Mathematical Modelling and Analysis
Volume 28, Issue 1, 118–145, 2023
https://doi.org/10.3846/mma.2023.14858

Approximation of Iterative Methods for Altering Points Problem with Applications

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Received May 4, 2021; revised December 13, 2022; accepted December 13, 2022

Abstract. In this paper, we consider and investigate an altering points problem involving generalized accretive mappings over closed convex subsets of a real uniformly smooth Banach space. Parallel Mann and parallel S-iterative methods are suggested to analyze the approximate solution of altering points problem. Consequently, some systems of generalized variational inclusions and generalized variational inequalities are also explored using the conceptual framework of altering points. Convergence of suggested iterative methods are verified by an illustrative numerical example.

Keywords: iterative methods, altering points problem.

AMS Subject Classification: 47H09; 49J4.

1 Introduction

Variational inequality theory was brought into existence by Hartman and Stampacchia [9] in 1966 as a tool to study partial differential equations, in particular, to study the theory of elliptical partial differential equations [23, 24]. In fact, this theory is a very natural generalization of the theory of boundary value problems which allows us to contemplate new problems arising from many fields of applied Mathematics, such as Mechanics, Physics, the theory of convex programming and the theory of control. In recent past, variational inclusions have been widely studied as the generalization of variational inequalities and optimization problems. Let H be a real Hilbert space, the variational inclusion problem is to find $\bar{x} \in \Omega$ such that

$$\bar{x} \in (A+M)^{-1}(0), \tag{1.1}$$

where $(A+M)^{-1}(0)$ is the set of zeros of $A+M$, $A : \Omega \to H$ and $M : H \to 2^H$ with $Dom(M) \subseteq \Omega$ are the monotone operators. Numerous problems emerging in diverse branches of mathematical analysis such as convex analysis, optimization, elasticity, image processing, biomedical sciences, mathematical physics, etc., can be formulated as variational inclusion problem (1.1). These applications drew the attention of many researchers and stipulated to suggest and analyze iterative methods.

In 1922, S. Banach gave a fundamental result known as Banach contraction principle which is stated as "every contraction mapping on a complete metric space has a unique fixed point". So far, numerous problems in nonlinear analysis involving contraction and nonexpansive mappings have been solved using this fundamental result. In recent past, the fixed point iterative schemes have been examined considerably to study monotone variational inequalities, problems from nonlinear analysis and applied mathematics such as initial and boundary value problems, image recovery problems, image restoration problems, image processing problems, for more details, see [12, 13, 25, 26, 32]. Note that, Mann-like iterative algorithms and its variant forms are efficiently applied for solving several nonlinear problems. Mann iteration method [14] is defined as follows: For a convex subset Ω of vector space X and self mapping A on Ω, the sequence $\{\bar{u}_n\}$ generated from an arbitrary initial point $\bar{u}_1 \in \Omega$ is estimated as:

$$\bar{u}_{n+1} = (1-\alpha_n)\bar{u}_n + \alpha_n A\bar{u}_n,$$

where $\{\alpha_n\}$ is a real sequence with assumptions $0 \leq \alpha_n \leq 1$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Later, Ishikawa [10] developed a generalized iterative method to approximate the fixed points of Lipschitz pseudocontractive mappings for compact convex subsets in Hilbert spaces. For a convex subset Ω of vector space X and self mapping A on Ω, for any arbitrary initial point $\bar{u}_1 \in \Omega$, the sequence $\{\bar{u}_n\}$ generated from Ishikawa iterative method is estimated as:

$$\begin{cases} \bar{u}_{n+1} = (1-\alpha_n)\bar{u}_n + \alpha_n A\bar{v}_n, \\ \bar{v}_n = (1-\beta_n)\bar{u}_n + \beta_n A\bar{u}_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of real numbers in $[0,1]$ with assumptions $0 \leq \alpha_n, \beta_n \leq 1$, $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n = \infty$. In 2007, Agarwal et al. [1] introduced and studied a more generalized iterative scheme for nearly asymptotically nonexpansive mappings in Banach spaces, namely S-iteration method.

Under some appropriate assumptions, for any arbitrary $\bar{u}_0 \in \Omega$ and the sequences $\{\alpha_n\}, \{\beta_n\}$ in (0,1), it is defined as follows:

$$\begin{cases} \bar{u}_{n+1} = (1-\alpha_n)A\bar{u}_n + \alpha_n A\bar{v}_n, \\ \bar{v}_n = (1-\beta_n)\bar{u}_n + \beta_n A\bar{u}_n. \end{cases}$$

Recently, Gursoy and Karakaya [7], introduced Picard S-iterative method to approximate the fixed points of contraction mappings. They investigated the solution of a delay differential equation using Picard S-iterative method. In [19], the author showed that S-iterative method has better rate of convergence than Picard and Mann iterative methods for a class of contraction mappings in metric spaces. He also studied minimization and split feasibility problems by using S-iteration process. After that number of problems have been solved using S-iterative method and its modified version, see [2, 3, 16, 22]. In recent past, Parallel iterative methods have been using by number of researchers with numerous applications, see, for example, [5, 6, 8, 11, 17, 31]. Following these ongoing research techniques, Sahu [20] studied the altering points problem of nonlinear mappings and presented the convergence analysis of the following parallel S-iterative method.

Let Ω_1 and Ω_2 be two nonempty closed convex subsets of a Banach space E; let $A_1 : \Omega_1 \to \Omega_2$ and $A_2 : \Omega_2 \to \Omega_1$ be two mappings. Then for any arbitrary $(\bar{u}_1, \bar{v}_1) \in \Omega_1 \times \Omega_2$ and $\alpha \in (0,1)$, the parallel S-iterative method is defined as follows:

$$\begin{cases} \bar{u}_{n+1} = A_2[(1-\alpha)\bar{v}_n + \alpha A_1\bar{u}_n], \\ \bar{v}_{n+1} = A_1[(1-\alpha)\bar{u}_n + \alpha A_2\bar{v}_n], \forall n \in \mathbb{N}. \end{cases}$$

Following the above stated facts and methodologies, it is worth to study system of altering points problem in real uniformly smooth Banach spaces. We propose parallel Mann and parallel S-iterative methods to establish convergence analysis of the considered iterative methods. The paper is arranged in the following order. Next section contains some fundamental results and preliminaries. Section 3 begins with formulation of the system of altering points problems followed by some special cases. We propose parallel Mann and parallel S-iterative methods and prove existence and convergence results under some suitable assumptions. In the last section, we explore some applications which include existence and convergence results for generalized systems of variational inclusions and inequalities. At last, an illustrative example is given to validate the existence result and convergence of the proposed iterative schemes.

2 Preludes and auxiliary results

Let E be a real Banach space with its dual space E^* and let $\mathcal{U} = \{\bar{x} \in E : \|\bar{x}\| = 1\}$ be the unit sphere. E is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists a constant $\delta > 0$ such that for any $\bar{x}, \bar{y} \in \mathcal{U}$, $\|\bar{x} - \bar{y}\| \geq \epsilon$ implies $\left\|\frac{\bar{x}-\bar{y}}{2}\right\| \leq 1 - \delta$. E is said to be smooth, if for each $\bar{x}, \bar{y} \in \mathcal{U}$,

$$\lim_{\tau \to 0} \frac{\|\bar{x} + \tau\bar{y}\| - \|\bar{x}\|}{\tau} \tag{2.1}$$

exists and the norm on E is called Gâteaux differentiable. The norm on E is said to be uniformly Frechet differentiable norm, if the limit (2.1) is attained uniformly for $\bar{x}, \bar{y} \in \mathcal{U}$. In this case, E is said to be uniformly smooth. The modulus of smoothness $\rho_E : [0, \infty) \to [0, \infty)$ is defined as

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2}(\|\bar{x} + \bar{y}\| + \|\bar{x} - \bar{y}\|) - 1 : \bar{x} \in \mathcal{U}, \|\bar{y}\| \leq \tau \right\}.$$

The Banach space E is known as uniformly smooth, if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$. For any fixed $q \in (1, 2]$, E is called q-uniformly smooth, if there exists a constant $c > 0$ satisfying $\rho_E(\tau) \leq c\tau^q$ for all $\tau > 0$. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(\bar{x}) = \{f \in E^* : \langle \bar{x}, f \rangle = \|\bar{x}\|^q, \|f\| = \|\bar{x}\|^{q-1}\} \text{ for all } \bar{x} \in E.$$

For $q = 2$, J_2 is called the normalized duality mapping which is usually denoted by J. It is well known that J is single-valued, if E is smooth. Xu [30] proved the following fundamental result in 2-uniformly smooth Banach spaces which is quite applicable in nonlinear analysis.

Lemma 1. *[30] Let E^* be a dual space of 2-uniformly smooth Banach space E. Then*

$$\|\bar{x} + \bar{y}\|^2 \leq \|\bar{x}\|^2 + 2c^2\|\bar{y}\|^2 + 2\langle \bar{y}, J(\bar{x}) \rangle, \forall \bar{x}, \bar{y} \in E,$$

where $c > 0$ is a real constant and $J : E \to E^*$ is a normalized duality mapping.

If E is a real Banach space with norm $\|\cdot\|$. Then the norm $\|\cdot\|_*$ on $E \times E$ defined by

$$\|(\bar{x}, \bar{y})\|_* = \|\bar{x}\| + \|\bar{y}\|, \forall \bar{x}, \bar{y} \in E \tag{2.2}$$

is a complete normed space.

Let Ω be a nonempty subset of a Banach space E. A mapping $Q_\Omega : E \to \Omega$ is said to be sunny, if

$$Q_\Omega(Q_\Omega(\bar{x}) + t(\bar{x} - Q_\Omega(\bar{x}))) = Q_\Omega(\bar{x}), \forall \bar{x} \in E, t \geq 0.$$

A mapping Q_Ω is called retraction, if $Q_\Omega(\bar{x}) = \bar{x}$ for all $\bar{x} \in \Omega$. Furthermore, Q_Ω is a sunny nonexpansive retraction from E onto Ω, if Q_Ω is a retraction from E onto Ω which is sunny as well as nonexpansive.

Lemma 2. *[18] Let Ω be a closed convex subset of a smooth Banach space E. Let $\mathcal{G}$ be a nonempty subset of Ω and $Q_\Omega : \Omega \to \mathcal{G}$ be a retraction. Then Q_Ω is sunny nonexpansive if and only if*

$$\langle \bar{x} - Q_\Omega(\bar{x}), J(\bar{z} - Q_\Omega(\bar{x})) \rangle \leq 0, \text{ for all } \bar{x} \in \Omega \text{ and } \bar{z} \in \mathcal{G}.$$

Lemma 3. *[29] Let $\{\vartheta_n\}$ be a nonnegative real sequence satisfying following inequality:*

$$\vartheta_{n+1} \leq (1 - \alpha_n)\vartheta_n + \bar{\varepsilon}_n, \quad \forall n \geq n_0,$$

where $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\bar{\varepsilon}_n = o(\alpha_n)$. Then $\lim_{n \to \infty} \vartheta_n = 0$.

Proposition 1. *[15] Let E be a uniformly smooth Banach space and $J : E \to 2^{E^*}$ be a normalized duality mapping. Then, for any $\bar{x}, \bar{y} \in E$,*

1. $\|\bar{x}+\bar{y}\|^2 \leq \|\bar{x}\|^2 + 2\langle \bar{y}, j(\bar{x}+\bar{y})\rangle, \ \forall j(\bar{x}+\bar{y}) \in J(\bar{x}+\bar{y})$;
2. $\langle \bar{x}-\bar{y}, j(\bar{x})-j(\bar{y})\rangle \leq 2\mathcal{C}^2\tau_E(4\|\bar{x}-\bar{y}\|/\mathcal{C})$, *where* $\mathcal{C} = \sqrt{(\|\bar{x}\|^2+\|\bar{y}\|^2)/2}$.

