Mathematical Modelling and Analysis
Volume 28, Issue 4, 653–672, 2023
https://doi.org/10.3846/mma.2023.17087

On Split Generalized Equilibrium and Fixed Point Problems of Bregman W-Mappings with Multiple Output Sets in Reflexive Banach Spaces

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Received June 3, 2022; accepted October 3, 2023

Abstract. In this paper, we introduce a Halpern iteration process for computing the common solution of split generalized equilibrium problem and fixed points of a countable family of Bregman W-mappings with multiple output sets in reflexive Banach spaces. We prove a strong convergence result for approximating the solutions of the aforementioned problems under some mild conditions. It is worth mentioning that the iterative algorithm employ in this article is designed in such a way that it does not require the prior knowledge of operator norm. We also provide some numerical examples to illustrate the performance of our proposed iterative method. The result discuss in this paper extends and complements many related results in literature.

Keywords: Bregman weak relatively nonexpansive mapping, Bregman W-mapping, Halpern method, iterative scheme, split generalized equilibrium problem.

AMS Subject Classification: 47H06; 47H09; 47J05; 47J25.

1 Introduction

Let Y be a reflexive Banach space with its dual Y^* and D be a nonempty, closed and convex subset of Y. The Generalized Equilibrium Problem (in brief, GEP) is to find $x^* \in D$ such that

$$G(x^*, x) + b(x^*, x) - b(x^*, x^*) \geq 0, \ \forall \ x \in D, \tag{1.1}$$

where $G : D \times D \to \mathbb{R}$ is a bifunction and $b : D \times D \to \mathbb{R}$ is a skew matrix. If $b \equiv 0$, then GEP (1.1) reduces to the following Equilibrium Problem (in brief, EP) which is to find $x^* \in D$ such that

$$G(x^*, x) \geq 0, \ \forall \ x \in D.$$

The Equilibrium Problem is known to include many mathematical problems, for example, variational inclusion problem, complementary problem, saddle point problem, Nash equilibrium problem in non-cooperative games, minimax inequality problem, minimization problem, variational inequality problem and fixed point problem, see [6,11,14,17,19,22,33,34,37]. Let D and E be nonempty, closed and convex subsets of two real Banach spaces Y_1 and Y_2 respectively. Let $A : Y_1 \to Y_2$ be a bounded linear operator. The Split Feasibility Problem (in brief, SFP) introduced by Censor and Elfving [15] is to find a point

$$x^* \in D \text{ such that } Ax^* \in E. \tag{1.2}$$

By combining SFP (1.2) and GEP (1.1), we have the Split Generalized Equilibrium Problem (in brief, SGEP), which is to

$$\text{find } x^* \in D \text{ such that } G_1(x^*, x) + b_1(x^*, x) - b_1(x^*, x^*) \geq 0, \ \forall \ x \in D, \tag{1.3}$$

and such that

$$y^* = Ax^* \in E \text{ solves } G_2(y^*, y) + b_2(y^*, y) - b_2(y^*, y^*) \geq 0, \ \forall \ y \in E. \tag{1.4}$$

We denote by

$$SGEP(G_1, b_1, G_2, b_2) := \{x^* \in D : x^* \in GEP(G_1, b_1) \text{ and } Ax^* \in GEP(G_2, b_2)\},$$

where $G_1, b_1 : D \times D \to \mathbb{R}$ and $G_2, b_2 : E \times E \to \mathbb{R}$ are bifunctions respectively. If b_1 and b_2 equal to zero in (1.3) and (1.4), we have the Split Equilibrium Problem (in brief, SEP) which is to

$$\text{find } x^* \in D \text{ such that } G_1(x^*, x) \geq 0, \ \forall \ x \in D, \tag{1.5}$$

that solves

$$y^* = Ax^* \in E \text{ solves } G_2(y^*, y) \geq 0, \ \forall \ y \in E. \tag{1.6}$$

We denote by $SEP(G_1, G_2)$ the solution set of (1.5)–(1.6). The Split Generalized Equilibrium Problem is very general in the sense that it includes as particular cases, split varaitional inequality problem and split minimization problem, to mention a few, (see [1, 2, 3, 4, 23, 24, 30, 31]).
To solve GEP (1.1), we need the following assumptions: Let $G : D \times D \to \mathbb{R}$.

Assumption 1.3:
(i) $G(x, x) = 0, \ \forall \ x \in D$;
(ii) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0, \ \forall \ x, y \in D$;
(iii) For each $x, y, z \in D$; $\limsup_{t \to 0} G(tz + (1 - t)x, y) \leq G(x, y)$;
(iv) For each $x \in D, y \longmapsto G(x, y)$ is convex and lower semicontinuous.

Assumption 1.4: Let $b : D \times D \to \mathbb{R}$.
(i) b is skew-symmetric, i.e., $b(x, x) - b(x, y) - b(y, x) - b(y, y) \geq 0, \ \forall \ x, y \in D$;
(ii) b is convex in the second argument; (iii) b is continuous.

In 2018, Phuengrattana and Lerkchayaphum [32] introduced a shrinking projection method for solving the common solution of split generalized equilibrium problem and fixed point problem of multivalued nonexpansive mappings in real Hilbert spaces. They proved that the sequence $\{x_n\}$ converges strongly to $proj^g_{\Delta} x_0$, where $\Delta := Sol(GEP(1.1) \cap F(T)$ is nonempty.

Our proposed method is endowed with the following characteristics:

(1) We extend the results of [1,2,32] from real Hilbert spaces to a more general space which is convex, continuous and strongly coercive Bregman function, which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets.

(2) Our method does not require computing the projection of the current iterate onto the intersection of sets C_n and Q_n which was used in [5, 18, 32].

(3) In the result of [2, 18, 25, 32] and other related results, we were able to dispense with one of the resolvents of the EP. Using the notion of multiple output sets, we were able to generalize some related results in literature without one of the resolvents.

(4) Our method uses self-adaptive stepsizes and the implementation of our method does not require prior knowledge of the norm of the bounded linear operator A, see [32].

(5) Our result also generalizes the results of [2,18,25,32] to a type of SGEP with multiple output sets.

2 Preliminaries

In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.
The notion of W-mapping was first introduced in 1999 by Atsushiba and Takahashi [8] and since then, it has been considered for a finite family of mappings, see ([19, 20, 27]). The notion was extended to a Banach space by Naraghirad and Timnak [29] as follows. Let D be a nonempty, closed and

convex subset of a reflexive Banach space Y. Let $\{S_n\}_{n\in\mathbb{N}}$ be an infinite family of Bregman weak relatively nonexpansive mappings of D into itself, and let $\{\mu_{n,t} : t, n \in \mathbb{N}, 1 \leq t \leq n\}$ be a sequence of real numbers such that $0 \leq \mu_{i,j} \leq 1$ for every $i, j \in \mathbb{N}$ with $i \geq j$. Then, for any $n \in \mathbb{N}$, we define a mapping W_n of D into itself as follows:

$$\begin{aligned}
&U_{n,n+1}x = x,\\
&U_{n,n}x = proj_D^g(\triangledown g^*[\mu_{n,n} \triangledown g(S_n, U_{n,n+1}x) + (1-\mu_{n,n}) \triangledown g(x)]),\\
&U_{n,n-1}x = proj_D^g(\triangledown g^*[\mu_{n,n-1} \triangledown g(S_{n-1}U_{n,n}x) + (1-\mu_{n,n-1}) \triangledown g(x)]),\\
&\qquad\vdots\\
&U_{n,t}x = proj_D^g(\triangledown g^*[\mu_{n,t} \triangledown g(S_tU_{n,t+1}x) + (1-\mu_{n,t}) \triangledown g(x)]),\\
&\qquad\vdots\\
&U_{n,2}x = proj_D^g(\triangledown g^*[\mu_{n,2} \triangledown g(S_2U_{n,3}x) + (1-\mu_{n,2}) \triangledown g(x)]),\\
&W_{n,x} = U_{n,1}x = \triangledown g^*[\mu_{n,1} \triangledown g(S_1U_{n,2}x) + (1-\mu_{n,1}) \triangledown g(x)],
\end{aligned}$$

for all $x \in D$, where $proj_D^g$ is the Bregman projection from Y onto D. Such a mapping W_n is called the Bregman W-mapping generated by $S_n, S_{n-1}, \dots, S_1$ and $\mu_{n,n}, \mu_{n,n-1}, \dots, \mu_{n,1}$.

