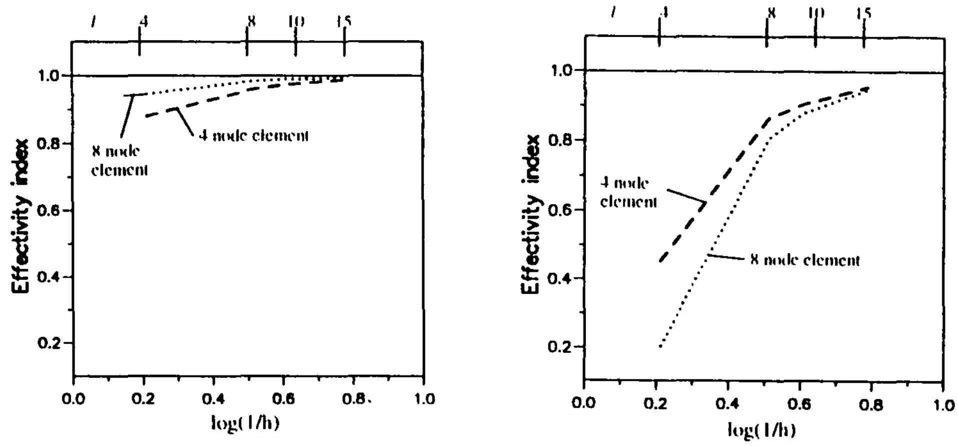
a) 1-st eigenfrequency

b) 6-th eigenfrequency

Figure 6. Rate of convergence of the eigenfrequencies using quadratic triangular elements

The error in eigenfrequencies of the original finite element solution (eq.12)), of the post-processed solution (eq.14)) and of the estimated error of the finite element solution (eq.10)) are presented in these figures. It is not difficult to observe that for sufficiently small h the recivered obtained by SPRD technique demonstrate approximately $O(h^{2(p+1)})$ rate of convergence for quadrilateral and triangular elements.

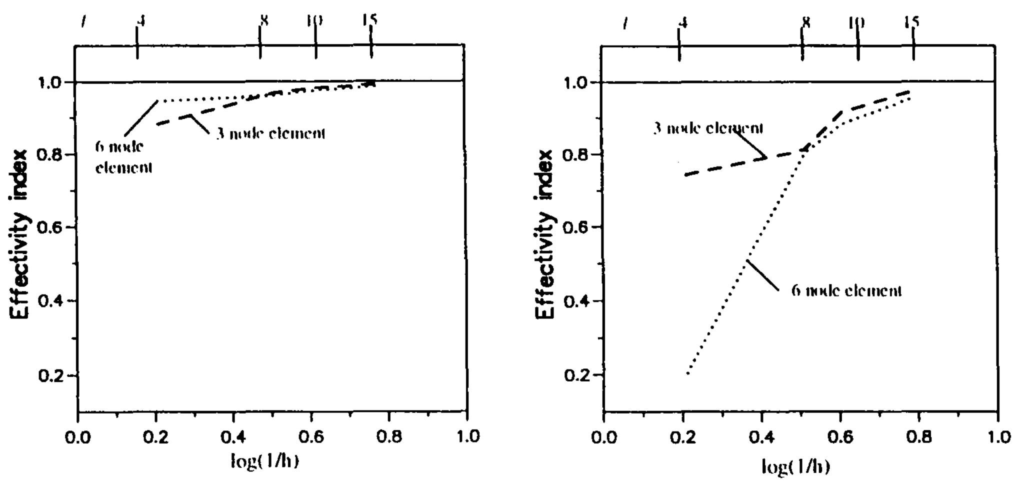
The convergence of the effectivity indices are plotted in Figures 7-8.



a) 1-st eigenfrequency

b) 6-th eigenfrequency

Figure 7. Effectivity indices for quadrilateral elements



a) 1-st eigenfrequency

b) 6-th eigenfrequency

Figure 8. Effectivity indices for triangular elements

We observe that the effectivity indices converge to one rapidly for all tested quadrilateral and triangular elements when the finite element mesh is refined. The numerical results show an asymptotic exactness of the proposed eigenfrequency error estimator.

6. Concluding remarks

The postprocessed error estimator introduced in this paper is applicable to problems other than free vibration analysis alone. The method can be used for almost any linear finite element discretization due to the fact that the proposed procedure is based on the ideas of approximation theory. The error estimation procedures are essential to construct adaptive finite element strategies. The mesh refinement strategies applied to elastostatics problems have been shown to be very effective [9,10]. Extension of such adaptive procedures to problems of free vibration analysis is in progress.

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PAKLAIDŲ NUSTATYMAS LAISVŲJŲ SVYRAVIMŲ UŽDAVINIUOSE

R.Baušys

S a n t r a u k a

Straipsnyje pateikiama skaičiavimo paklaidų nustatymo procedūra, skirta laisvųjų svyravimų analizės uždaviniams. Skaičiavimo paklaidos atsiranda dėl dviejų faktorių: a) baigtinių elementų diskretizacijos paklaidos; b) nuosavų reikšmių uždavinio sprendimo algoritmų paklaidos. Literatūroje plačiausiai yra tyrinėtas nuosavųjų reikšmių uždavinio sprendimo algoritmų tikslumas, tuo tarpu baigtinių elementų diskretizacijos įtaka gautų rezultatų tikslumui nėra plačiai nagrinėta.

Baigtinių elementų diskretizacijos paklaidos gali būti įvertintos išankstiniais paklaidų nustatymo būdais. Šie būdai, besiremiantys bendromis sprendinių bei diskretizacijos metodų aproksimacinėmis savybėmis, pateikia tiktai sprendinio kokybinį įvertinimą ir asimptotinį konvergavimo greitį, kai diskretinio modelio

laisvės laipsnių skaičius artėja į begalybę. Tačiau išankstiniai paklaidų nustatymo būdai nepateikia jokios informacijos apie diskretinės aproksimacijos tikrąją paklaidą. Tikroji diskretinio modelio paklaida gali būti įvertinta tiktai poprocesorinėmis procedūromis, kuriose yra tyrinėjamas gautasis baigtinių elementų sprendinys.

Nagrinėjama skaičiavimo paklaidų nustatymo procedūra paremta tuo faktu, jog pasinaudojant pradinio baigtinių elementų sprendinio superkonvergencinėmis savybėmis, galima gauti aukštesnės tikslumo klasės poprocesorinį sprendinį. Šio poprocesorinio sprendinio pagalba gali būti įvertinta tikroji baigtinių elementų sprendinio paklaida bei jos pasiskirstymas tyrinėjamoje konstrukcijoje.

Atlikti skaitiniai eksperimentai patvirtina, jog gautasis poprocesorinis sprendinys yra aukštesnės tikslumo klasės ir pateiktoji paklaidų nustatymo procedūra yra patikima ir efektyvi visai tyrinėtai dvimačių baigtinių elementų klasei.