2.1 Altering points

DEFINITION 1. [20] Let Ω_1 and Ω_2 be two nonempty subsets of a metric space E; let $S : \Omega_1 \to \Omega_2$ and $T : \Omega_2 \to \Omega_1$ be the single-valued mappings. Then $\bar{x} \in \Omega_1$ and $\bar{y} \in \Omega_2$ are called altering points of S and T, if

$$S(\bar{x}) = \bar{y}, \quad T(\bar{y}) = \bar{x}.$$

We designate the set of altering points of the mappings $S : \Omega_1 \to \Omega_2$ and $T : \Omega_2 \to \Omega_1$ by

$$Alt(S,T) = \{(\bar{x},\bar{y}) \in \Omega_1 \times \Omega_2 : S(\bar{x}) = \bar{y} \text{ and } T(\bar{y}) = \bar{x}\}.$$

Example 1. Let $E = R, \Omega_1 = \Omega_2 = R_+$. Define $S : \Omega_1 \to \Omega_2$ and $T : \Omega_2 \to \Omega_1$ as $S(\bar{x}) = e^{\bar{x}}$ and $T(\bar{y}) = \ln \bar{y}$. Then $ST : \Omega_2 \to \Omega_2$ and $TS : \Omega_1 \to \Omega_1$ are self mappings such that each point of Ω_1 is a fixed point of TS and each point of Ω_2 is a fixed point of ST. Thus, $(\bar{x},\bar{y})$ are altering points of S and T.

In what follows, we establish an equivalence between altering points problem and system of generalized variational inequalities.

Lemma 4. *Let Ω_1 and Ω_2 be two nonempty closed convex subsets of a real smooth Banach space E. Let $Q_{\Omega_1} : E \to \Omega_1$ and $Q_{\Omega_2} : E \to \Omega_2$ be the sunny nonexpansive retractions. Let $S_1, T_1 : \Omega_1 \to E$ and $S_2, T_2 : \Omega_2 \to E$ be nonlinear mappings. Let $P_1, P_2 : E \to E$ be the single-valued mappings and $\lambda_1, \lambda_2 > 0$ be positive real numbers. Then the following assertions are equivalent:*

(i) $\bar{x} \in \Omega_1$ *and* $\bar{y} \in \Omega_2$ *are altering points of* $Q_{\Omega_2}[P_1 - \lambda_1(S_1 - T_1)]$ *and* $Q_{\Omega_1}[P_2 - \lambda_2(S_2 - T_2)]$.

(ii) $(\bar{x},\bar{y}) \in \Omega_1 \times \Omega_2$ *is a solution of the following system of generalized variational inequalities:*

Find $(\bar{x},\bar{y}) \in \Omega_1 \times \Omega_2$ *such that*

$$\begin{cases} \langle \lambda_1(S_1(\bar{x}) - T_1(\bar{x})) + \bar{y} - P_1(\bar{x}), J(P_1(y) - \bar{y})\rangle \geq 0, \quad \forall y \in \Omega_2, \\ \langle \lambda_2(S_2(\bar{y}) - T_2(\bar{y})) + \bar{x} - P_2(\bar{y}), J(P_2(x) - \bar{x})\rangle \geq 0, \quad \forall x \in \Omega_1. \end{cases} \tag{2.3}$$

Proof. $(i) \Rightarrow (ii)$. Assume that $(\bar{x},\bar{y}) \in \Omega_1 \times \Omega_2$ are altering points of $Q_{\Omega_2}[P_1 - \lambda_1(S_1 - T_1)]$ and $Q_{\Omega_1}[P_2 - \lambda_2(S_2 - T_2)]$. Therefore, we have

$$Q_{\Omega_2}[P_1 - \lambda_1(S_1 - T_1)](\bar{x}) = \bar{y}, \;\; Q_{\Omega_1}[P_2 - \lambda_2(S_2 - T_2)](\bar{y}) \quad = \bar{x}.$$

It follows from Lemma 2 that

$$\langle \lambda_1(S_1(\bar{x}) - T_1(\bar{x})) + \bar{y} - P_1(\bar{x}), J(P_1(y) - \bar{y})\rangle \geq 0, \quad \forall y \in \Omega_2,$$
$$\langle \lambda_2(S_2(\bar{y}) - T_2(\bar{y})) + \bar{x} - P_2(\bar{y}), J(P_2(x) - \bar{x})\rangle \geq 0, \quad \forall x \in \Omega_1.$$

$(ii) \Rightarrow (i)$. Assume that $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ is a solution of the system of generalized variational inequalities (2.3). Then

$$\langle \lambda_1(S_1(\bar{x}) - T_1(\bar{x})) + \bar{y} - P_1(\bar{x}), J(P_1(y) - \bar{y})\rangle \geq 0, \quad \forall y \in \Omega_2,$$
$$\langle \lambda_2(S_2(\bar{y}) - T_2(\bar{y})) + \bar{x} - P_2(\bar{y}), J(P_2(x) - \bar{x})\rangle \geq 0, \quad \forall x \in \Omega_1.$$

Again, it follows from Lemma 2 that

$$Q_{\Omega_2}[P_1 - \lambda_1(S_1 - T_1)](\bar{x}) = \bar{y}, \;\; Q_{\Omega_1}[P_2 - \lambda_2(S_2 - T_2)](\bar{y}) \;\; = \bar{x}.$$

Hence $\bar{x} \in \Omega_1$ and $\bar{y} \in \Omega_2$ are altering points of $Q_{\Omega_2}[P_1 - \lambda_1(S_1 - T_1)]$ and $Q_{\Omega_1}[P_2 - \lambda_2(S_2 - T_2)]$. $\square$

2.2 $\mathbf{H}(\cdot,\cdot)$-accretive mapping

DEFINITION 2. [27] A single-valued mapping $A : E \to E$ is said to be

(i) accretive, if $\langle A(x) - A(y), J(x - y)\rangle \geq 0, \forall x, y \in E$;

(ii) strictly accretive, if A is accretive and the equality holds if and only if $x = y$;

(iii) α-strongly accretive, if there exists a constant $\alpha > 0$ such that

$$\langle A(x) - A(y), J(x - y)\rangle \geq \alpha\|x - y\|^2, \; \forall x, y \in E;$$

(iv) relaxed (ϑ, ω)-cocoercive, if there exist constants $\vartheta, \omega > 0$ such that

$$\langle A(x) - A(y), J(x - y)\rangle \geq (-\vartheta)\|A(x) - A(y)\|^2 + \omega\|x - y\|^2, \; \forall x, y \in E;$$

(v) δ_A-Lipschitz continuous, if there exists a constant $\delta_A > 0$ such that

$$\|A(x) - A(y)\| \leq \delta_A\|x - y\|, \; \forall x, y \in E;$$

(vi) κ-contraction, if there exists a constant $0 < \kappa < 1$ such that

$$\|A(x) - A(y)\| \leq \kappa\|x - y\|, \; \forall x, y \in E.$$

DEFINITION 3. [27] Let $A, B : E \to E; H : E \times E \to E$ be the single-valued mappings. Then $H(\cdot, \cdot)$ is said to be

(i) ρ-mixed Lipschitz continuous with respect to A and B, if there exists a constat $\rho > 0$ such that

$$\|H(A(x), B(x)) - H(A(y), B(y))\| \leq \rho\|x - y\|, \; \forall x, y \in E;$$

(ii) α-generalized accretive with respect to A, if there exists a constant $\alpha \in \mathbb{R}$ such that

$$\langle H(A(x), u) - H(A(y), u), J(x-y)\rangle \geq \alpha \|x-y\|^2, \ \forall x, y, u \in E.$$

Similarly, we can define β-generalized accretivity of the mapping $H(\cdot,\cdot)$ with respect to B.

Remark 1. $H(A,\cdot)$ is said to be α-strongly accretive if $\alpha > 0$. If $E = \mathcal{H}$, the real Hilbert space then α-generalized accretivity coincides with α-generalized monotonicity.

DEFINITION 4. Let $A, B : \Omega \to E; H : E \times E \to E$ be three single-valued mappings. Let $M : E \to 2^E$ be a set-valued mapping. Then M is said to be

(i) accretive, if $\langle u - v, J(x-y)\rangle \geq 0$, $\forall x, y \in \Omega, \forall u \in M(x), v \in M(y)$;

(ii) m-accretive, if M is accretive and $(I + \lambda M)(E) = E, \ \forall \lambda > 0$;

(iii) $H(\cdot,\cdot)$-accretive with respect to A and B (or simply $H(\cdot,\cdot)$-accretive in the sequel), if M is accretive and $[H(A,B) + \lambda M](E) = E, \ \forall \lambda > 0$.

Remark 2. If $H(A,B) = H$, then $H(\cdot,\cdot)$-accretivity with respect to the mappings A and B coincides to H-accretivity. If $H = I$, the identity mapping then $H(\cdot,\cdot)$-accretivity with respect to the mappings A and B becomes m-accretivity.

Proposition 2. *[27] Let $A, B : \Omega \to E$ and $H : E \times E \to E$ be the single-valued mappings such that $H(\cdot,\cdot)$ is α, β-generalized accretive mapping with respect to A, B, respectively with $\alpha + \beta \neq 0$. Let $M : E \to 2^E$ be an $H(\cdot,\cdot)$-accretive mapping with respect to A and B. Then the operator $[H(A,B) + \lambda M]^{-1}$ is single-valued.*

DEFINITION 5. [27] Let $A, B : \Omega \to E$ and $H : E \times E \to E$ be the single-valued mappings such that $H(\cdot,\cdot)$ is α, β-generalized accretive mapping with respect to A, B, respectively with $\alpha + \beta \neq 0$. Let $M : E \to 2^E$ be an $H(\cdot,\cdot)$-accretive mapping with respect to A and B. For each $\lambda > 0$, the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M} : E \to E$ is defined by

$$R^{H(\cdot,\cdot)}_{\lambda,M}(x) = [H(A,B) + \lambda M]^{-1}(x), \ \forall x \in E.$$

Proposition 3. *[27] Let $A, B : \Omega \to E$ and $H : E \times E \to E$ be the single-valued mappings such that $H(\cdot,\cdot)$ is α, β-generalized accretive mapping with respect to A, B, respectively with $\alpha + \beta > 0$. Let $M : E \to 2^E$ be an $H(\cdot,\cdot)$-accretive mapping with respect to A and B. For each $\lambda > 0$, the resolvent operator $R^{H(\cdot,\cdot)}_{\lambda,M} : E \to E$ is $\frac{1}{\alpha+\beta}$-Lipschitz continuous, i.e.,*

$$\|R^{H(\cdot,\cdot)}_{\lambda,M}(x) - R^{H(\cdot,\cdot)}_{\lambda,M}(y)\| \leq \frac{1}{\alpha+\beta}\|x-y\|, \ \forall x, y \in E.$$

Lemma 5. *Let Ω_1 and Ω_2 be two nonempty closed convex subsets of a real 2-uniformly smooth Banach space E. Let $\Pi_{\Omega_1} : E \to \Omega_1$ be $\delta_{\Pi_{\Omega_1}}$-Lipschitz continuous mapping and $A_2, B_2, S_2, T_2 : \Omega_2 \to E$ be the single-valued mappings such that S_2 is δ_{S_2}-Lipschitz continuous, τ_2-strongly accretive and T_2 is δ_{T_2}-Lipschitz continuous. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively such that G is ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exists a constant $\lambda_2 > 0$ satisfying following condition:*

$$\begin{aligned}0 <&\delta_{\Pi_{\Omega_1}}(\sqrt{1-2(\alpha_2+\beta_2)+2c^2\rho_2^2}+\sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2}+\lambda_2\delta_{T_2})<1,\\ &1+2c^2\rho_2^2>2(\alpha_2+\beta_2), 1+2c^2\lambda_2^2\delta_{S_2}^2>2\lambda_{2\tau_2}.\end{aligned} \tag{2.4}$$

Then $\Pi_{\Omega_1}[G(A_2,B_2)-\lambda_2(S_2-T_2)] : \Omega_2 \to \Omega_1$ is a κ_1-contraction mapping, where $\kappa_1 = \delta_{\Pi_{\Omega_1}}(\sqrt{1-2(\alpha_2+\beta_2)+2c^2\rho_2^2}+\sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2}+\lambda_2\delta_{T_2})$.