Let Y be a reflexive Banach space with Y^* its dual and Q be a nonempty closed and convex subset of Y. Let $g : Y \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of g denoted as $g^* : Y^* \to (-\infty, +\infty]$ is defined as

$$g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in Y\},\ x^* \in Y^*.$$

Let the domain of g be denoted as $dom(g) = \{x \in Y : g(x) < +\infty\}$, hence for any $x \in intdom(g)$ and $y \in Y$, we define the right-hand derivative of g at x in the direction of y by

$$g^o(x, y) = \lim_{t\to 0^+} \frac{g(x + ty) - g(x)}{t}.$$

Let $g : Y \to (-\infty, +\infty]$ be a function, then g is said to be:
(i) Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x, y)$ coincides with $\nabla g(x)$ (the value of the gradient ∇g of g at x);
(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in intdomg$;
(iii) Fréchet differentiable at x, if its limit is attained uniformly in $\|y\| = 1$;
(iv) Uniformly Fréchet differentiable on a subset Q of Y, if the above limit is attained uniformly for $x \in Q$ and $\|y\| = 1$.
(v) Essentially smooth, if the subdifferential of g denoted as ∂g is both locally bounded and single-valued on its domain, where $\partial g(x) = \{w \in Y : g(x)-g(y) \geq \langle w, y - x\rangle,\ y \in Y\}$;
(vi) Essentially strictly convex, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $dom\ \partial g$;
(vii) Legendre, if it is both essentially smooth and essentially strictly convex. See [9, 10] for more details on Legendre functions.

Alternatively, a function g is said to be Legendre if it satisfies the following conditions:
(i) The $intdom(g)$ is nonempty, g is Gâteaux differentiable on $intdom(g)$ and $dom\nabla g = intdom(g)$;
(ii) The $intdomg^*$ is nonempty, g^* is Gâteaux differentiable on $intdomg^*$ and $dom\nabla g^* = intdom(g)$.
Let E be a Banach space and $B_s := \{z \in Y : \|z\| \leq s\}$ for all $s > 0$. Then, a function $g : Y \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of Y, [see pp. 203 and 221] [39] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, with ρ_s denoting the gauge of uniform convexity of g. The function g is also said to be uniformly smooth on bounded subsets of Y, [see pp. 221] [39], if $\lim_{t \downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x \in B, y \in S_Y, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)ty) + (1-\alpha)g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, and uniformly convex if the function $\delta g : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta g(t) := \sup \left\{ \frac{1}{2} g(x) + \frac{1}{2} g(y) - g(\frac{x+y}{2}) : \|y - x\| = t \right\},$$

satisfies $\lim_{t \downarrow 0} \frac{\delta g(t)}{t} = 0$.

Definition 1. [12] Let Y be a Banach space. A function $g : Y \to (-\infty, \infty]$ is said to be proper if the interior of its domain $dom(g)$ is nonempty. Let $g : Y \to (-\infty, \infty]$ be a convex and Gâteaux differentiable function. Then the Bregman distance corresponding to g is the function $D_g : dom(g) \times intdom(g) \to \mathbb{R}$ defined by

$$D_g(x, y) := g(x) - g(y) - \langle x - y, \nabla_Y^g(y) \rangle, \ \forall \ x, y \in Y, \tag{2.1}$$

where ∇_Y^g is the gradient function of Y dependent on g. It is clear that $D_g(x, y) \geq 0$ for all $x, y \in Y$.

It is well-known that Bregman distance D_g does not satisfy all the properties of a metric function because D_g fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in dom(g)$ and $y, z \in intdom(g)$,

$$D_g(x, z) = D_g(x, y) + D_g(y, z) + \langle x - y, \nabla_Y^g(y) - \nabla_Y^g(z) \rangle.$$

In particular,

$$D_g(x, y) = -D_g(y, x) + \langle y - x, \nabla_Y^g(y) - \nabla_Y^g(x) \rangle, \ \forall \ x, y \in Y.$$

The relationship between D_g and $\|.\|$ is guaranteed when g is strongly convex with strong convexity constant $\rho > 0$, i.e.,

$$D_g(x,y) \geq \frac{\rho}{2}\|x-y\|^2,\ \forall\ x \in dom(g),\ y \in intdom(g).$$

Let $g : Y \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function and $T : Q \to intdom(g)$ be a mapping, a point $x \in Q$ is called a fixed point of T, if for all $x \in Q,\ Tx = x$. We denote by $F(T)$ the set of all fixed points of T. Furthermore, a point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A point $p \in Q$ is called a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote the set of strong asymptotic fixed points of T by $\tilde{F}(T)$. It follows from the definition that $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$.

Let Q be a nonempty closed and convex subset of int(dom g), then we define an operator $T : Q \to int(domg)$ to be:

(i) Bregman relatively nonexpansive, if $F(T) \neq \emptyset$, and

$$D_g(p,Tx) \leq D_g(p,x),\ \forall\ p \in F(T),\ x \in Q \text{ and } F(\hat{T}) = F(T).$$

(ii) Bregman weak relatively nonexpansive, if $\tilde{F}(T) \neq \emptyset$, and

$$D_g(p,Tx) \leq D_g(p,x),\ \forall\ p \in F(T),\ x \in Q \text{ and } F(\tilde{T}) = F(T).$$

(iii) Bregman quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$D_g(p,Tx) \leq D_g(p,x), \forall\ x \in Q \text{ and } p \in F(T).$$

(iv) Bregman firmly nonexpansive (BFNE), if

$$\langle \nabla_Y^g(Tx) - \nabla_Y^g(Ty), Tx - Ty\rangle \leq \langle \nabla_Y^g(x) - \nabla_Y^g(y), Tx - Ty\rangle,\ \forall\ x,y \in Y.$$

Example 1. [16] Let $Y = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty} |\sigma_i|^2 < \infty\},\ \|\sigma\| = \big(\sum_{i=1}^{\infty} |\sigma_i|\big)^{\frac{1}{2}}\ \forall\ \sigma \in H$ and let $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in Y$. Let $\{x_n\} \subset Y$ be a sequence defined by $x_0 = (1,0,0,0,\ldots),\ x_1 = (1,1,0,0,\ldots),\ x_2 = (1,0,1,0,\ldots), \ldots, x_n = (\sigma_{n,1}, \sigma_{n,2}, \sigma_{n,3}, \ldots), \ldots$, where

$$\sigma_{n,k} = \begin{cases} 1, & \text{if } k = 1, n+1, \\ 0, & \text{if otherwise},\ \forall\ n \in \mathbb{N}, \end{cases}$$

$n \in \mathbb{N}$. Define the mapping $T : H \to H$ by

$$Tx = \begin{cases} \frac{n}{n+1}x, & \text{if } x = x_n, \\ -x, & \text{if } x \neq x_n. \end{cases}$$

We define a countable family $S_j : H \to H$ by

$$S_j(x) = \begin{cases} \frac{n}{n+1}x, & \text{if } x = x_n, \\ \frac{-j}{j+1}x, & \text{if } x \neq x_n, \end{cases}$$

for all $j \geq 1$ and $n \geq 0$. It is clear that $F(S_j) = \{0\}$ for all $j \geq 1$.

It can be shown that T and S_j are Bregman quasi-nonexpansive, precisely Bregman weak relatively nonexpansive (see [16,28]).

DEFINITION 2. [21] Let Q be a nonempty, closed and convex subset of a reflexive Banach space Y and $g : Y \to (-\infty, +\infty]$ be a strongly coercive Bregman function. Let β and γ be real numbers with $\beta \in (-\infty, 1)$ and $\gamma \in [0, \infty)$, respectively. Then a mapping $T : Q \to Y$ with $F(T) \neq \emptyset$ is called Bregman (β, γ)-demigeneralized if for any $x \in Q$ and $p \in F(T)$,

$$\langle x-p, \nabla_Y^g(x)-\nabla_Y^g(Tx)\rangle \geq (1-\beta)D_g(x, Tx) + \gamma D_g(Tx, x), \forall\, x \in Y, p \in F(T).$$

DEFINITION 3. [9,13] A function $g : Y \to \mathbb{R}$ is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} g(x)/\|x\| = \infty.$$

DEFINITION 4. A mapping $T : Q \to Y$ is said to be demiclosed at p if $\{x_n\}$ is a sequence in Q such that $\{x_n\}$ converges weakly to some $x^* \in Q$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Lemma 1. *[38] Let Y be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : Y \to \mathbb{R}$ be a strongly coercive Bregman function. Then,*
(i) For any $x, y \in B_s$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned} &D_g\big(x, \nabla_{Y^*}^{g^*}[\alpha\nabla_Y^g(y) + (1-\alpha)\nabla_Y^g(z)]\big) \\ &\quad \leq \alpha D_g(x, y) + (1-\alpha)D_g(x, z) - \alpha(1-\alpha)\rho_s(\|\nabla_Y^g(y) - \nabla_Y^g(z)\|); \end{aligned}$$

(ii) For any $x, y \in B_s := \{z \in Y : \|z\| \leq s\},\ s > 0$,

$$\rho_s(\|x-y\|) \leq D_g(x, y).$$

Lemma 2. *[13] Let Y be a reflexive Banach space, $g : Y \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*),\ x \in Y,\ x^* \in Y^*.$$

The following assertions also hold:

$$D_g(x, \nabla_{Y^*}^{g^*}(x^*)) = V(x, x^*),\ \textit{for all } x \in Y \textit{ and } x^* \in Y^*,$$
$$V(x, x^*) + \langle \nabla_{Y^*}^{g^*}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*) \textit{ for all } x \in Y \textit{and } x^*, y^* \in Y^*.$$

Also, following a similar approach as in Lemma 2 and for any $x \in Y, y^*,\ z^* \in B_r$ and $\alpha \in (0, 1)$, we have

$$V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x, y^*) + (1-\alpha)V_g(x, z^*) - \alpha(1-\alpha)\rho_r^*(\|y^* - x^*\|).$$

The resolvent of $G : D\times D\to\mathbb{R}$ with respect to b is the operator $res^g_{G,b} : Y\to 2^D$ defined as follows:

$$\begin{aligned} res^g_{G,b}(u) =\{&u_0 \in D : G(u_0, v) + \langle\nabla g(u_0) - \nabla g(u), v - u_0\rangle \\ &+ b(u_0, v) - b(u_0, u_0) \geq 0, \forall\ v \in D\},\ \forall\ u \in Y. \end{aligned} \tag{2.2}$$

We obtain some properties of the resolvent operator $res^g_{G,b}$.

Lemma 3. *[29] Let $g : Y \to (-\infty, +\infty]$ be a Gâteaux differentiable and coercive function. Let $G, b : D \times D \to \mathbb{R}$ satisfy Assumptions 1.3 and 1.4, respectivley, and let $res^g_{G,b} : Y \to 2^D$ be defined by (2.2). Then, the following hold: (i) $dom(res^g_{G,b}) = Y$, (ii) $res^g_{G,b}$ is single-valued,*
(iii) $res^g_{G,b}$ is a Bregman firmly nonexpansive type mapping, that is, $\forall\ u, v \in Y$,

$$\begin{aligned} &\langle\nabla g(res^g_{G,b}u) - \nabla g(res^g_{G,b}v), res^g_{G,b}u - res^g_{G,b}v\rangle \\ &\leq \langle\nabla g(u) - \nabla g(v), res^g_{G,b}u - res^g_{G,b}v\rangle, \end{aligned}$$

(iv) $F(res^g_{G,b}) = Sol(GEP(1.1))$ is closed and convex,
(v) $D_g(q, res^g_{G,b}u) + D_g(res^g_{G,b}u, u) \leq D_g(q, u),\ \forall\ q \in F(res^g_{G,b})$,
(vi) $res^g_{G,b}$ is Bregman quasi-nonexpansive.

Lemma 4. *[21] Let Y_1 and Y_2 be two Banach spaces. Let $F : Y_1 \to Y_2$ be a bounded linear operator and $T : Y_2 \to Y_2$ be a Bregman (ϕ, σ)-demigeneralized for some $\phi \in (-\infty, 1)$ and $\sigma \in [0, \infty)$. Suppose that $K = ran(A) \cap F(T) \neq \emptyset$ (where $ran(A)$ denotes the range of (A). Then for any $(x, q) \in Y_1 \times K$,*

$$\begin{aligned} \langle x - q, F^*(\nabla^{g_2}_{Y_2}(T(Fx)))\rangle &\geq (1 - \phi)D_{g_2}(Fx, T(Fx)) + \sigma D_{g_2}(T(Fx), Fx) \\ &\geq (1 - \phi)D_{g_2}(Fx, T(Fx)). \end{aligned}$$

So, given any real numbers ξ_1 and ξ_2, the mapping $L_1 : Y_1 \to [0, \infty)$ and $L_2 : Y_2 \to [0.\infty)$ formulated for $x \in Y_1$ as

$$L_1(x) = \begin{cases} \frac{D_{g_2}(Fx,TFx)}{D^*_{g_1}(F^*(\nabla^{g_2}_{Y_2}(Fx)),F^*(\nabla^{g_2}_{Y_2}(TFx))}, & \text{if } (I - T)Fx \neq 0, \\ \xi_1, & \text{otherwise}, \end{cases}$$

and

$$L_2(x) = \begin{cases} \frac{D^*_{g_1}(\nabla^{g_1}_{Y_1}(x)-\gamma F^*(\nabla^{g_2}_{Y_2}(Fx)-\nabla^{g_2}_{Y_2}(TFx)),\nabla^{g_1}_{Y_1}(x))}{D^*_{g_1}(F^*(\nabla^{g_2}_{Y_2}(Fx)),F^*(\nabla^{g_2}_{Y_2}(TFx))}, & \text{if } (I - T)Fx \neq 0, \\ \xi_2, & \text{otherwise}, \end{cases}$$

are well-defined, where γ is any nonnegative real number. Moreover, for any $(x, p) \in E_1 \times K$, we have

$$\begin{aligned} D_{g_1}(q, y) \leq &D_{g_1}(q, x) \\ &- (\gamma(1 - \phi)L_1(x) - L_2(x))D_{g_1^*}(F^*(\nabla^{g_2}_{Y_2}(Fx)), F^*(\nabla^{g_2}_{Y_2}(TFx)), \end{aligned} \tag{2.3}$$

where

$$y = (\nabla^{g_1}_{Y_1})^{-1}[\nabla^{g_1}_{Y_1}(x) - \gamma F^*(\nabla^{g_2}_{Y_2}(Fx) - \nabla^{g_2}_{Y_2}(TFx))].$$

Remark 1. It is easy to see from [21] that $res^g_{G,b}$ *is* $(0,1)$*-demigeneralized. Therefore, we conclude from* (2.3) *that*

$$D_{g_1}(q,y)\leq D_{g_1}(q,x)-(\gamma L_1(x)-L_2(x))D_{g_1^*}(F^*(\nabla^{g_2}_{Y_2}(Fx)),F^*(\nabla^{g_2}_{Y_2}((res^g_{G,b})x), \tag{2.4}$$

where $T = res^g_{G,b}$.

Lemma 5. *[13] Let* Y *be a Banach space and* $g : Y \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* Y*. Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be bounded sequences in* Y*. Then,*

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 \Longleftrightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 6. *[29] Let* Y *be a Banach space and* $g : Y \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* Y*. Let* D *be a nonempty, closed and convex subset of* Y *and* $S_1, S_2, \dots, S_n$ *be Bregman weak relatively nonexpansive mappings of* D *into itself such that* $\Gamma : \cap_{i=1}^N F(S_i) \neq \emptyset$*. Let* $\{\mu_{n,t} : t, n \in \mathbb{N}, 1 \leq t \leq n\}$ *be a sequence of real numbers such that* $0 < \mu_{n,1} \leq 1$ *and* $0 < \mu_{n,i} < 1$ *for every* $i = 2, 3, \dots, n$*. Let* W_n *be the Bregman* W*-maping generated by* $S_n, S_{n-1}, \dots, S_1$ *and* $\mu_{n,n}, \mu_{n,n-1}, \dots, \mu_{n,1}$*. Then, the following assertions holds:*

(i) $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$,

(ii) for every $t = 1, 2, \dots, n,\ x \in D$ *and* $z \in F(W_n),\ D_g(z, U_{n,t}x) \leq D_g(z, x)$ *and* $D_g(z, S_tU_{n,t+1}x) \leq D_g(z, x)$,

(iii) for every $n \in \mathbb{N}$, W_n *is a Bregman weak relatively nonexpansive mapping.*

Lemma 7. *[36] Let* $g : Y \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in Y$ *and the sequence* $\{D_g(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Definition 5. Let Q be a nonempty closed and convex subset of a reflexive Banach space Y and $g : Y \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A Bregman projection of $x \in int(dom(g))$ onto $Q \subset int(domg)$ is the unique vector $proj^g_Q(x) \in Q$ satisfying

$$D_g(proj^g_Q(x), x) = int\{D_g(y, x) : y \in Q\}.$$

Lemma 8. *[35] Let* Q *be a nonempty closed and convex subset of a reflexive Banach space* Y *and* $x \in Y$*. Let* $g : Y \to \mathbb{R}$ *be a strongly coercive Bregman function. Then,*

(i) $z = proj^g_Q(x)$ *if and only if* $\langle\nabla^g_Y(x) - \nabla^g_Y(z), y - z\rangle \leq 0,\ \forall\ y \in Q.$

(ii) $D_g(y, proj^g_Q(x)) + D_g(proj^g_Q(x), x) \leq D_g(y, x),\ \forall\ y \in Q.$

Lemma 9. *[7, 26] Let* $\{a_n\}$ *be a sequence of non-negative real numbers,* $\{\gamma_n\}$ *be a sequence of real numbers in* $(0,1)$ *with conditions* $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\{d_n\}$ *be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n d_n, \quad n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying the condition:* $\limsup_{k\to\infty}(a_{n_k} - a_{n_k+1}) \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

3 Main result

Throughout this section, we assume that

Assumption

(1) Let $Y_j,\ j=0,1,2,\dots,m$ be reflexive Banach spaces with $Y_0=Y$ and $D_j\subseteq Y_j, j=0,1,2,\dots,m$ be nonempty, closed and convex sets with $D_j\subseteq int(domg_j)$, where $g_j:Y_j\to(-\infty,+\infty]$ is a coercive Bregman functions which are bounded, uniformly Frechet differentiable and totally convex on bounded subsets of $Y_j, j=0,1,2,\dots,m$.