Proof. Let $x, y \in \Omega_2$, then we have

$$\begin{aligned}\|[G(A_2,B_2)-&\lambda_2(S_2-T_2)](x)-[G(A_2,B_2)-\lambda_2(S_2-T_2)](y)\|\\ &\le \|x-y-(G(A_2(x),B_2(x))-G(A_2(y),B_2(y)))\| \\ &\quad + \|(x-y)-\lambda_2(S_2(x)-S_2(y))\|+\lambda_2\|T_2(x)-T_2(y)\|.\end{aligned} \tag{2.5}$$

Since G is α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively, ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Then from Lemma 1, we have

$$\begin{aligned}&\|x-y-(G(A_2(x),B_2(x))-G(A_2(y),B_2(y)))\|^2\le\|x-y\|^2-2\langle G(A_2(x),B_2(x))\\ &\quad -G(A_2(y),B_2(y)),J(x-y)\rangle+2c^2\|G(A_2(x),B_2(x))-G(A_2(y),B_2(y))\|^2\\ &=\|x-y\|^2-2\langle G(A_2(x),B_2(x))-G(A_2(y),B_2(x)),J(x-y)\rangle\\ &\quad -2\langle G(A_2(y),B_2(x))-G(A_2(y),B_2(y)),J(x-y)\rangle+2c^2\|G(A_2(x),B_2(x))\\ &\quad -G(A_2(y),B_2(y))\|^2\le\|x-y\|^2-2(\alpha_2+\beta_2)\|x-y\|^2+2c^2\rho_2^2\|x-y\|^2\\ &=[1-2(\alpha_2+\beta_2)+2c^2\rho_2^2]\|x-y\|^2.\end{aligned} \tag{2.6}$$

Since S_2 is δ_{S_2}-Lipschitz continuous and τ_2-strongly accretive mapping. Then from Lemma 1, we have

$$\begin{aligned}&\|x-y-\lambda_2(S_2(x)-S_2(y))\|^2\le\|x-y\|^2-2\lambda_2\langle S_2(x)\\ &\quad -S_2(y),J(x-y)\rangle+2c^2\lambda_2^2\|S_2(x)-S_2(y)\|^2\le(1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2)\|x-y\|^2.\end{aligned} \tag{2.7}$$

Using δ_{T_2}-Lipschitz continuity of T_2, we have

$$\|T_2(x)-T_2(y)\|\le\delta_{T_2}\|x-y\|. \tag{2.8}$$

Combining (2.5)–(2.8), we obtain

$$\begin{aligned}&\|[G(A_2,B_2)-\lambda_2(S_2-T_2)](x)-[G(A_2,B_2)-\lambda_2(S_2-T_2)](y)\|\\ &\le\Big[\sqrt{1-2(\alpha_2+\beta_2)+2c^2\rho_2^2}+\sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2}+\lambda_2\delta_{T_2}\Big]\|x-y\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|\varPi_{\Omega_1}[G(A_2,B_2)-\lambda_2(S_2-T_2)](x)-\varPi_{\Omega_1}[G(A_2,B_2)-\lambda_2(S_2-T_2)](y)\|\\&\quad\le\delta_{\varPi_{\Omega_1}}(\sqrt{1-2(\alpha_2+\beta_2)+2c^2\rho_2^2}+\sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2}\\&\qquad+\lambda_2\delta_{T_2})\|x-y\|=\kappa_1\|x-y\|.\end{aligned}$$

It follows from (2.4) that $0<\kappa_1<1$. Therefore, the mapping $\varPi_{\Omega_1}[G(A_2,B_2)-\lambda_2(S_2-T_2)]:\Omega_2\to\Omega_1$ is κ_1-contraction. □

Lemma 6. *Let Ω_1 and Ω_2 be two nonempty closed convex subsets of a real 2-uniformly smooth Banach space E. Let $A_1,B_1:\Omega_1\to E$ and $A_2,B_2:\Omega_2\to E$ be the single-valued mappings; $H:E\times E\to E$ be α_1,β_1-generalized accretive mapping with respect to A_1,B_1, respectively such that $\alpha_1+\beta_1\neq 0$. Let $G:E\times E\to E$ be α_2,β_2-generalized accretive mapping with respect to A_2,B_2, respectively and ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Let $M_1:E\to 2^E$ be an $H(\cdot,\cdot)$-accretive mapping with respect to A_1 and B_1 such that $\overline{Dom(M_1)}\subseteq\Omega_1$. Let $S_2,T_2:\Omega_2\to E$ be the single-valued mappings such that S_2 is δ_{S_2}-Lipschitz continuous, τ_2-strongly accretive and T_2 is δ_{T_2}-Lipschitz continuous. Suppose that there exists a positive constant λ_2 satisfying following condition:*

$$0<\varDelta(G)+\sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2}+\lambda_2\delta_{T_2}<\alpha_1+\beta_1,\ 1+2c^2\lambda_2^2\delta_{S_2}^2>2\lambda_2\tau_2.\quad(2.9)$$

Then $R^{H(\cdot,\cdot)}_{\lambda_1,M_1}[G(A_2,B_2)-\lambda_2(S_2-T_2)]:\Omega_2\to\Omega_1$ is L_1^*-contraction mapping, where

$$L_1^*=\frac{\varDelta(G)+\sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2}+\lambda_2\delta_{T_2})}{\alpha_1+\beta_1},\ \varDelta(G)=\sqrt{1-2(\alpha_2+\beta_2)+64\mathcal{C}\rho_2^2}.$$

Proof. Let $x,y\in\Omega_2$, then we have

$$\begin{aligned}&\|[G(A_2,B_2)-\lambda_2(S_2-T_2)](x)-[G(A_2,B_2)-\lambda_2(S_2-T_2)](y)\|\\&\qquad\le\|G(A_2(x),B_2(x))-G(A_2(y),B_2(y))-(x-y)\|\\&\qquad\quad+\|(x-y)-\lambda_2(S_2(x)-S_2(y))\|+\lambda_2\|T_2(x)-T_2(y)\|.\end{aligned}\tag{2.10}$$

Since G is α_2,β_2-generalized accretive mapping with respect to A_2,B_2, respectively and ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2, using the techniques of Alber and Yao [4] and Proposition 1, we have

$$\begin{aligned}&\|G(A_2(x),B_2(x))-G(A_2(y),B_2(y))-(x-y)\|^2\le\|x-y\|^2-2\langle G(A_2(x),B_2(x))\\&\quad-G(A_2(y),B_2(y)),J(x-y-(G(A_2(x),B_2(x))-G(A_2(y),B_2(y))))\rangle\\&=\|x-y\|^2-2\langle G(A_2(x),B_2(x))-G(A_2(y),B_2(y)),J(x-y)\rangle\\&\quad-2\langle G(A_2(x),B_2(x))-G(A_2(y),B_2(y)),J(x-y-(G(A_2(x),B_2(x))\\&\quad-G(A_2(y),B_2(y)))-J(x-y)\rangle\le\|x-y\|^2-2(\alpha_2+\beta_2)\|x-y\|^2\\&\quad+4\mathcal{C}^2\tau_E(4\|G(A_2(x),B_2(x))-G(A_2(y),B_2(y))\|)/\mathcal{C}\\&\le\|x-y\|^2-2(\alpha_2+\beta_2)\|x-y\|^2+64\mathcal{C}\|G(A_2(x),B_2(x))-G(A_2(y),B_2(y))\|^2\\&\le\|x-y\|^2-2(\alpha_2+\beta_2)\|x-y\|^2+64\mathcal{C}\rho_2^2\|x-y\|^2=\varDelta^2(G)\|x-y\|^2,\end{aligned}\tag{2.11}$$

where $\Delta(G) = \sqrt{1 - 2(\alpha_2 + \beta_2) + 64\mathcal{C}\rho_2^2}$. Since S_2 is δ_{S_2}-Lipschitz continuous and τ_2-strongly accretive mapping. Then from Lemma 1, we have

$$\begin{aligned}\|x - y - \lambda_2(S_2(x) - S_2(y))\|^2 &\le \|x - y\|^2 - 2\lambda_2\langle S_2(x) - S_2(y), J(x - y)\rangle \\ &+ 2c^2\lambda_2^2\|S_2(x) - S_2(y)\|^2 \le (1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2)\|x - y\|^2. \end{aligned} \tag{2.12}$$

Using δ_{T_2}-Lipschitz continuity of T_2, we have

$$\|T_2(x) - T_2(y)\| \le \delta_{T_2}\|x - y\|. \tag{2.13}$$

By combining (2.10)–(2.13), we have

$$\begin{aligned}&\|[G(A_2, B_2) - \lambda_2(S_2 - T_2)](x) - [G(A_2, B_2) - \lambda_2(S_2 - T_2)](y)\| \\ &\le \left[\Delta(G) + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2}\right]\|x - y\|.\end{aligned}$$

Thus, it follows from Lipschitz continuity of resolvent operator $R^{H(\cdot,\cdot)}_{\lambda_1,M_1}$ that

$$\begin{aligned}&\|R^{H(\cdot,\cdot)}_{\lambda_1,M_1}[G(A_2, B_2) - \lambda_2(S_2 - T_2)](x) - R^{H(\cdot,\cdot)}_{\lambda_1,M_1}[G(A_2, B_2) - \lambda_2(S_2 - T_2)](y)\| \\ &\le L_1^*\|x - y\|.\end{aligned}$$

It follows from (2.9) that $0 < L_1^* < 1$. Therefore, the mapping $R^{H(\cdot,\cdot)}_{\lambda_1,M_1}[G(A_2, B_2) - \lambda_2(S_2 - T_2)] : \Omega_2 \to \Omega_1$ is L_1^*-contraction. □

3 Formulation of the problem and convergence results

Let Ω_1 and Ω_2 be two nonempty closed convex subsets of a real smooth Banach space E. Let $\Pi_{\Omega_1} : E \to \Omega_1$ and $\Pi_{\Omega_2} : E \to \Omega_2$ be operators. Let $S_1, T_1 : \Omega_1 \to E$ and $S_2, T_2 : \Omega_2 \to E$ be the single-valued mappings. Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1 and $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2. Suppose that there exist constants $\lambda_1, \lambda_2 > 0$. We consider the following altering points problem (in short, APP): Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases}\Pi_{\Omega_2}[H(A_1, B_1) - \lambda_2(S_1 - T_1)](\bar{x}) = \bar{y}, \\ \Pi_{\Omega_1}[G(A_2, B_2) - \lambda_1(S_2 - T_2)](\bar{y}) = \bar{x}.\end{cases} \tag{3.1}$$

If $H(\cdot,\cdot) = H, G(\cdot,\cdot) = G, S_1 - T_1 = S$ and $S_2 - T_2 = T, \Omega_1 = C$ and $\Omega_2 = D$, then the problem (3.1) coincides with the following altering points problem of finding $(\bar{x}, \bar{y}) \in C \times D$ such that

$$\begin{cases}\Pi_D(H - \eta S)(\bar{x}) = \bar{y}, \\ \Pi_C(G - \rho T)(\bar{y}) = \bar{x}.\end{cases} \tag{3.2}$$

If $H = G = I$, then the problem (3.2) reduces to the following altering points problem of finding $(\bar{x}, \bar{y}) \in C \times D$ such that

$$\begin{cases}\Pi_D(I - \eta S)(\bar{x}) = \bar{y}, \\ \Pi_C(I - \rho T)(\bar{y}) = \bar{x}.\end{cases}$$

Note that APP (3.1) is more general in nature and for suitable choices of the mappings involved in the formulation, it include many problems existing in the literature as specialization. Some particular cases of APP (3.1) are listed below:

1. If $\Pi_{\Omega_1} = R^{H(\cdot,\cdot)}_{\lambda_1,M_1}$, where $M_1 : E \to 2^E$ is $H(\cdot,\cdot)$-generalized accretive mapping with respect to A_1, B_1 such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $\Pi_{\Omega_2} = R^{G(\cdot,\cdot)}_{\lambda_2,M_2}$, where $M_2 : E \to 2^E$ is $G(\cdot,\cdot)$-generalized accretive mapping with respect to A_2, B_2 such that $\overline{Dom(M_2)} \subseteq \Omega_2$, then APP (3.1) reduces to the following system of generalized variational inclusions ($SGVI$): Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that
$$\begin{cases} 0 \in G(A_2(\bar{y}), B_2(\bar{y})) - H(A_1(\bar{x}), B_1(\bar{x})) + \lambda_2(M_2(\bar{y}) + S_1(\bar{x}) - T_1(\bar{x})), \\ 0 \in H(A_1(\bar{x}), B_1(\bar{x})) - G(A_2(\bar{y}), B_2(\bar{y})) + \lambda_1(M_1(\bar{x}) + S_2(\bar{y}) - T_2(\bar{y})). \end{cases} \tag{3.3}$$

2. If $\Pi_{\Omega_1} = R^H_{\lambda_1,M_1}$, where $M_1 : E \to 2^E$ is H-accretive mapping such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $\Pi_{\Omega_2} = R^G_{\lambda_2,M_2}$, where $M_2 : E \to 2^E$ is G-accretive mapping such that $\overline{Dom(M_2)} \subseteq \Omega_2$, then the system of generalized variational inclusions (3.3) reduces to the following system of variational inclusions (SVI): Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that
$$\begin{cases} 0 \in G(\bar{y}) - H(\bar{x}) + \lambda_2(M_2(\bar{y}) + S_1(\bar{x}) - T_1(\bar{x})), \\ 0 \in H(\bar{x}) - G(\bar{y}) + \lambda_1(M_1(\bar{x}) + S_2(\bar{y}) - T_2(\bar{y})). \end{cases} \tag{3.4}$$

3. If $T_1 = T_2 = 0$, then the system of variational inclusions (3.4) reduces to the following system of variational inclusions investigated by Zhao et al. [33]: Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that
$$\begin{cases} 0 \in G(\bar{y}) - H(\bar{x}) + \lambda_2(M_2(\bar{y}) + S_1(\bar{x})), \\ 0 \in H(\bar{x}) - G(\bar{y}) + \lambda_1(M_1(\bar{x}) + S_2(\bar{y})). \end{cases} \tag{3.5}$$

4. If $\Pi_{\Omega_1} = R^{M_1}_{\lambda_1}$ and $\Pi_{\Omega_2} = R^{M_2}_{\lambda_2}$, where $M_1, M_2 : E \to 2^E$ are m-accretive mappings such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $\overline{Dom(M_2)} \subseteq \Omega_2$, then the system of variational inclusions (3.5) coincides with the following system of variational inclusions (SVI): Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that
$$\begin{cases} 0 \in \bar{y} - \bar{x} + \lambda_2(M_2(\bar{y}) + S_1(\bar{x})), \\ 0 \in \bar{x} - \bar{y} + \lambda_1(M_1(\bar{x}) + S_2(\bar{y})). \end{cases}$$

5. If $\Pi_{\Omega_1} = Q_{\Omega_1}$ and $\Pi_{\Omega_2} = Q_{\Omega_2}$, the sunny nonexpansive retractions onto Ω_1 and Ω_2, respectively such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $\overline{Dom(M_2)} \subseteq \Omega_2$, then APP (3.1) reduces to the following system of generalized variational inequalities ($SGVIneq$): Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that
$$\begin{cases} \langle \lambda_1(S_1(\bar{x}) - T_1(\bar{x})) + \bar{y} - H(A_1(\bar{x}), B_1(\bar{x})), J(P_1(\hat{y}) - \bar{y}) \rangle \geq 0, \ \forall \hat{y} \in \Omega_2, \\ \langle \lambda_2(S_2(\bar{y}) - T_2(\bar{y})) + \bar{x} - G(A_2(\bar{y}), B_2(\bar{y})), J(P_2(\hat{x}) - \bar{x}) \rangle \geq 0, \ \forall \hat{x} \in \Omega_1. \end{cases} \tag{3.6}$$

6. If $E = H$ is a real Hilbert space, then the system of generalized variational inequalities (3.6) coincides to the following system of generalized variational inequalities ($SGVIneq$): Find $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases} \langle \lambda_1(S_1(\bar{x}) - T_1(\bar{x})) + \bar{y} - H(A_1(\bar{x}), B_1(\bar{x})), P_1(\hat{y}) - \bar{y} \rangle \geq 0, \ \forall \hat{y} \in \Omega_2, \\ \langle \lambda_2(S_2(\bar{y}) - T_2(\bar{y})) + \bar{x} - G(A_2(\bar{y}), B_2(\bar{y})), P_2(\hat{x}) - \bar{x} \rangle \geq 0, \ \forall \hat{x} \in \Omega_1. \end{cases} \tag{3.7}$$

Remark 3. For $T_1 = T_2 = 0, A_1 = A_2 = B_1 = B_2 = I, H(\cdot, \cdot) = H$ and $G(\cdot, \cdot) = G$, $SGVIneq$ (3.7) is identical to the problem studied in [21]. Further, if $S_1 = S_2 = S, H = G = I$ and $\Omega_1 = \Omega_2 = \Omega$, then the problem (3.7) is analogous to the problem examined in [28].

Proposition 4. *Let Ω_1 and Ω_2 be two nonempty subsets of a Banach space E. Let $S : \Omega_1 \to \Omega_2$ be κ_1-contraction and $T : \Omega_2 \to \Omega_1$ be κ_2-contraction mappings. Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves the following system of altering points problem: Find $\bar{x} \in \Omega_1$ and $\bar{y} \in \Omega_2$ such that*

$$S(\bar{x}) = \bar{y}, \quad T(\bar{y}) = \bar{x}.$$

Proof. Since $S : \Omega_1 \to \Omega_2$ is κ_1-contraction and $T : \Omega_2 \to \Omega_1$ is κ_2-contraction mapping. Thus $TS : \Omega_1 \to \Omega_1$ is a contraction mapping. Hence TS has a unique point $\bar{x} \in \Omega_1$ such that $\bar{x} = TS(\bar{x})$. Further, there exists a unique point $\bar{y} \in \Omega_2$ such that $\bar{y} = S(\bar{x})$. Thus, we have $\bar{x} = T(\bar{y})$. □

Lemma 7. *Let Ω_1 and Ω_2 be two nonempty closed convex subsets of a real 2-uniformly smooth Banach space E. Let $A_1, B_1, S_1, T_1 : \Omega_1 \to E$ and A_2, B_2, S_2, $T_2 : \Omega_2 \to E$ be the single-valued mappings. Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1 and $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2. Let $M_1 : E \to 2^E$ be $H(\cdot, \cdot)$-generalized accretive mapping with respect to A_1, B_1 such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $M_2 : E \to 2^E$ be $G(\cdot, \cdot)$-generalized accretive mapping with respect to A_2, B_2 such that $\overline{Dom(M_2)} \subseteq \Omega_2$. Suppose that there exist constants $\lambda_1, \lambda_2 > 0$. Then SGVI (3.3) has a solution $(\bar{x}, \bar{y})$, if and only if $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ satisfies following system of altering points problem:*

$$\begin{aligned} R^{G(\cdot,\cdot)}_{\lambda_2, M_2}[H(A_1, B_1) - \lambda_2(S_1 - T_1)](\bar{x}) &= \bar{y}, \\ R^{H(\cdot,\cdot)}_{\lambda_1, M_1}[G(A_2, B_2) - \lambda_1(S_2 - T_2)](\bar{y}) &= \bar{\bar{x}}. \end{aligned}$$

Now, we propose parallel Mann and parallel S-iteration processes to solve APP (3.1).

Algorithm 1. Let Ω_1 and Ω_2 be closed convex subsets of a 2-uniformly smooth Banach space E. Let $S_1 : \Omega_1 \to \Omega_2$ and $S_2 : \Omega_2 \to \Omega_1$ be two mappings. Then for $\alpha_n \in [0, 1]$ and initial point $(\bar{x}_0, \bar{y}_0) \in \Omega_1 \times \Omega_2$, the sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ produced by parallel Mann iterative method [21] is defined as:

$$\begin{cases} \bar{x}_{n+1} = (1 - \alpha_n)\bar{x}_n + \alpha_n S_2(\bar{y}_n), \\ \bar{y}_{n+1} = (1 - \alpha_n)\bar{y}_n + \alpha_n S_1(\bar{x}_n). \end{cases} \tag{3.8}$$

Algorithm 2. Let Ω_1 and Ω_2 be closed convex subsets of a 2-uniformly smooth Banach space E. Let $S_1 : \Omega_1 \to \Omega_2$ and $S_2 : \Omega_2 \to \Omega_1$ be two mappings. Then for $\alpha_n, \beta_n \in (0,1)$ and initial point $(\bar{x}_0, \bar{y}_0) \in \Omega_1 \times \Omega_2$, the sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ produced by parallel S-iterative method [33] is defined as:

$$\begin{cases} \bar{x}_{n+1} = S_2[(1-\alpha_n)\bar{y}_n + \alpha_n S_1(\bar{x}_n)], \\ \bar{y}_{n+1} = S_1[(1-\beta_n)\bar{x}_n + \beta_n S_2(\bar{y}_n)]. \end{cases} \tag{3.9}$$

Now, we are ready to prove convergence results for APP (3.1). Now onward, unless otherwise specified, for each $i \in \{1,2\}$, we assume Ω_i be nonempty closed convex subsets of a real 2-uniformly smooth Banach space E.

Theorem 1. *Let $\Pi_{\Omega_i} : E \to \Omega_i$ be $\delta_{\Pi_{\Omega_i}}$-Lipschitz continuous mappings and $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1, respectively and ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively and ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exist constants $\lambda_i > 0$ satisfying following conditions:*

$$\begin{aligned} &0 < \delta_{\Pi_{\Omega_i}}\left(\sqrt{1-2(\alpha_i+\beta_i)+2c^2\rho_i^2} + \sqrt{1-2\lambda_i\tau_i+2c^2\lambda_i^2\delta_{S_i}^2} + \lambda_i\delta_{T_i}\right) < 1, \\ &1+2c^2\rho_i^2 > 2(\alpha_i+\beta_i), 1+2c^2\lambda_i^2\delta_{S_i}^2 > 2\lambda_i\tau_i. \end{aligned} \tag{3.10}$$

If for any arbitrary $(\bar{x}_0, \bar{y}_0) \in \Omega_1 \times \Omega_2$, let $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ be any sequence generated by parallel Mann iterative method (3.8) with $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves APP (3.1).

(ii) The sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel Mann iterative method (3.8) converges strongly to $(\bar{x}, \bar{y})$.

Proof. (i) Evidently from Lemma 5, $\Pi_{\Omega_1}[G(A_2,B_2) - \lambda_2(S_2 - T_2)] : \Omega_2 \to \Omega_1$ is a κ_1-contraction and $\Pi_{\Omega_2}[H(A_1,B_1) - \lambda_1(S_1 - T_1)] : \Omega_1 \to \Omega_2$ is a κ_2-contraction mapping. Hence, the proof follows immediately from Proposition 4.