(2) Suppose $\nabla^{g_j}_{Y_j}, j=0,1,2,\dots,m$ be the gradients of Y_j dependent on g_j and $K_j:Y\to Y_j, j=1,2,\dots m$ be bounded linear operators. Let $G_j,b_j: D_j\times D_j\to\mathbb{R}, j=0,1,2,\dots,m$ satisfy Assumptions 1.3 and 1.4, respectively, with $G_0=G$ and $b_0=b$.

(3) Let $\{S_n\}$ be family of Bregman weak relatively nonexpansive mappings of D into itself and let $\{\mu_{n,k}:k,n\in\mathbb{N},\ 1\le k\le n\}$ be a sequence of real numbers such that $0<\mu_{i,t}\le 1$ for all $i\in\mathbb{N}$ and every $t=2,3,\dots,n$. Let W_n be the Bregman W-mapping generated by $S_n,S_{n-1},\dots,S_1$ and $\mu_{n,n},\mu_{n,n-1},\dots,\mu_{n,1}$.

(4) Assume that $\Omega:=x^*\in GEP(G,b)\bigcap\bigcap\limits_{k=1}^{n}F(S_k):K_jx^*\in\bigcap\limits_{j=1}^{m}GEP(G_j,b_j)$ is nonempty. Let $\gamma>0$ be a real number and $\{\alpha_n\}_{n\in\mathbb{N}},\ \{\theta_{j,m}\}$ be two sequences in $(0,1)$ with $\sum_{j=0}^{m}\theta_{j,m}=1$ satisfying the following control conditions:

(i) $\lim\limits_{n\to\infty}\alpha_n=0,\ \sum\limits_{n=1}^{\infty}\alpha_n=\infty,$

(ii) $\beta_n\in[0,1)$ and $0<\liminf\limits_{n\to\infty}\beta_n\le\limsup\limits_{n\to\infty}\beta_n<1.$

Algorithm 1. Define a sequence $\{x_n\}_{n=1}^{\infty}$ generated arbitrarily by chosen $x_1\in E$ and any fixed $u\in E$, such that

$$\begin{cases} u_n=(\nabla^g_Y)^{-1}\big[\sum\limits_{j=0}^{m}\theta_{j,m}(\nabla^g_Y(x_n)-\gamma K_j^*(\nabla^{g_j}_{Y_j}(I^{Y_j}-(res^{g_j}_{G_j,b_j})K_jx_n))\big],\\ z_n=(\nabla^g_Y)^{-1}\big[\beta_n\nabla^g_Y(u_n)+(1-\beta_n)\nabla^g_Y(W_nu_n)\big],\\ x_{n+1}=(\nabla^g_Y)^{-1}\big[\alpha_n\nabla^g_Y(u)+(1-\alpha_n)\nabla^g_Y(z_n)\big].\end{cases}\tag{3.1}$$

Let the sequences $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ satisfy the following condition: there exists a positive real number ρ such that

$$0<\rho<\liminf_{n\to\infty}\xi_{2,n}/\xi_{1,n}<\gamma,$$

where

$$\xi_{1,n}=\begin{cases}\dfrac{D_{g_j}(K_jx_n,(res^{g_j}_{G_j,b_j})K_jx_n)}{D_g^*(K_j^*(\nabla^{g_j}_{Y_j}(K_jx_n)),K_j^*(\nabla^{g_j}_{E_j}((res^{g_j}_{G_j,b_j})K_jx_n))}, & \text{if } (I-(res^{g_j}_{G_j,b_j}))K_jx_n\neq 0,\\ \xi_1, & \text{otherwise,}\end{cases}$$

and

$$\xi_{2,n}=\begin{cases}\dfrac{D_g^*(\nabla^g_Y(x_n)-\gamma K_j^*(\nabla^{g_j}_{Y_j}(K_jx_n)-\nabla^{g_j}_{Y_j}(res^{g_j}_{G_j,b_j})K_jx_n)),\nabla^g_Y(x_n))}{D_g^*(K_j^*(\nabla^{g_j}_{E_j}(K_jx_n)),K_j^*(\nabla^{g_j}_{E_j}((res^{g_j}_{G_j,b_j})K_jx_n))}, & \text{if}\\ (I-(res^{g_j}_{G_j,b_j}))K_jx_n\neq 0, & \\ \xi_2, & \text{otherwise.}\end{cases}$$

Then, the sequence $\{x_n\}$ generated iteratively by Algorithm 1 converges strongly to $z = proj^g_\Omega u$, where $proj^g_\Omega$ is the Bregman projection of Y onto Ω.

We proceed with the following two steps.
Step 1: Boundedness of the iterative method.

Proof. Let $x^* \in \Omega$, then, from (2.4) and Algorithm 1, we obtain that

$$D_g(x^*, u_n) = D_g\Big(x^*, (\nabla^g_Y)^{-1}\big[\sum_{j=0}^{m} \theta_{j,m}(\nabla^g_Y(x_n) - \gamma K^*_j(I^{Y_j} - (res^{g_j}_{G_j,b_j})K_jx_n))\big]\Big)$$
$$\leq D_g(x^*, x_n) - \sum_{j=0}^{m} \theta_{j,m}(\gamma\xi_{1,n} - \xi_{2,n})D^*_g\Big(K^*_j(\nabla^{g_j}_{Y_j}(K_jx_n)),$$
$$K^*_j(\nabla^{g_j}_{Y_j}(res^{g_j}_{G_j,b_j}K_jx_n))\Big) \leq D_g(x^*, x_n). \tag{3.2}$$

From Algorithm 1, Lemma 2 and (3.2), we obtain that

$$\begin{aligned} D_g(x^*, z_n) =& D_g\Big(x^*, (\nabla^g_Y)^{-1}[\beta_n \nabla^g_Y (u_n) + (1-\beta_n) \nabla^g_Y (W_nu_n)]\Big) \\ =& V_g\Big(x^*, \beta_n \nabla^g_Y (u_n) + (1-\beta_n) \nabla^g_Y (W_nu_n)\Big) \\ =& g(x^*) - \langle x^*, \beta_n \nabla^g_Y (u_n) + (1-\beta_n) \nabla^g_Y (W_nu_n)\rangle \\ &+ g^*(\beta_n \nabla^g_Y (u_n) + (1-\beta_n) \nabla^g_Y (W_nu_n)) \\ \leq& \beta_n g(x^*) + (1-\beta_n)g(x^*) + \beta_n g^*(\nabla^g_Y(u_n)) \\ &+ (1-\beta_n)g^*(\nabla^g_Y)g^*(\nabla^g_Y(W_nu_n)) \\ =& \beta_n V_g(x^*, \nabla^g_Y(u_n)) + (1-\beta_n)V_g(x^*, \nabla^g_Y(W_nu_n)) \\ =& \beta_n D_g(x^*, u_n) + (1-\beta_n)D_g(x^*, W_nu_n) \\ \leq& \beta_n D_g(x^*, u_n) + (1-\beta_n)D_g(x^*, u_n) \\ =& D_g(x^*, u_n) \leq D_g(x^*, x_n). \end{aligned} \tag{3.3}$$

We conclude from (3.1) and (3.2)–(3.3) that

$$\begin{aligned} &D_g(x^*, x_{n+1}) = D_g\Big(x^*, (\nabla^g_Y)^{-1}[\alpha_n \nabla^g_Y (u) + (1-\alpha_n) \nabla^g_Y (z_n)]\Big) \\ &\leq \alpha_n D_g(x^*, u) + (1-\alpha_n)D_g(x^*, z_n) \leq \alpha_n D_g(x^*, u) + (1-\alpha_n)D_g(x^*, u_n) \\ &\leq \alpha_n D_g(x^*, u) + (1-\alpha_n)D_g(x^*, x_n) \leq \max\{D_g(x^*, u), D_g(x^*, x_n)\} \\ &\vdots \\ &\leq \max\{D_g(x^*, u), D_g(x^*, x_1)\}, \ \forall\, n \geq 1. \end{aligned}$$

Thus, we obtain that the sequence $\{D_g(x^*, x_n)\}_{n\in\mathbb{N}}$ is bounded.