(ii) Define $S_1 =: \Pi_{\Omega_2}[H(A_1,B_1) - \lambda_1(S_1 - T_1)]$ and $S_2 =: \Pi_{\Omega_1}[G(A_2,B_2) - \lambda_2(S_2 - T_2)]$. Then $S_1 : \Omega_1 \to \Omega_2$ is κ_2-contraction mapping and $S_2 : \Omega_2 \to \Omega_1$ is κ_1-contraction mapping. Then, it follows from (3.8) that

$$\begin{aligned} \|\bar{x}_{n+1} - \bar{x}\| &= \|(1-\alpha_n)\bar{x}_n + \alpha_n S_2(\bar{y}_n) - \bar{x}\| \le (1-\alpha_n)\|\bar{x}_n - \bar{x}\| \\ &+ \alpha_n\|S_2(\bar{y}_n) - \bar{x}\| \le (1-\alpha_n)\|\bar{x}_n - \bar{x}\| + \alpha_n\|S_2(\bar{y}_n) - S_2(\bar{y})\| \\ &\le (1-\alpha_n)\|\bar{x}_n - \bar{x}\| + \alpha_n\kappa_1\|\bar{y}_n - \bar{y}\|. \end{aligned}$$

Since

$$\|\bar{y}_n - \bar{y}\| = \|S_1(\bar{x}_n) - S_1(\bar{x})\| \le \kappa_2\|\bar{x}_n - \bar{x}\|. \tag{3.11}$$

Thus, we acquire

$$\|\bar{x}_{n+1} - \bar{x}\| \le (1 - \alpha_n(1 - \kappa_1\kappa_2)\|\bar{x}_n - \bar{x}\|.$$

Hence, it follows from Lemma 3 that $\{\bar{x}_n\}$ converges to $\bar{x}$ and from (3.11), it is easy to see that $\{\bar{y}_n\}$ converges to $\bar{y}$. This completes the proof. □

Theorem 2. *Let $\Pi_{\Omega_i} : E \to \Omega_i$ be $\delta_{\Pi_{\Omega_i}}$-Lipschitz continuous mappings and $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1, respectively and ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively and ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exist constants $\lambda_i > 0$ satisfying condition (3.10). If for any arbitrary element $(\bar{x}_0, \bar{y}_0) \in \Omega_1 \times \Omega_2$; let $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ be any sequence generated by parallel S-iterative method (3.9), where $\alpha_n, \beta_n \in (0, 1)$ satisfying the following condition:*

$$\beta_n > \alpha_n\kappa_2, \alpha_n > \beta_n\kappa_1, \forall n \in N. \tag{3.12}$$

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves APP (3.1).

(ii) The sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel S-iteration process (3.9) converges strongly to $(\bar{x}, \bar{y})$.

Proof. (i) Proof follows from part (i) of Theorem 1.

(ii) Define $S_1 =: \Pi_{\Omega_2}[H(A_1, B_1) - \lambda_1(S_1 - T_1)]$ and $S_2 =: \Pi_{\Omega_1}[G(A_2, B_2) - \lambda_2(S_2 - T_2)]$. Then from Lemma 5, we know that $S_1 : \Omega_1 \to \Omega_2$ is κ_2-contraction mapping and $S_2 : \Omega_2 \to \Omega_1$ is κ_1-contraction mapping. Thus from (3.9), we obtain

$$\begin{aligned}\|\bar{x}_{n+1} - \bar{x}\| &= \|S_2[(1 - \alpha_n)\bar{y}_n + \alpha_n S_1(x_n)] - \bar{x}\| \le \kappa_1\|(1 - \alpha_n)\bar{y}_n \\ &+ \alpha_n S_1(\bar{x}_n) - \bar{y}\| \le \kappa_1[(1 - \alpha_n)\|\bar{y}_n - \bar{y}\| + \alpha_n\|S_1(\bar{x}_n) - \bar{y}\|] \\ &\le \kappa_1[(1 - \alpha_n)\|\bar{y}_n - \bar{y}\| + \alpha_n\kappa_2\|\bar{x}_n - \bar{x}\|].\end{aligned}$$

Following the same steps as above, we have

$$\|\bar{y}_{n+1} - \bar{y}\| \le \kappa_2[(1 - \beta_n)\|\bar{x}_n - \bar{x}\| + \beta_n\kappa_1\|\bar{y}_n - \bar{y}\|].$$

Thus, we infer that

$$\begin{aligned}\|\bar{x}_{n+1} - \bar{x}\| + \|\bar{y}_{n+1} - \bar{y}\| &\le \kappa_2[1 - (\beta_n - \alpha_n\kappa_1)]\|\bar{x}_n - \bar{x}\| \\ &+ \kappa_1[1 - (\alpha_n - \beta_n\kappa_2)]\|\bar{y}_n - \bar{y}\|.\end{aligned} \tag{3.13}$$

Choose $\kappa = \max(\kappa_1, \kappa_2)$, then from (3.12) and (3.13), we have

$$\|\bar{x}_{n+1} - \bar{x}\| + \|\bar{y}_{n+1} - \bar{y}\| \le \kappa[\|\bar{x}_n - \bar{x}\| + \|\bar{y}_n - \bar{y}\|]. \tag{3.14}$$

From (2.2) and (3.14), we deduce that

$$\|(\bar{x}_{n+1}, \bar{y}_{n+1}) - (\bar{x}, \bar{y})\|_* \leq \kappa \|(\bar{x}_n, \bar{y}_n) - (\bar{x}, \bar{y})\|_*.$$

Now utilizing Lemma 3, we have

$$\lim_{n\to\infty} \|(\bar{x}_n, \bar{y}_n) - (\bar{x}, \bar{y})\|_* = \lim_{n\to\infty} \|\bar{x}_n - \bar{x}\| + \lim_{n\to\infty} \|\bar{y}_n - \bar{y}\| = 0.$$

Thus, the sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ converges strongly to $(\bar{x}, \bar{y})$. □

4 Applications

Next, we prove the convergence results for the system of generalized variational inclusions and inequalities as applications of conceptual framework of altering points. Some special cases are also discussed.

Theorem 3. *Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1, respectively such that $\alpha_1 + \beta_1 > 0$, ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively such that $\alpha_2 + \beta_2 > 0$, ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Let $M_1 : E \to 2^E$ be $H(\cdot,\cdot)$-accretive mapping with respect to A_1 and B_1 such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $M_2 : E \to 2^E$ be $G(\cdot,\cdot)$-accretive mapping with respect to A_2 and B_2 such that $\overline{Dom(M_2)} \subseteq \Omega_2$. Let $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Suppose that there exist positive constants λ_i and $\alpha_n \in [0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ satisfying following conditions:*

$$0 < \Delta(G) + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2} < \alpha_1 + \beta_1, 1 + 2c^2\lambda_2^2\delta_{S_2}^2 > 2\lambda_2\tau_2.$$

$$0 < \Delta(H) + \sqrt{1 - 2\lambda_1\tau_1 + 2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1} < \alpha_2 + \beta_2, 1 + 2c^2\lambda_1^2\delta_{S_1}^2 > 2\lambda_1\tau_1.$$

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves SGVI (3.3).

(ii) The sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel Mann iterative method:

$$\begin{cases} \bar{x}_{n+1} = (1-\alpha_n)\bar{x}_n + \alpha_n R^{H(\cdot,\cdot)}_{\lambda_1, M_1}[G(A_2, B_2) - \lambda_1(S_2 - T_2)](\bar{y}_n), \\ \bar{y}_{n+1} = (1-\alpha_n)\bar{y}_n + \alpha_n R^{G(\cdot,\cdot)}_{\lambda_2, M_2}[H(A_1, B_1) - \lambda_2(S_1 - T_1)](\bar{x}_n). \end{cases} \tag{4.1}$$

converges strongly to $(\bar{x}, \bar{y})$.

Proof. (i) Define $S =: R^{G(\cdot,\cdot)}_{\lambda_2, M_2}[H(A_1, B_1) - \lambda_2(S_1 - T_1)]$, $T =: R^{H(\cdot,\cdot)}_{\lambda_1, M_1}$ $[G(A_2, B_2) - \lambda_1(S_2 - T_2)]$. Consequently, from Lemma 6, $T : \Omega_2 \to \Omega_1$ is L_1^*-contraction mapping, where $L_1^* = \frac{1}{\alpha_1+\beta_1}\big(\Delta(G) + \sqrt{1-2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} +$

$\lambda_2\delta_{T_2}\big)$ and $\varDelta(G) = \sqrt{1-2(\alpha_2+\beta_2)+64\mathcal{C}\rho_2^2}$. Similarly $S:\varOmega_1\to\varOmega_2$ is L_2^*-contraction mapping, where $L_2^* = \frac{1}{\alpha_2+\beta_2}\big(\varDelta(H) + \sqrt{1-2\lambda_1\tau_1+2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1}\big)$ and $\varDelta(H) = \sqrt{1-(\alpha_1+\beta_1)+64\mathcal{C}\rho_1^2}$. By utilizing Proposition 4, one can conclude that there exists a unique point $(\bar{x},\bar{y})\in\varOmega_1\times\varOmega_2$ such that $S(\bar{x})=\bar{y}$ and $T(\bar{y})=\bar{x}$. Therefore, we get

$$\begin{cases} R^{G(\cdot,\cdot)}_{\lambda_2,M_2}[H(A_1,B_1)-\lambda_2(S_1-T_1)](\bar{x})=\bar{y}, \\ R^{H(\cdot,\cdot)}_{\lambda_1,M_1}[G(A_2,B_2)-\lambda_1(S_2-T_2)](\bar{y})=\bar{x}. \end{cases}$$

Thus from Lemma 7, we deduce that $(\bar{x},\bar{y})\in\varOmega_1\times\varOmega_2$ is a solution of $SGVI$ (3.3).

(ii) Since $S:\varOmega_1\to\varOmega_2$ is L_2^*-contraction mapping and $T:\varOmega_2\to\varOmega_1$ is L_1^*-contraction mapping. Then from (4.1), we obtain

$$\begin{aligned} \|\bar{x}_{n+1}-\bar{x}\| &= \|(1-\alpha_n)\bar{x}_n+\alpha_n T(\bar{y}_n)-\bar{x}\| \le (1-\alpha_n)\|\bar{x}_n-\bar{x}\| + \alpha_n\|T(\bar{y}_n)-\bar{x}\| \\ &\le (1-\alpha_n)\|\bar{x}_n-\bar{x}\|+\alpha_n\|T(\bar{y}_n)-T(\bar{y})\| \le (1-\alpha_n)\|\bar{x}_n-\bar{x}\|+\alpha_n L_1^*\|\bar{y}_n-\bar{y}\|. \end{aligned}$$

Since

$$\|\bar{y}_n-\bar{y}\| = \|S(\bar{x}_n)-S(\bar{x})\| = L_2^*\|\bar{x}_n-\bar{x}\|. \tag{4.2}$$

Thus, we obtain

$$\|\bar{x}_{n+1}-\bar{x}\| \le (1-\alpha_n(1-L_2^*L_1^*))\|\bar{x}_n-\bar{x}\|.$$

Hence, it follows from Lemma 3 that $\{\bar{x}_n\}$ converges to $\bar{x}$ and from (4.2), it is easy to see that $\{\bar{y}_n\}$ converges to $\bar{y}$. This completes the proof. □

Corollary 1. Let $H:E\to E$ be γ_1-strongly accretive and ρ_1-Lipschitz continuous mapping and $G:E\to E$ be γ_2-strongly accretive and ρ_2-Lipschitz continuous mapping. Let $M_1:E\to 2^E$ be H-accretive mapping such that $\overline{Dom(M_1)}\subseteq\varOmega_1$ and $M_2:E\to 2^E$ be G-accretive mapping such that $\overline{Dom(M_2)}\subseteq\varOmega_2$. Let $S_i,T_i:\varOmega_i\to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Suppose that there exist constants $\lambda_i>0$ and $\alpha_n\in[0,1]$ with $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$ satisfying following conditions:

$$\begin{aligned} 0 <& \varDelta(G) + \sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2} < \alpha_1+\beta_1, 1+2c^2\lambda_2^2\delta_{S_2}^2 > 2\lambda_2\tau_2, \\ 0 <& \varDelta(H) + \sqrt{1-2\lambda_1\tau_1+2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1} < \alpha_2+\beta_2, 1+2c^2\lambda_1^2\delta_{S_1}^2 > 2\lambda_1\tau_1, \end{aligned}$$

where, $\varDelta(G)=\sqrt{1-2\gamma_2+64\mathcal{C}\rho_2^2}$ and $\varDelta(H)=\sqrt{1-2\gamma_1+64\mathcal{C}\rho_1^2}$.