Using Lemma 7, then we conclude that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ and $\{z_n\}$ are bounded.

Step 2: Convergence analysis of the sequences $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$. From (3.1), (3.2) and Lemma 1, we obtain

$$D_g(x^*, z_n) = D_g\Big(x^*, (\nabla^g_Y)^{-1}\big[\beta_n \nabla^g_Y (u_n) + (1-\beta_n) \nabla^g_Y (W_nu_n)\big]\Big)$$

$$
\begin{aligned}
&\leq \beta_n D_g(x^*, u_n) + (1-\beta_n) D_g(x^*, W_n u_n) \\
&\quad - \beta_n(1-\beta_n)\rho_s\big(\|\nabla_Y^g(u_n) - \nabla_Y^g(W_n u_n)\|\big) \\
&\quad \leq D_g(x^*, x_n) - \sum_{j=0}^{m} \theta_{j,m}(\gamma\xi_{1,n} - \xi_{2,n}) D_g^*\Big(K_j^*(\nabla_{Y_j}^{g_j}(K_j x_n)), \\
&\quad K_j^*(\nabla_{Y_j}^{g_j}(res_{G_j,b_j}^{g_j} K_j x_n))\Big) - \beta_n(1-\beta_n)\rho_s^*\big(\|\nabla_Y^g(u_n) - \nabla_Y^g(W_n u_n)\|\big). \qquad (3.4)
\end{aligned}
$$

By applying (3.1) and Lemma 2, we get

$$
\begin{aligned}
D_g(x^*, x_{n+1}) &= D_g\Big(x^*, (\nabla_Y^g)^{-1}[\alpha_n \nabla_Y^g(u) + (1-\alpha_n)\nabla_Y^g(z_n)]\Big) \\
&= V_g(x^*, \alpha_n \nabla_Y^g + (1-\alpha_n)\nabla_Y^g(z_n)) \leq V_g(x^*, \alpha_n \nabla_Y^g(y) \\
&\quad + (1-\alpha_n)\nabla_Y^g(z_n) - \alpha_n(\nabla_Y^g(u) - \nabla_Y^g(x^*))) \\
&\quad - \langle \nabla_{Y^*}^{g^*}(\alpha_n \nabla_Y^g(u) + (1-\alpha_n)\nabla_Y^g(z_n)) - x^*, -\alpha_n(\nabla_Y^g(u) - \nabla_Y^g(x^*))\rangle \\
&= V_g(x^*, \alpha_n \nabla_Y^g(x^*) + (1-\alpha_n)\nabla_Y^g(z_n)) + \alpha_n\langle x_{n+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle \\
&= D_g(x^*, (\nabla_{Y^*}^{g^*})[\alpha_n \nabla_Y^g(x^*) + (1-\alpha_n)\nabla_Y^g(z_n)]) \\
&\quad + \alpha_n\langle x_{n+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle \\
&\leq \alpha_n D_g(x^*, x^*) + (1-\alpha_n) D_g(x^*, z_n) + \alpha_n\langle x_{n+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle \\
&= (1-\alpha_n) D_g(x^*, z_n) + \alpha_n\langle x_{n+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle \\
&\leq (1-\alpha_n) D_g(x^*, x_n) - (1-\alpha_n) \\
&\quad \times \sum_{j=0}^{m} \theta_{j,m}(\gamma\xi_{1,n} - \xi_{2,n}) D_g^*\Big(K_j^*(\nabla_{Y_j}^{g_j}(K_j x_n)), K_j^*(\nabla_{Y_j}^{g_j}(res_{G_j,b_j}^{g_j} K_j x_n))\Big) \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)\rho_s^*\big(\|\nabla_Y^g(u_n) - \nabla_Y^g(W_n u_n)\|\big) \\
&\quad + \alpha_n\langle x_{n+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle \\
&\leq (1-\alpha_n) D_g(x^*, x_n) + \alpha_n\langle x_{n+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle. \qquad (3.5)
\end{aligned}
$$

In view of Lemma 9, we need to show that $\langle x_{n_k+1} - x^*, \nabla_Y^g(u) - \nabla_Y^g(x^*)\rangle \leq 0$ for every subsequence $\{D_g(x^*, x_{n_k})\}$ of $\{D_g(x^*, x_n)\}$ satisfying the condition

$$
\limsup_{k\to\infty}\{D_g(x^*, x_{n_k}) - D_g(x^*, x_{n_{n_k}+1})\} \leq 0. \qquad (3.6)
$$

Applying (3.4) and (3.6), we get that

$$
\begin{aligned}
&\limsup_{k\to\infty}\Big((1-\alpha_{n_k})\beta_{n_k}(1-\beta_{n_k})\rho_s^*\big(\|\nabla_Y^g(u_{n_k}) - \nabla_Y^g(W_{n_k} u_{n_k})\|\big)\Big) \\
&\leq \limsup_{k\to\infty}\Big(\alpha_{n_k} D_{g_1}(x^*, u) + (1-\alpha_{n_k}) D_g(x^*, x_{n_k}) - D_g(x^*, x_{n_{k+1}})\Big) \\
&= \limsup_{k\to\infty}\Big(D_g(x^*, x_{n_k}) - D_g(x^*, x_{n_{k+1}})\Big) \leq 0. \qquad (3.7)
\end{aligned}
$$

Following the same process as in (3.7), we obtain from (3.5) and (3.6) that

$$
\limsup_{k\to\infty}\Big((1-\alpha_{n_k})\sum_{j=0}^{m}\theta_{j,m}(\gamma\xi_{1,n_k} - \xi_{2,n_k}) D_g^*\Big(K_j^*(\nabla_{Y_j}^{g_j}(K_j x_{n_k})),
$$

$$K_j^*(\nabla_{Y_j}^{g_j}(res_{G_j,b_j}^{g_j}K_jx_{n_k}))\Big)\Big)$$
$$\leq \limsup_{k\to\infty}\Big(\alpha_{n_k}D_g(x^*,u)+(1-\alpha_{n_k})D_g(x^*,x_{n_k})-D_{g_1}(x^*,x_{n_{k+1}})\Big)$$
$$=\limsup_{k\to\infty}\Big(D_{g_1}(x^*,x_{n_k})-D_g(x^*,x_{n_{k+1}})\Big)\leq 0. \tag{3.8}$$

Therefore, we conclude from (3.7) and (3.8) that

$$\lim_{k\to\infty}\rho_s^*\big(||\nabla_Y^g(u_{n_k})-\nabla_Y^g(W_{n_k}u_{n_k})||\big)=0,$$

then

$$\lim_{k\to\infty}\big(||\nabla_Y^g(u_{n_k})-\nabla_Y^g(W_{n_k}u_{n_k})||\big)=0, \tag{3.9}$$
$$\lim_{k\to\infty}D_g^*\Big(K_j^*(\nabla_{Y_j}^{g_j}(K_jx_{n_k})),K_j^*(\nabla_{Y_j}^{g_j}(res_{G_j,b_j}^{g_j}K_jx_{n_k}))\Big)=0,\ j=0,1,2,\ldots,m. \tag{3.10}$$

So, from Lemma 5 and the properties of ρ_s^*, D_g^* and K_j, we obtain

$$\lim_{k\to\infty}||K_jx_{n_k}-(res_{G_j,b_j}^{g_j}K_jx_{n_k})||=\lim_{k\to\infty}D_{g_j}(K_jx_{n_k},(res_{G_j,b_j}^{g_j}K_jx_{n_k}))=0,$$
$$j=0,1,2,\ldots m, \tag{3.11}$$
$$\lim_{k\to\infty}\|u_{n_k}-W_{n_k}u_{n_k}\|=0. \tag{3.12}$$

On applying Lemma 5, we get

$$\lim_{k\to\infty}D_g(u_{n_k},W_{n_k}u_{n_k})=0. \tag{3.13}$$

We observe from (3.1), (3.10), (3.13) and applying Lemma 5 that

$$\begin{cases}\lim\limits_{k\to\infty}D_g(u_{n_k},x_{n_k})=\lim\limits_{k\to\infty}||u_{n_k}-x_{n_k}||=0,\\ \lim\limits_{k\to\infty}D_g(z_{n_k},u_{n_k})=\lim\limits_{k\to\infty}||z_{n_k}-u_{n_k}||=0,\\ \lim\limits_{k\to\infty}D_g(x_{n_k+1},z_{n_k})=\lim\limits_{k\to\infty}||x_{n_k+1}-z_{n_k}||=0.\end{cases} \tag{3.14}$$

Using (3.14), we obtain

$$\lim_{k\to\infty}||z_{n_k}-x_{n_k}||=0\lim_{k\to\infty}||x_{n_{k+1}}-x_{n_k}||=0. \tag{3.15}$$

Since ∇_Y^g is uniformly norm-to-norm continuous on any bounded subset of Y, then we obtain from (3.14), (3.15) and Lemma 5 that

$$\begin{cases}\lim\limits_{k\to\infty}||\nabla_Y^g(u_{n_k})-\nabla_Y^g(x_{n_k})||=\lim\limits_{k\to\infty}D_g(u_{n_k},x_{n_k})=0,\\ \lim\limits_{k\to\infty}||\nabla_Y^g(z_{n_k})-\nabla_Y^g(u_{n_k})||=\lim\limits_{k\to\infty}D_g(z_{n_k},u_{n_k})=0,\\ \lim\limits_{k\to\infty}||\nabla_Y^g(x_{n_{k+1}})-\nabla_Y^g(z_{n_k})||=\lim\limits_{k\to\infty}D_g(x_{n_{k+1}},z_{n_k})=0,\\ \lim\limits_{k\to\infty}||\nabla_Y^g(z_{n_k})-\nabla_Y^g(x_{n_k})||=\lim\limits_{k\to\infty}D_g(z_{n_k},x_{n_k})=0,\\ \lim\limits_{k\to\infty}||\nabla_Y^g(x_{n_{k+1}})-\nabla_Y^g(x_{n_k})||=\lim\limits_{k\to\infty}D_g(x_{n_{k+1}},x_{n_k})=0.\end{cases}$$

From (2.1) and (3.1), we obtain that

$$
\begin{aligned}
& D_g(x^*, U_{n,t}u_n){=}D_g(x^*, proj_C^g(\nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t}u_n){+}(1{-}\mu_{n,t}) \\
& \times \nabla_Y^g (u_n)])){\leq}D_g(x^*, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t}u_n){+}(1{-}\mu_{n,t} \nabla_Y^g(u_n))]) \\
& - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) + (1-\mu_{n,t}) \nabla_Y^g (u_n)]) \\
& \quad = g(x^*) - \langle x^*, \nabla_Y^g(S_tU_{n,t+1}u_n) + (1-\mu_{n,t}) \nabla_Y^g (u_n)\rangle \\
& \quad\ + g^*(\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) + (1-\mu_{n,t}) \nabla_Y^g (u_n)) \\
& \ - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n){+}(1{-}\mu_{n,t}) \nabla_Y^g (u_n)]) \\
& \ \leq \mu_{n,t}g(x^*) + (1-\mu_{n,1})g(x^*) + \mu_{n,t}g^*(\nabla_Y^g(S_tU_{n,3}u_n)) \\
& \ {+}(1{-}\mu_{n,1})g^*(\nabla_Y^g(u_n)){-}\mu_{n,t}(1{-}\mu_{n,t})\rho_{s_1}^*(\| \nabla_Y^g (S_tU_{n,2}u_n) \\
& - \nabla_Y^g(u_n)\|) - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) \\
& \quad + (1-\mu_{n,t}) \nabla_Y^g (u_n)]) = \mu_{n,t}V_g(x^*, \nabla_Y^g(S_tU_{n,2}u_n)){+}(1{-}\mu_{n,1}) \\
& \times V_g(x^*, \nabla_Y^g(u_n)) - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) \\
& \quad + (1{-}\mu_{n,t}) \nabla_Y^g (u_n)]){-}\mu_{n,t}(1{-}\mu_{n,t}) \\
& \quad \times \rho_{s_1}^*(\| \nabla_Y^g (S_tU_{n,2}u_n){-} \nabla_Y^g (u_n)\|) \\
& - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) + (1-\mu_{n,t}) \nabla_Y^g (u_n)]) \\
& \ \leq \mu_{n,t}D_g(x^*, U_{n,t+1}u_n) + (1-\mu_{n,1})D_g(x^*, u_n) \\
& \quad - \mu_{n,t}(1-\mu_{n,t})\rho_{s_1}^*(\| \nabla_Y^g (S_tU_{n,2}u_n) - \nabla_Y^g(u_n)\|) \\
& - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) + (1-\mu_{n,t}) \nabla_Y^g (u_n)]) \\
& \leq \mu_{n,t}D_g(x^*, U_{n,t+1}u_n) + (1-\mu_{n,t})D_g(x^*, u_n) \\
& \quad - \mu_{n,t}(1-\mu_{n,t})\rho_{s_1}^*(\| \nabla_Y^g (S_tU_{n,2}u_n) - \nabla_Y^g(u_n)\|) \\
& - D_g(U_{n,t}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,t} \nabla_Y^g (S_tU_{n,t+1}u_n) + (1-\mu_{n,t}) \nabla_Y^g (u_n)]).
\end{aligned}
$$

Also,

$$
\begin{aligned}
& D_g(x^*, W_nu_n) = D_g(x^*, U_{n,1}u_n) \leq \mu_{n,1}D_g(x^*, U_{n,2}u_n) + (1-\mu_{n,1}) \\
& \quad \times D_g(x^*, u_n) - \mu_{n,1}(1-\mu_{n,1})\rho_{s_2}^*(\| \nabla_Y^g (S_1U_{n,2}u_n) - \nabla_Y^g(u_n)\|) \\
& \ \leq \mu_{n,1}[\mu_{n,2}D_g(x^*, U_{n,3}) + (1-\mu_{n,2})D_g(x^*, u_n) \\
& - \mu_{n,2}(1-\mu_{n,2})\rho_{s_2}^*(\| \nabla_Y^g (S_2U_{n,3}u_n) - \nabla_Y^g(u_n)\|) - D_g(U_{n,2}u_n, \\
& \ \nabla_{Y^*}^{g^*} [\mu_{n,2} \nabla_Y^g (S_2U_{n,3}u_n) + (1-\mu_{n,2}) \nabla_Y^g (u_n)])] + (1-\mu_{n,1}) \\
& \ \times D_g(x^*, u_n) - \mu_{n,1}(1-\mu_{n,1})\rho_{s_2}^*(\| \nabla_Y^g (S_1U_{n,2}u_n) - \nabla_Y^g(u_n)\|) \\
& \leq \ldots \leq D_g(x^*, u_n){-}\mu_{n,1}(1{-}\mu_{n,1})\rho_{s_2}^*(\| \nabla_Y^g (S_1U_{n,2}u_n){-} \nabla_Y^g (u_n)\|) \\
& - \mu_{n,1}\mu_{n,2}(1-\mu_{n,2})\rho_{s_2}^*(\| \nabla_Y^g (S_2U_{n,3}u_n) - \nabla_Y^g(u_n)\|) \\
& - \mu_{n,1}D_g(U_{n,2}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,2} \nabla_Y^g (S_2U_{n,3}u_n) + (1-\mu_{n,2}) \nabla_Y^g (u_n)]) \\
& - \ldots - \mu_{n,1}\mu_{n,2}\cdots\mu_{n,n-1}D_g(U_{n,n}u_n, \nabla_{Y^*}^{g^*}[\mu_{n,n} \nabla_Y^g (S_nU_{n,n+1}u_n) \\
& + (1-\mu_{n,n}) \nabla_Y^g u_n]),
\end{aligned} \tag{3.16}
$$

for all $n \in \mathbb{N}$. Since ∇_Y^g is uniformly norm-to-norm continuous on bounded subsets of Y, by following the same approach as in (3.7) and applying (3.9), (3.13), we obtain

$$\lim_{k\to\infty}\mu_{n_k,1}||\bigtriangledown^g_Y(S_1U_{n_k,2}u_{n_k})-\bigtriangledown^g_Y(u_{n_k})||$$
$$=\lim_{k\to\infty}||\bigtriangledown^g_Y(W_{n_K}u_{n_k})-\bigtriangledown^g_Y(u_{n_k})||=0.$$

This implies that

$$\lim_{k\to\infty}||\bigtriangledown^g_Y(S_1U_{n_{k,2}}u_{n_k})-\bigtriangledown^g_Y u_{n_k}||=0.$$

From (3.13) and (3.16), we deduce that

$$\lim_{k\to\infty}||\bigtriangledown^g_Y(S_tU_{n_k,t+1}u_{n_k})-\bigtriangledown^g_Y u_{n_k}||=0,\forall\ t\in\mathbb{N}.$$

Since $\bigtriangledown^{g^*}_{Y^*}$ is uniformly norm-to-norm continuous on bounded subsets of Y^*, we deduce that

$$\lim_{k\to\infty}||S_tU_{n_k,t+1}u_{n_k}-u_{n_k}||=0,\ \forall\ t\in\mathbb{N}. \tag{3.17}$$