(i) Then there exists a unique element $(\bar{x},\bar{y})\in\varOmega_1\times\varOmega_2$ such that $(\bar{x},\bar{y})$ solves SVI (3.4).

(ii) The sequence $\{(\bar{x}_n,\bar{y}_n)\}\in\varOmega_1\times\varOmega_2$ generated by parallel Mann iterative method:

$$\begin{cases} \bar{x}_{n+1} = (1-\alpha_n)\bar{x}_n + \alpha_n R^H_{\lambda_1,M_1}[G-\lambda_1(S_2-T_2)](\bar{y}_n), \\ \bar{y}_{n+1} = (1-\alpha_n)\bar{y}_n + \alpha_n R^G_{\lambda_2,M_2}[H-\lambda_2(S_1-T_1)](\bar{x}_n) \end{cases}$$

converges strongly to $(\bar{x}, \bar{y})$.

Theorem 4. *Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1, respectively such that $\alpha_1 + \beta_1 \neq 0$, ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively such that $\alpha_2 + \beta_2 \neq 0$, ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Let $M_1 : E \to 2^E$ be $H(\cdot,\cdot)$-accretive mapping with respect to A_1 and B_1 such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $M_2 : E \to 2^E$ be $G(\cdot,\cdot)$-accretive mapping with respect to A_2 and B_2 such that $\overline{Dom(M_2)} \subseteq \Omega_2$. Let $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Suppose that there exist positive constants λ_i and the sequences $\{\alpha_n\}, \{\beta_n\}$ in (0,1) with $\beta_n > \alpha_n L_1^*$ and $\alpha_n > \beta_n L_2^*$ satisfying following conditions:*

$$0 < \Delta(G) + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2} < \alpha_1 + \beta_1, 1 + 2c^2\lambda_2^2\delta_{S_2}^2 > 2\lambda_2\tau_2,$$

$$0 < \Delta(H) + \sqrt{1 - 2\lambda_1\tau_1 + 2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1} < \alpha_2 + \beta_2, 1 + 2c^2\lambda_1^2\delta_{S_1}^2 > 2\lambda_1\tau_1.$$

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves SGVI (3.3).

(ii) For any arbitrary $(\bar{x}_0, \bar{y}_0) \in \Omega_1 \times \Omega_2$, there exists a sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel S-iterative method:

$$\begin{aligned}\bar{x}_{n+1} =& R^{H(\cdot,\cdot)}_{\lambda_1, M_1}[G(A_2,B_2) - \lambda_1(S_2 - T_2)]\Big[(1-\alpha_n)\bar{y}_n + \alpha_n R^{G(\cdot,\cdot)}_{\lambda_2, M_2}[H(A_1,B_1) \\ &- \lambda_2(S_1 - T_1)](\bar{x}_n)\Big], \\ \bar{y}_{n+1} =& R^{G(\cdot,\cdot)}_{\lambda_2, M_2}[H(A_1,B_1) - \lambda_2(S_1 - T_1)]\Big[(1-\beta_n)\bar{x}_n + \beta_n R^{H(\cdot,\cdot)}_{\lambda_1, M_1}[G(A_2,B_2) \\ &- \lambda_1(S_2 - T_2)](\bar{y}_n)\Big]\end{aligned}$$

converges strongly to $(\bar{x}, \bar{y})$.

Proof. (i) Define $S =: R^{G(\cdot,\cdot)}_{\lambda_2, M_2}[H(A_1, B_1) - \lambda_2(S_1 - T_1)]$ and $T =: R^{H(\cdot,\cdot)}_{\lambda_1, M_1}[G(A_2, B_2) - \lambda_1(S_2 - T_2)]$. In consequence of Lemma 6, $T : \Omega_2 \to \Omega_1$ is L_1^*-contraction mapping, where $L_1^* = \frac{1}{\alpha_1+\beta_1}\big(\Delta(G) + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2}\big)$ and $\Delta(G) = \sqrt{1 - 2(\alpha_2 + \beta_2) + 64\mathcal{C}\rho_2^2}$. Similarly $S : \Omega_1 \to \Omega_2$ is L_2^*-contraction mapping, where $L_2^* = \frac{1}{\alpha_2+\beta_2}\big(\Delta(H) + \sqrt{1 - 2\lambda_1\tau_1 + 2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1}\big)$ and $\Delta(H) = \sqrt{1 - 2(\alpha_1 + \beta_1) + 64\mathcal{C}\rho_1^2}$. Then from Proposition 4, we know that $S(\bar{x}) = \bar{y}$ and $T(\bar{y}) = \bar{x}$. Thus, the conclusion follows from Lemma 7.

(ii) Since $S : \Omega_1 \to \Omega_2$ is a L_2^*-contraction mapping and $T : \Omega_2 \to \Omega_1$ is a L_1^*-contraction mapping. Then from (3.9), we have

$$\begin{aligned}\|\bar{x}_{n+1} - \bar{x}\| &= \|T[(1-\alpha_n)\bar{y}_n + \alpha_n S(\bar{x}_n)] - \bar{x}\| \leq L_1^*\|(1-\alpha_n)\bar{y}_n \\ &\quad + \alpha_n S(\bar{x}_n) - \bar{y}\| \leq L_1^*[(1-\alpha_n)\|\bar{y}_n - \bar{y}\| + \alpha_n\|S(\bar{x}_n) - \bar{y}\|] \\ &\leq L_1^*[(1-\alpha_n)\|\bar{y}_n - \bar{y}\| + \alpha_n L_2^*\|\bar{x}_n - \bar{x}\|].\end{aligned}$$

Following the same steps as above, we have

$$\|\bar{y}_{n+1} - \bar{y}\| \leq L_2^*[(1-\beta_n)\|\bar{x}_n - \bar{x}\| + \beta_n L_1^* \|\bar{y}_n - \bar{y}\|].$$

Thus, we have

$$\begin{aligned} \|\bar{x}_{n+1} - \bar{x}\| + \|\bar{y}_{n+1} - \bar{y}\| \leq L_2^*[1 - (\beta_n - \alpha_n L_1^*)]\|\bar{x}_n - \bar{x}\| \\ + L_1^*[1 - (\alpha_n - \beta_n L_2^*)]\|\bar{y}_n - \bar{y}\|. \end{aligned} \tag{4.3}$$

Choose $L^* = \max\{L_1^*, L_2^*\}$, then from (4.3) and assumptions of the theorem, we have

$$\|\bar{x}_{n+1} - \bar{x}\| + \|\bar{y}_{n+1} - \bar{y}\| \leq L^*[\|\bar{x}_n - \bar{x}\| + \|\bar{y}_n - \bar{y}\|]. \tag{4.4}$$

From (2.2) and (4.4), we infer that

$$\|(\bar{x}_{n+1}, \bar{y}_{n+1}) - (\bar{x}, \bar{y})\|_* \leq L^* \|(\bar{x}_n, \bar{y}_n) - (\bar{x}, \bar{y})\|_*.$$

Now utilizing Lemma 3, we acquire

$$\lim_{n\to\infty} \|(\bar{x}_n, \bar{y}_n) - (\bar{x}, \bar{y})\|_* = \lim_{n\to\infty} \|\bar{x}_n - \bar{x}\| + \lim_{n\to\infty} \|\bar{y}_n - \bar{y}\| = 0.$$

Thus, the sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ converges strongly to $(\bar{x}, \bar{y})$. $\square$

Corollary 2. Let $H : E \to E$ be γ_1-strongly accretive and ρ_1-Lipschitz continuous and $G : E \to E$ be γ_2-strongly accretive and ρ_2-Lipschitz continuous. Let $M_1 : E \to 2^E$ be H-accretive mapping such that $\overline{Dom(M_1)} \subseteq \Omega_1$ and $M_2 : E \to 2^E$ be G-accretive mapping such that $\overline{Dom(M_2)} \subseteq \Omega_2$. Let $S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Suppose that there exist positive constants λ_i and the sequences $\{\alpha_n\}, \{\beta_n\}$ in (0,1) with $\beta_n > \alpha_n L_1^*$ and $\alpha_n > \beta_n L_2^*$ satisfying following conditions:

$$0 < \Delta(G) + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2} < \alpha_1 + \beta_1, 1 + 2c^2\lambda_2^2\delta_{S_2}^2 > 2\lambda_2\tau_2,$$
$$0 < \Delta(H) + \sqrt{1 - 2\lambda_1\tau_1 + 2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1} < \alpha_2 + \beta_2, 1 + 2c^2\lambda_1^2\delta_{S_1}^2 > 2\lambda_1\tau_1,$$

where $\Delta(G) = \sqrt{1 - 2\gamma_2 + 64\mathcal{C}\rho_2^2}$ and $\Delta(H) = \sqrt{1 - 2\gamma_1 + 64\mathcal{C}\rho_1^2}$.

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves SVI (3.4).

(ii) For any arbitrary $(\bar{x}_0, \bar{y}_0) \in \Omega_1 \times \Omega_2$, there exists a sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel S-iterative method:

$$\bar{x}_{n+1} = R^H_{\lambda_1, M_1}[G - \lambda_1(S_2 - T_2)]\Big[(1-\alpha_n)\bar{y}_n + \alpha_n R^G_{\lambda_2, M_2}[H - \lambda_2(S_1 - T_1)](\bar{x}_n)\Big],$$
$$\bar{y}_{n+1} = R^G_{\lambda_2, M_2}[H - \lambda_2(S_1 - T_1)]\Big[(1-\beta_n)\bar{x}_n + \beta_n R^H_{\lambda_1, M_1}[G - \lambda_1(S_2 - T_2)](\bar{y}_n)\Big]$$

converges strongly to $(\bar{x}, \bar{y})$.

Theorem 5. *Let Q_{Ω_i} be sunny nonexpansive retractions from E onto Ω_i and $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1, respectively and ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively and ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exist constants $\lambda_i > 0$ such that $\alpha_n \in [0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ satisfying the following conditions:*

$$\begin{aligned} \sqrt{1-2(\alpha_i+\beta_i)+2c^2\rho_i^2} + \sqrt{1-2\lambda_i\tau_i+2c^2\lambda_i^2\delta_{S_i}^2} + \lambda_i\delta_{T_i} < 1 \\ 1+2c^2\rho_i^2 > 2(\alpha_i+\beta_i), 1+2c^2\lambda_i^2\delta_{S_i}^2 > 2\lambda_i\tau_i. \end{aligned} \tag{4.5}$$

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves SGVIneq (3.6).

(ii) The sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel Mann iterative method:

$$\begin{cases} \bar{x}_{n+1} = (1-\alpha_n)\bar{x}_n + \alpha_n Q_{\Omega_1}[G(A_2,B_2) - \lambda_1(S_2-T_2)](\bar{y}_n), \\ \bar{y}_{n+1} = (1-\alpha_n)\bar{y}_n + \alpha_n Q_{\Omega_2}[H(A_1,B_1) - \lambda_2(S_1-T_1)](\bar{x}_n) \end{cases} \tag{4.6}$$

converges strongly to $(\bar{x}, \bar{y})$.

Proof.