On the other hand, we obtain

$$\lim_{k\to\infty}D_g(U_{n_k,t}u_{n_k},\bigtriangledown^{g^*}_{Y^*}[\mu_{n_k,t}\bigtriangledown^g_Y(S_tU_{n_k,t+1},u_{n_k})+(1-\mu_{n_k,t})\bigtriangledown^g_Y(u_{n_{k_t}})])=0,$$

$\forall\ t\in\mathbb{N}$ with $t\geq 2$. This together with Lemma 5 implies that

$$\lim_{k\to\infty}\|U_{n_k,t}u_{n_k}\bigtriangledown^{g^*}_{Y^*}[(\mu_{n_k,t+1}u_{n_k})+(1-\mu_{n,t})\bigtriangledown^g_Y u_{n_k}]\|=0, \tag{3.18}$$

$\forall t\in\mathbb{N}$ with $k\geq 2$. In view of (3.17), we get

$$\lim_{k\to\infty}||[\mu_{n,t}\bigtriangledown^g_Y(S_tU_{n_k,t+1}u_{n_k})+(1-\mu_{n,t})\bigtriangledown^g_Y(u_{n_t})]||=0,\ \forall\ t\in\mathbb{N}.$$

Therefore,

$$\lim_{k\to\infty}\|\bigtriangledown^{g^*}_{Y^*}[\bigtriangledown^g_Y(S_tU_{n_k,t+1},u_{n_k})+(1-\mu_{n,t})\bigtriangledown^g_Y(u_{n_k})]-u_{n_k}\|=0,\ \forall\ t\in\mathbb{N}.$$

From (3.13) and (3.17), we get

$$\lim_{k\to\infty}||U_{n_k,t}u_{n_k}-u_{n_k}||=0,\ \forall\ t\in\mathbb{N}.$$

This together with (3.18) implies that

$$\lim_{k\to\infty}||S_tU_{n_k,t+1}u_{n_k}-U_{n_k,t+1}u_{n_k}||=0,\ \forall\ t\in\mathbb{N}.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to $z\in\Omega$. Also, from (3.14) and (3.15), there exist $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ and $\{z_{n_{k_j}}\}$ of $\{z_{n_k}\}$ which converge weakly to z respectively. Thus, for each $j=0,1,2,\ldots m$, K_j is a bounded linear operator, then it follows that $K_jx_{n_k}\rightharpoonup K_jz\in Y_j$ as $k\to\infty$.

From (3.11), $K_jz \in F(res^{g_j}_{G_j,b_j}) = Sol(GEP), j = 0,1,2,\dots m$. More so, since $U_{n_k,t+1}u_{n_k} \rightharpoonup z$ and S_k is Bregman weak relatively nonexpansive, we obtain from Lemma 6 and (3.13) that $z \in F(S_k)$ for every $k \in \mathbb{N}$. Hence, we conclude that $z \in \Omega$.

Next our aim is to show that $\langle x_{n_k+1} - z, \nabla^g_Y(u) - \nabla^g_Y(z)\rangle \leq 0$.

$$\begin{aligned}\limsup_{k\to\infty}\langle x_{n_{k+1}} - x^*, \nabla^g_Y(u) - \nabla^g_Y(x^*)\rangle &= \lim_{j\to\infty}\langle x_{n_{k_j}+1} - x^*, \nabla^g_Y(u) - \nabla^g_Y(x^*)\rangle \\ &\leq \langle z - x^*, \nabla^g_Y(u) - \nabla^g_Y(x^*)\rangle.\end{aligned}$$

Hence, we obtain that

$$\limsup_{k\to\infty}\langle x_{n_k+1} - z, \nabla^g_Y(u) - \nabla^g_Y(z)\rangle \leq \langle z - x^*, \nabla^g_Y(u) - \nabla^g_Y(x^*)\rangle \leq 0. \quad (3.19)$$

On substituting (3.19) and Lemma 9 into (3.5), we conclude that $\{x_n\}$ converges strongly to z. $\square$

4 Numerical example

Example 2. Let $Y, Y_j = \ell(\mathbb{R})$ for $j = 0,1$ with $Y_0 = Y$ be the linear spaces whose elements are all 2-summable sequences $\{x_t\}_{t=1}^{\infty}$ of scalars in $\mathbb{R}$, that is $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \dots, x_t, \dots), x_t \in \mathbb{R}\}$, with inner product $\langle ., .\rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y\rangle := \sum_{t=1}^{\infty} x_t y_t$ and the norm $\|.\| : \ell_2 \to \ell_2$ defined by $\|x\| := \sqrt{\sum_{t=1}^{\infty}|x_t|^2}$, where $x = \{x_t\}_{t=1}^{\infty}, y = \{y_t\}_{t=1}^{\infty}$.

Let $K_j : \ell_2 \to \ell_2$ be given by $K_jx = (\frac{jx_1}{5}, \frac{jx_2}{den}, \cdots, \frac{jx_t}{5}, \cdots,)$ for all $x = \{x_t\}_{t=1}^{\infty} \in \ell_2$. Define the set $D := \{x \in \ell_2 : \|x\| \leq 1\}$ and $D_1 := \{y \in \ell_2 : \|y\| \leq 1\}$. We define the mapping $G = G_0 : D \times D \to \mathbb{R}$, $G_1 : D_1 \times D_1 \to \mathbb{R}$ by $G(x, y) = x(y - x)\ \forall\, x, y \in D$ and $G_1(x, y) = (x - 1)(y - x)\ \forall\, x, y \in D_1$, let $b_0 = b : D \times D \to \mathbb{R}$ and $b_1 : D_1 \times D_1 \to \mathbb{R}$ be defined by $b(x, y) = b_1(x, y) = xy\ \forall\, x, y \in D$. We observe that G, G_1, b and b_1 satisfy Assumptions 1.3 and 1.4, respectively with $Sol(GEP(G, b)) = \{0\} \neq \emptyset$ and $Sol(GEP(G_1, b_1)) = \{\frac{1}{2}\} \neq \emptyset$. For $x \in D$, let S_k be as defined in Example 1.

For this experiment, let $\alpha_n = \frac{1}{n+10}, \beta_n = \frac{1}{n^3}, \theta_{j,m} = \frac{1}{2^j}(1 - \frac{m}{2m+1})\ \forall\ j \geq 1,\ n \geq 1, \gamma = 0.25, res^g_{G,b}(x_n) = \frac{2x_5}{5}$ and $res^g_{G_1,b_1}(x_n) = \frac{4x_n+3}{10}$. We consider the following cases for initial values of x_1 :

Case 1: $x_1 = (0.09, 0.45, 0, \dots, 0)$;

Case 2: $x_1 = (0.5, 0.5, 0, \dots, 0)$;

Case 3: $x_1 = (-0.96, 0.85, \dots, 0)$;

Case 4: $x_1 = (1, 1, 0, \dots 0, 1)$.

The results of this experiment are reported in Figure 1 below.

Acknowledgements

The first author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Post-Doctoral Fellowship. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

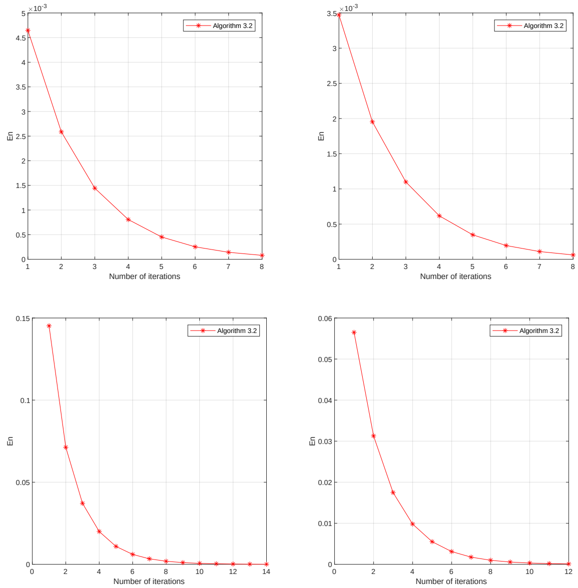

Figure 1. Example 2. Top left: Case 1, top right: Case 2, bottom left: Case 3, bottom right: Case 4.

References

[1] H.A. Abass, F.U. Ogbuisi and O.T. Mewomo. On split equality mixed equilibrium and fixed point problems of generalized k_i-strictly pseudo-contractive multivalued mappings. *Dyn. Contin. Discrete Impuls. Syst., Series B: Applications and Algorithms.*, **25**(6):369–395, 2018.

[2] H.A. Abass, C.C. Okeke and O.T. Mewomo. Common solution of split equilibrium problem with no prior knowledge of operator norm. *U. P. B Sci. Bull.*, **80**(1), 2018.

[3] K. Afassinou, O.K. Narain and O.E. Otunuga. Iterative algorithm for approximating solutions of split monotone variational inclusion, variational inequality and fixed point problems in real Hilbert space. *Nonlinear Funct. Anal. and Appl.*, **25**(3):491–510, 2020.

[4] F. Akutsah, O.K. Narain, H.A. Abass and A.A. Mebawondu. Shrinking approximation method for solution of split monotone variational inclusion and fixed point problems in Banach spaces. *International J. Nonlinear Anal. Appl.*, **12**(2):825–842, 2020. https://doi.org/10.1155/2021/9421449.

[5] M. Alansari, M. Farid and R. Ali. An inertial iterative algorithm for generalized equilibrium problems and Bregman relatively nonexpansive mappings in Banach spaces. *J. Inequal. Appll.*, **2022**(11), 2022.

[6] S.F. Aldosary, W. Cholamjiak, R. Ali and M. Farid. Strong convergence of an inertial iterative algorithm for generalized mixed variational-like inequality problem and Bregman relatively nonexpansive mapping in reflexive Banach space. *J. Math.*, **2021**:Art. ID 9421449, 2021. https://doi.org/10.1155/2021/9421449.

[7] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyodau. Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. *Nonlinear Anal.*, **67**:2350–2360, 2007. https://doi.org/10.1016/j.na.2006.08.032.

[8] S. Atsushiba and W. Takahashi. Strong convergence theorems for a finite family of nonexpansive mappings and application. *Indian J. Math.*, **41**(2):435–453, 1999.

[9] H.H. Bauschke and J.M. Borwein. Legendre functions and method of random Bregman functions. *J. Convex Anal*, **4**:27–67, 1997.

[10] H.H. Bauschke, J.M. Borwein and P.L. Combettes. Essentially smoothness, essentially strict convexity and Legendre functions in Banach spaces. *Commun. Contemp. Math.*, **3**:615–647, 2001. https://doi.org/10.1142/S0219199701000524.

[11] E. Blum and W. Oettli. From optimization and variational inequalities to equilibrium problems. *Math. Stud.*, **63**:123–145, 1994.

[12] L.M. Bregman. The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming. *U.S.S.R Comput. Math. Phys.*, **7**:200–217, 1967. https://doi.org/10.1016/0041-5553(67)90040-7.

[13] D. Butnairu and E. Resmerita. Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces. *Abstract and Applied Analysis.*, **2006**(2):1–39, Art. ID 84919, 2006. https://doi.org/10.1155/AAA/2006/84919.

[14] L.C. Ceng and J.C. Yao. SA hybrid iterative scheme for mixed equilibrium problems and fixed point problems. *J. Comput. App. Math.*, **214**:186–201, 2008. https://doi.org/10.1016/j.cam.2007.02.022.

[15] Y. Censor and T. Elfving. A multiprojection algorithms using Bregman projections in a product space. *Numer. Algor.*, **8**:221–239, 1994. https://doi.org/10.1007/BF02142692.

[16] J. Chen, Z. Wan, L. Yuan and Y. Zhang. Approximation of fixed points of weak relatively nonexpansive mappings in Banach spaces. *IJMMS*, p. Art. ID 420192, 2011. https://doi.org/10.1155/2011/420192.

[17] P.L. Combettes and S.A. Hirstoaga. Equilibrium programming using proximal-like algorithm. *J. Nonlinear Convex Anal.*, **6**:117–136, 2005.

[18] J. Deepho, W. Kumam and P. Kumam. A new hybrid projection algorithm for solving the generalized equilibrium problems and system of variational inequality problems. *J. Math. Model. Algorithms Oper. Res.*, **13**(4):404–423, 2014. https://doi.org/10.1007/s10852-014-9261-0.

[19] M. Farid, R. Ali and W. Cholamjiak. An inertial iterative algorithm to find common solution of a split generalized equilibrium and a variational inequality problem in Hilbert spaces. *J. Math.*, **2021**(157):Art. ID 3653807, 2021. https://doi.org/10.1155/2021/3653807.

[20] M. Farid and K.R. Kazmi. A new mapping for finding a common solution of split generalized equilibrium problem, variational inequality problem and fixed point problem. *The Korean J. Math.*, **27**(2):297–327, 2019.

[21] H. Gazmeh and E. Naraghirad. The split common null point problem for Bregman generalized resolvents in two Banach spaces. *Optimization.*, 2020. https://doi.org/10.1080/02331934.2020.1751157.

[22] N. Hussain, E. Naraghirad and A. Alotaibi. Existence of common fixed points using Bregman nonexpansive retracts and Bregman functions in Banach spaces. *Fixed Point Theory Appl.*, **113**:1–19, 2013. https://doi.org/10.1186/1687-1812-2013-113.

[23] C. Izuchukwu, C.C. Okeke and F. O. Isiogugu. A viscosity iterative technique for split variational inclusion and fixed point problems between a Hilbert and a Banach space. *J. Fixed Point Theory Appl.*, **20**(157), 2018. https://doi.org/10.1007/s11784-018-0632-4.

[24] K.R. Kazmi, R. Ali and S. Yousuf. Generalized equilibrium and fixed point problems for Bregman relatively nonexpansive mappings in Banach spaces. *J. Fixed Point Theory Appl.*, **20**(151), 2018. https://doi.org/10.1007/s11784-018-0627-1.

[25] K.R. Kazmi and S.H. Rizvi. Iterative approximation of a common solution of a split generalized equilibrium problem and a fixed problem for nonexpansive semigroup. *Math. Sci.*, **7**(1), 2013.

[26] Y. Kimura and S. Saejun. Strong convergence for a common fixed points of two different generalizations of cutter operators. *Linear Nonlinear Anal.*, **1**:53–65, 2015.

[27] S.Y. Matsushika, K. Nakajo and W. Takahashi. Strong convergence theorems obtained by a generalized projections hybrid method for families of mappings in Banach spaces. *Nonlinear Anal.*, **73**:1466–1480, 2010. https://doi.org/10.1016/j.na.2010.04.007.

[28] E. Naraghirad and J.C. Yao. Bregman weak relatively nonexpansive mappings in Banach space. *Fixed Point Theory Appl.*, 2013. https://doi.org/10.1186/1687-1812-2013-141.

[29] N. Naraghirad and S. Timnak. Strong convergence theorems for Bregman W-mappings with applications to convex feasibility problems in Banach spaces. *Fixed Point Theory App.*,**149**,2015. https://doi.org/10.1186/s13663-015-0395-1.

[30] F.U. Ogbuisi and O.T. Mewomo. Iterative solution of split variational inclusion problem in a real Banach spaces. *Afr. Mat.*, **28**:295–309, 2017. https://doi.org/10.1007/s13370-016-0450-z.

[31] O.K. Oyewole, H.A. Abass and O.T. Mewomo. A strong convergence algorithm for a fixed point constraint split null point problem. *Rend. Circ. Mat.*, pp. 1–20, 2020.

[32] W. Phuengrattana and K. Lerkchaiyaphum. On solving split generalized equilibrium problem and the fixed point problem for a countable family of nonexpansive multivalued mappings. *Fixed Point Theory Appl.*, **6**, 2018. https://doi.org/10.1186/s13663-018-0631-6.

[33] S. Reich. Weak convergence theorems for nonexpansive mappings in Banach spaces,. *J. Math. Anal. Appl.*, **67**(2):274–276, 1979. https://doi.org/10.1016/0022-247X(79)90024-6.

[34] S. Reich. On the asymptotic behavior of nonlinear semigroups and the range of accretive operators. *J. Math. Anal. Appl.*, **79**(1):113–126, 1981. https://doi.org/10.1016/0022-247X(81)90013-5.

[35] S. Reich and S. Sabach. S strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces. *J. Nonlinear Convex Anal.*, **10**:471–485, 2009.

[36] S. Reich and S. Sabach. Two strong convergence theorems for a proximal method in reflexive Banach spaces. *Numer. Funct. Anal. Optim.*, **31**(1):24–44, 2010. https://doi.org/10.1080/01630560903499852.

[37] S. Reich and A.J. Zaslavski. Existence of a unique fixed point for nonlinear contractive mappings. *Math.*, 2020. https://doi.org/10.3390/math8010055.

[38] S. Timnak, E. Naraghirad and N. Hussain. Strong convergence of Halpern iteration for products of finitely many resolvents of maximal monotone operators in Banach spaces. *Filomat*, **31**(15):4673–4693, 2017.

[39] C. Zalinescu. *Convex Analysis in General Vector spaces.* World Scientific Publishing Co. Inc., River Edge NJ, 2002.