(i) Define $\psi =: Q_{\Omega_1}[G(A_2,B_2) - \lambda_1(S_2-T_2)]$ and $\varphi =: Q_{\Omega_2}[H(A_1,B_1) - \lambda_2(S_1-T_1)]$. Since Q_{Ω_1} is sunny nonexpansive, then it follows from Lemma 5 and (4.5) that $\psi : \Omega_2 \to \Omega_1$ is $L(\psi)$-contraction mapping, where

$$L(\psi) = \sqrt{1-2(\alpha_2+\beta_2)+2c^2\rho_2^2} + \sqrt{1-2\lambda_2\tau_2+2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2}.$$

Similarly, Q_{Ω_2} is sunny nonexpansive and $\varphi : \Omega_1 \to \Omega_2$ is $L(\varphi)$-contraction mapping, where

$$L(\varphi) = \sqrt{1-2(\alpha_1+\beta_1)+2c^2\rho_1^2} + \sqrt{1-2\lambda_1\tau_1+2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1}.$$

From Proposition 4, it follows that there exists unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $\varphi(\bar{x}) = \bar{y}$ and $\psi(\bar{y}) = \bar{x}$. Thus, the conclusion follows immediately from the Lemma 4.

(ii) Since $\varphi : \Omega_1 \to \Omega_2$ is a $L(\varphi)$-contraction mapping and $\psi : \Omega_2 \to \Omega_1$ is a $L(\psi)$-contraction mapping. Then from (4.6), we have

$$\begin{aligned} \|\bar{x}_{n+1} - \bar{x}\| &= \|(1-\alpha_n)\bar{x}_n + \alpha_n\psi(y_n) - \bar{x}\| \le (1-\alpha_n)\|\bar{x}_n - \bar{x}\| \\ &+ \alpha_n\|\psi(\bar{y}_n) - \bar{x}\| \le (1-\alpha_n)\|\bar{x}_n - \bar{x}\| + \alpha_n\|\psi(\bar{y}_n) - \psi(\bar{y})\| \\ &\le (1-\alpha_n)\|\bar{x}_n - \bar{x}\| + \alpha_n L(\psi)\|\bar{y}_n - \bar{y}\|. \end{aligned}$$

Since

$$\|\bar{y}_n - \bar{y}\| = \|\varphi(\bar{x}_n) - \varphi(\bar{x})\| = L(\varphi)\|\bar{x}_n - \bar{x}\|. \quad (4.7)$$

Thus, we obtain

$$\|\bar{x}_{n+1} - \bar{x}\| \le (1 - \alpha_n(1 - L(\psi)L(\varphi)))\|\bar{x}_n - \bar{x}\|.$$

Hence, it follows from Lemma 3 that $\{\bar{x}_n\}$ converges to $\bar{x}$ and from (4.7), it is easy to see that $\{\bar{y}_n\}$ converges to $\bar{y}$. This completes the proof. □

Corollary 3. For each $i \in \{1,2\}$; let Ω_i be nonempty closed convex subsets of a real Hilbert space H. Let Π_{Ω_i} be metric projections from E onto Ω_i and $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly monotone and T_i are δ_{T_i}-Lipschitz continuous. Let $H : E \times E \to E$ be α_1, β_1-generalized monotone mapping with respect to A_1, B_1, respectively such that H is ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized monotone mapping with respect to A_2, B_2, respectively such that G is ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exist constants $\lambda_i > 0$ such that $\alpha_n \in [0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ satisfying the following conditions:

$$\sqrt{1 - 2(\alpha_i + \beta_i) + 2c^2\rho_i^2} + \sqrt{1 - 2\lambda_i\tau_i + 2c^2\lambda_i^2\delta_{S_i}^2} + \lambda_i\delta_{T_i} < 1, \ \forall i \in I,$$
$$1 + 2c^2\rho_i^2 > 2(\alpha_i + \beta_i), 1 + 2c^2\lambda_i^2\delta_{S_i}^2 > 2\lambda_i\tau_i.$$

(i) Then there exists a unique element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x}, \bar{y})$ solves $SGVIneq$ (3.7).

(ii) The sequence $\{(\bar{x}_n, \bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel Mann iterative method:

$$\begin{cases} \bar{x}_{n+1} = (1 - \alpha_n)\bar{x}_n + \alpha_n Q_{\Omega_1}[G(A_2, B_2) - \lambda_1(S_2 - T_2)](\bar{y}_n), \\ \bar{y}_{n+1} = (1 - \alpha_n)\bar{y}_n + \alpha_n Q_{\Omega_2}[H(A_1, B_1) - \lambda_2(S_1 - T_1)](\bar{x}_n) \end{cases}$$

converges strongly to $(\bar{x}, \bar{y})$.

Theorem 6. *Let Q_{Ω_i} be sunny nonexpansive retractions from E onto Ω_i and $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly accretive and T_i are δ_{T_i}-Lipschitz continuous. Let $H : E \times E \to E$ be α_1, β_1-generalized accretive mapping with respect to A_1, B_1, respectively such that H is ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized accretive mapping with respect to A_2, B_2, respectively such that G is ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exist constants $\lambda_i > 0$ and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1) with $\beta_n > \alpha_n L(\psi)$ and $\alpha_n > \beta_n L(\varphi)$ satisfying the following conditions:*

$$\sqrt{1 - 2(\alpha_i + \beta_i) + 2c^2\rho_i^2} + \sqrt{1 - 2\lambda_i\tau_i + 2c^2\lambda_i^2\delta_{S_i}^2} + \lambda_i\delta_{T_i} < 1$$
$$1 + 2c^2\rho_i^2 > 2(\alpha_i + \beta_i), 1 + 2c^2\lambda_i^2\delta_{S_i}^2 > 2\lambda_i\tau_i. \quad (4.8)$$

(i) Then there exists a unique element $(\bar{x},\bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x},\bar{y})$ solves SGVIneq (3.6).

(ii) The sequence $\{(\bar{x}_n,\bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel S-iterative method:

$$\begin{cases} \bar{x}_{n+1} = Q_{\Omega_1}[G(A_2,B_2) - \lambda_1(S_2 - T_2)]\big[(1-\alpha_n)\bar{y}_n \\ \qquad\qquad +\alpha_n Q_{\Omega_2}[H(A_1,B_1) - \lambda_2(S_1-T_1)](\bar{x}_n)\big], \\ \bar{y}_{n+1} = Q_{\Omega_2}[H(A_1,B_1) - \lambda_2(S_1-T_1)]\big[(1-\beta_n)\bar{x}_n \\ \qquad\qquad +\beta_n Q_{\Omega_1}[G(A_2,B_2) - \lambda_1(S_2-T_2)](\bar{y}_n)\big] \end{cases}$$

converges strongly to $(\bar{x},\bar{y})$.

Proof. (i) Proof can be obtained by following the proof (i) of Theorem 5.

(ii) Define $\psi =: Q_{\Omega_1}[G(A_2,B_2) - \lambda_1(S_2-T_2)]$ and $\varphi =: Q_{\Omega_2}[H(A_1,B_1) - \lambda_2(S_1-T_1)]$. Since Q_{Ω_1} is sunny nonexpansive, then it follows from Lemma 5 and (4.8) that $\psi : \Omega_2 \to \Omega_1$ is $L(\psi)$-contraction mapping, where

$$L(\psi) = \sqrt{1 - 2(\alpha_2+\beta_2) + 2c^2\rho_2^2} + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2}.$$

Similarly, Q_{Ω_2} is sunny nonexpansive and $\varphi : \Omega_1 \to \Omega_2$ is $L(\varphi)$-contraction mapping, where

$$L(\varphi) = \sqrt{1 - 2(\alpha_1+\beta_1) + 2c^2\rho_1^2} + \sqrt{1 - 2\lambda_1\tau_1 + 2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1}.$$

From (3.9), we infer

$$\begin{aligned} &\|\bar{x}_{n+1} - \bar{x}\| = \|\psi[(1-\alpha_n)\bar{y}_n + \alpha_n\varphi(\bar{x}_n)] - \bar{x}\| \\ &\quad \le L(\psi)\|(1-\alpha_n)\bar{y}_n + \alpha_n\varphi(\bar{x}_n) - \bar{y}\| \le L(\psi)[(1-\alpha_n)\|\bar{y}_n - \bar{y}\| \\ &\quad + \alpha_n\|\varphi(\bar{x}_n) - \bar{y}\|] \le L(\psi)[(1-\alpha_n)\|\bar{y}_n - \bar{y}\| + \alpha_n L(\varphi)\|\bar{x}_n - \bar{x}\|]. \end{aligned}$$

Following the same steps as above, we obtain

$$\|\bar{y}_{n+1} - \bar{y}\| \le L(\varphi)[(1-\beta_n)\|\bar{x}_n - \bar{x}\| + \beta_n L(\psi)\|\bar{y}_n - \bar{y}\|].$$

Thus, we have

$$\begin{aligned} \|\bar{x}_{n+1} - \bar{x}\| + \|\bar{y}_{n+1} - \bar{y}\| \le L(\varphi)[1 - (\beta_n - \alpha_n L(\psi))]\|\bar{x}_n - \bar{x}\| \\ + L(\psi)[1 - (\alpha_n - \beta_n L(\varphi))]\|\bar{y}_n - \bar{y}\|. \end{aligned} \tag{4.9}$$

Choose $L(\triangle) = \max\{L(\psi), L(\varphi)\}$, then from (4.9) and assumptions of the theorem, we have

$$\|\bar{x}_{n+1} - \bar{x}\| + \|\bar{y}_{n+1} - \bar{y}\| \le L(\triangle)[\|\bar{x}_n - \bar{x}\| + \|\bar{y}_n - \bar{y}\|]. \tag{4.10}$$

It follows from (2.2) and (4.10) that

$$\|(\bar{x}_{n+1},\bar{y}_{n+1}) - (\bar{x},\bar{y})\|_* \le L(\triangle)\|(\bar{x}_n,\bar{y}_n) - (\bar{x},\bar{y})\|_*.$$

Now utilizing Lemma 3, we have

$$\lim_{n\to\infty} \|(\bar{x}_n,\bar{y}_n) - (\bar{x},\bar{y})\|_* = \lim_{n\to\infty} \|\bar{x}_n - \bar{x}\| + \lim_{n\to\infty} \|\bar{y}_n - \bar{y}\| = 0.$$

Thus, the sequence $\{(\bar{x}_n,\bar{y}_n)\} \in \Omega_1 \times \Omega_2$ converges strongly to $(\bar{x},\bar{y})$. □

Corollary 4. For each $i \in \{1,2\}$; let Ω_i be nonempty closed convex subsets of a real Hilbert space H. Let Π_{Ω_i} be metric projections from E onto Ω_i and $A_i, B_i, S_i, T_i : \Omega_i \to E$ be the single-valued mappings such that S_i are δ_{S_i}-Lipschitz continuous, τ_i-strongly monotone and T_i are δ_{T_i}-Lipschitz continuous. Let $H : E \times E \to E$ be α_1, β_1-generalized monotone mapping with respect to A_1, B_1, respectively such that H is ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1. Let $G : E \times E \to E$ be α_2, β_2-generalized monotone mapping with respect to A_2, B_2, respectively such that G is ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2. Suppose that there exist constants $\lambda_i > 0$ and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in (0,1) with $\beta_n > \alpha_n L(\psi)$ and $\alpha_n > \beta_n L(\varphi)$ satisfying the following conditions:

$$
\begin{aligned}
&\sqrt{1-2(\alpha_i+\beta_i)+2c^2\rho_i^2} + \sqrt{1-2\lambda_i\tau_i+2c^2\lambda_i^2\delta_{S_i}^2} + \lambda_i\delta_{T_i} < 1,\\
&1+2c^2\rho_i^2 > 2(\alpha_i+\beta_i), 1+2c^2\lambda_i^2\delta_{S_i}^2 > 2\lambda_i\tau_i.
\end{aligned}
$$

(i) Then there exists a unique element $(\bar{x},\bar{y}) \in \Omega_1 \times \Omega_2$ such that $(\bar{x},\bar{y})$ solves $SGVIneq$ (3.7).

(ii) The sequence $\{(\bar{x}_n,\bar{y}_n)\} \in \Omega_1 \times \Omega_2$ generated by parallel S-iterative method:

$$
\begin{cases}
\bar{x}_{n+1} = Q_{\Omega_1}[G(A_2,B_2)-\lambda_1(S_2-T_2)]\big[(1-\alpha_n)\bar{y}_n \\
\qquad +\alpha_n Q_{\Omega_2}[H(A_1,B_1)-\lambda_2(S_1-T_1)](\bar{x}_n)\big],\\
\bar{y}_{n+1} = Q_{\Omega_2}[H(A_1,B_1)-\lambda_2(S_1-T_1)]\big[(1-\beta_n)\bar{x}_n \\
\qquad +\beta_n Q_{\Omega_1}[G(A_2,B_2)-\lambda_1(S_2-T_2)](\bar{y}_n)\big]
\end{cases}
$$

converges strongly to $(\bar{x},\bar{y})$.

5 Numerical example

Example 2. Let $E=\mathbb{R}, \Omega_1=\Omega_2=[0,\infty)$. Let $\Pi_{\Omega_1} : E \to \Omega_1$ and $\Pi_{\Omega_2} : E \to \Omega_2$ be the single-valued mappings defined by

$$
\Pi_{\Omega_1}(x) = \frac{1}{3}(x+1) \ \text{ and } \ \Pi_{\Omega_2}(x) = \frac{1}{6}(2x-3), \ \ \forall x \in E.
$$

Then the mappings Π_{Ω_1} and Π_{Ω_2} are Lipschitz continuous with constants $\delta_{\Pi_{\Omega_1}} = \frac{1}{3}$ and $\delta_{\Pi_{\Omega_2}} = \frac{1}{3}$, respectively. Suppose that $A_1, B_1, S_1, T_1 : \Omega_1 \to E$ and $A_2, B_2, S_2, T_2 : \Omega_2 \to E$ are the single-valued mappings. Let $H : E \times E \to E$ be α_1, β_1 and $G : E \times E \to E$ be α_2, β_2-generalized accretive mappings. We define all the mappings mentioned above as follows:

$$
\begin{aligned}
A_1(x) = &\frac{-x}{2} + \frac{1}{3}, \quad B_1(x) = \frac{2x}{3} - \frac{1}{2}, \text{ for all } x \in \Omega_1,\\
A_2(x) = &-x + \frac{3}{2}, \quad B_2(x) = x+1, \text{ for all } x \in \Omega_2,
\end{aligned}
$$

$$S_1(x) = \frac{x+1}{3}, \quad T_1(x) = \frac{x}{6} + \frac{2}{3}, \text{ for all } x \in \Omega_1,$$
$$S_2(x) = \frac{x}{2} - 1, \quad T_2(x) = \frac{-x}{4} + \frac{1}{2}, \text{ for all } x \in \Omega_2,$$
$$H(A_1(x), B_1(x)) = (A_1(x) - B_1(x)), \text{ for all } x \in \Omega_1,$$
$$G(A_2(x), B_2(x)) = \frac{A_2(x) - B_2(x)}{2}, \text{ for all } x \in \Omega_2.$$

Then, it is easily verified that

1. S_1 and S_2 are Lipschitz continuous with constants $\delta_{S_1} = \frac{1}{3}, \delta_{S_2} = \frac{1}{2}$ and strongly accretive with constants $\tau_1 = \frac{1}{3}, \tau_2 = \frac{1}{2}$, respectively.
2. T_1 and T_2 are Lipschitz continuous with constants $\delta_{T_1} = \frac{1}{6}$ and $\delta_{T_2} = \frac{1}{4}$, respectively.
3. H is α_1, β_1-generalized accretive mapping and ρ_1-mixed Lipschitz continuous with respect to A_1 and B_1 with constants $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{2}{3}$ and $\rho_1 = \frac{7}{6}$.
4. G is α_2, β_2-generalized accretive mapping and ρ_2-mixed Lipschitz continuous with respect to A_2 and B_2 with constants $\alpha_2 = \frac{1}{2}, \beta_2 = \frac{1}{2}$ and $\rho_2 = 1$.

If, we choose $c = 1$, then condition (3.10) is satisfied for $i = 1, 2$. That is,

$$\begin{aligned} 0 &< \delta_{\Pi_{\Omega_2}}(\sqrt{1 - 2(\alpha_1 + \beta_1) + 2c^2\rho_1^2} + \sqrt{1 - 2\lambda_1\tau_1 + 2c^2\lambda_1^2\delta_{S_1}^2} + \lambda_1\delta_{T_1}) \\ &= 0.70365 < 1, \\ 0 &< \delta_{\Pi_{\Omega_1}}(\sqrt{1 - 2(\alpha_2 + \beta_2) + 2c^2\rho_2^2} + \sqrt{1 - 2\lambda_2\tau_2 + 2c^2\lambda_2^2\delta_{S_2}^2} + \lambda_2\delta_{T_2}) \\ &= 0.65237 < 1. \end{aligned}$$

Define $U_1 =: \Pi_{\Omega_2}[H(A_1, B_1) - \lambda_1(S_1 - T_1)]$ and $U_2 =: \Pi_{\Omega_1}[G(A_2, B_2) - \lambda_2(S_2 - T_2)]$. Then U_1 is a contraction mapping with constant $\kappa_1 = \frac{2}{3}$ and U_2 is a contraction mapping with constant $\kappa_2 = \frac{7}{12}$. Thus, all the suppositions and conditions of Theorem 1 and Theorem 2 are satisfied. Hence $x = -1.51136$ and $y = 1.590909$ are altering points of U_1 and U_2. That is, $x = -1.51136$ and $y = 1.590909$ solves APP (3.1).

Now, we shall present the convergence of sequences generated by parallel iterative scheme (3.8). For arbitrary $(x_0, y_0) \in \Omega_1 \times \Omega_2$ and $\alpha_n = \frac{n}{n+1}$, we have

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n U_2(y_n) = \frac{1}{(n+1)}x_n + \frac{n}{12(n+1)}[-7y_n - 7] \\ &= \frac{1}{12(n+1)}[12x_n - 7ny_n - 7n], \ \forall n \in \mathbb{N}, \\ y_{n+1} &= (1 - \alpha_n)y_n + \alpha_n U_1(x_n) = \frac{1}{(n+1)}y_n + \frac{n}{12(n+1)}[-8x_n + 7] \\ &= \frac{1}{12(n+1)}[12y_n - 8nx_n + 7n], \ \forall n \in \mathbb{N}. \end{aligned}$$

That is,

$$x_{n+1} = \frac{1}{12(n+1)}[12x_n - 7ny_n - 7n], \ \forall n \in \mathbb{N},$$
$$y_{n+1} = \frac{1}{12(n+1)}[12y_n - 8nx_n + 7n], \ \forall n \in \mathbb{N}.$$

Next, we shall present the convergence of sequences generated by parallel iterative scheme (3.9). For arbitrary $(x_0, y_0) \in \Omega_1 \times \Omega_2$ and $\alpha_n = \frac{n}{n+1} = \beta_n$, we have

$$\begin{aligned} x_{n+1} &= U_2[(1-\alpha_n)y_n + \alpha_n U_1(x_n)] = U_2\Big[\frac{1}{(n+1)}y_n + \frac{n(-8x_n+7)}{12(n+1)}\Big] \\ &= U_2\Big[\frac{12y_n - 8nx_n + 7n}{12(n+1)}\Big] = \Big[\frac{-84y_n + 56nx_n - 49n - 84}{144(n+1)}\Big], \ \forall n \in \mathbb{N}, \\ y_{n+1} &= U_1[(1-\beta_n)x_n + \beta_n U_2(y_n)] = U_1\Big[\frac{1}{(n+1)}x_n + \frac{n(-7y_n-7)}{12(n+1)}\Big] \\ &= U_1\Big[\frac{12x_n - 7ny_n - 7n}{12(n+1)}\Big] = \Big[\frac{-96x_n + 56ny_n + 56n + 84}{144(n+1)}\Big], \ \forall n \in \mathbb{N}. \end{aligned}$$

Thus, we have

$$x_{n+1} = \Big[\frac{-84y_n + 56nx_n - 49n - 84}{144(n+1)}\Big], \ \forall n \in \mathbb{N},$$
$$y_{n+1} = \Big[\frac{-96x_n + 56ny_n + 56n + 84}{144(n+1)}\Big], \ \forall n \in \mathbb{N}.$$

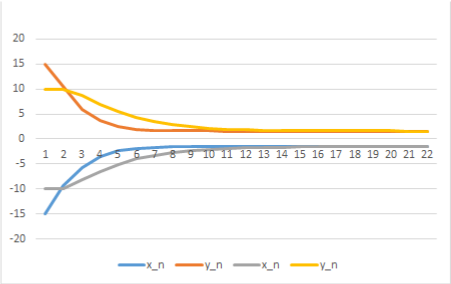

Figure 1. Convergence of parallel Mann iterative method (3.13) and parallel S-iterative method (3.14).

The convergence of sequences $\{x_n\}$ and $\{y_n\}$ is plotted in Figures 1 and 2 using different initial values and from Table 1 and Table 2, we infer that the sequences $\{x_n\}$ and $\{y_n\}$ produced by the presented iterative methods converge to the altering points $x = -1.51136$ and $y = 1.590909$.

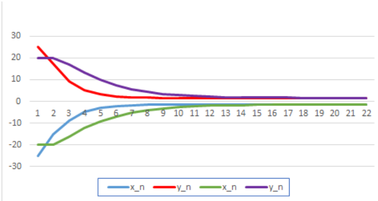

Figure 2. Convergence of parallel Mann iterative method (3.13) and parallel S-iterative method (3.14).

Table 1. Convergence of parallel Mann iterative method (3.13) and parallel S-iterative method (3.14).

	Parallel Mann iterations		Parallel S-iterations	
n	x_n	y_n	x_n	y_n
1	-10	10	-15	15
5	-4.0053	4.257716	-2.37071	2.509074
10	-1.85357	1.956734	-1.52311	1.603469
15	-1.55196	1.634313	-1.5115	1.591052
20	-1.51588	1.595741	-1.51137	1.590911
25	-1.51185	1.591428	-1.51136	1.590909
30	-1.51141	1.590964	-1.51136	1.590909
35	-1.51137	1.590915	-1.51136	1.590909
40	-1.51136	1.59091	-1.51136	1.590909
45	-1.51136	1.590909	-1.51136	1.590909
50	-1.51136	1.590909	-1.51136	1.590909
60	-1.51136	1.590909	-1.51136	1.590909
100	-1.51136	1.590909	-1.51136	1.590909

Table 2. Convergence of parallel Mann iterative method (3.13) and parallel S-iterative method (3.14).

	Parallel Mann iterations		Parallel S-iterations	
n	x_n	y_n	x_n	y_n
1	-20	20	-25	25
5	-6.95669	7.41358	-3.00961	3.19173
10	-2.25854	2.389654	-1.53185	1.612807
15	-1.60001	1.685677	-1.5116	1.591159
20	-1.52123	1.60146	-1.51137	1.590912
25	-1.51242	1.592042	-1.51136	1.590909
30	-1.51147	1.591028	-1.51136	1.590909
35	-1.51138	1.590921	-1.51136	1.590909
40	-1.51136	1.59091	-1.51136	1.590909
45	-1.51136	1.590909	-1.51136	1.590909
50	-1.51136	1.590909	-1.51136	1.590909
60	-1.51136	1.590909	-1.51136	1.590909
100	-1.51136	1.590909	-1.51136	1.590909

6 Conclusions

In this paper, we investigated an altering points problem involving generalized accretive mappings over closed convex subsets of a real uniformly smooth Banach space. Parallel Mann and parallel S-iterative methods are suggested to analyze the approximate solution of altering points problem. As a consequence, some systems of generalized variational inclusions and generalized variational inequalities are also explored using the conceptual framework of altering points. The existence result and convergence analysis is validated by an illustrative example.

Acknowledgements

The authors are grateful to the editor and the anonymous referees for their valuable comments and suggestions. This research is supported by the Deanship of Scientific Research at Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia under the research project (PSAU-2022/01/20271).

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