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On the Functional Independence of the Riemann Zeta-Function

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Abstract. In 1973, Voronin proved the functional independence of the Riemann zeta-function $\zeta(s)$, i.e., that $\zeta(s)$ and its derivatives do not satisfy a certain equation with continuous functions. In the paper, we obtain a joint version of the Voronin theorem.
Keywords: functional independence, Riemann zeta-function, universality of zeta-functions.

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1 Introduction

Hölder was the first who began to study the algebraic-differential independence of functions. In [4], he proved the latter property for the Euler gamma-function $\Gamma(s)$, $s = \sigma + it$. Recall that $\Gamma(s)$ is the integral

$$\int_0^\infty \mathrm{e}^{-u} u^{s-1} \, \mathrm{d}u, \quad \sigma > 0,$$

and its meromorphic continuation to the whole complex plane. More precisely, Hölder obtained that there is no any polynomial $p(s_1, \dots, s_r) \not\equiv 0$ such that

$$p\left(\Gamma(s), \Gamma'(s), \dots, \Gamma^{(r-1)}(s)\right) \equiv 0.$$

This beautiful Hölder's result has been observed by Hilbert who stated the most important problems of mathematics at the International Congress of Mathematicians in Paris (1900). In the description of the 18th problem, Hilbert mentioned that the famous Riemann zeta-function $\zeta(s)$ which is the sum of the series

$$\sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and its meromorphic continuation with the unique simple pole at the point $s = 1$ with residue 1 is also algebraically-differentially independent, and this is implied by the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

and the algebraic-differential independence of the function $\Gamma(s)$. Additionally, Hilbert raised a conjecture on the algebraic-differential independence of a more general function

$$\zeta(s,x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}.$$

The latter conjectures were proved independently by Mordukhai-Boltovskoi [24] and Ostrowski [28]. Postnikov considered [29] the function $\zeta(s,x)$ twisted with the Dirichlet character χ

$$L(s,x,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)x^m}{m^s}$$

and obtained its algebraic-differential independence. In the case of functions $\zeta(s,x)$ and $L(s,x,\chi)$, in the relation of independence the derivatives with respect to x are involved, i.e., the equality

$$p\left(s, x, \frac{\partial^{k+l} L(s,x,\chi)}{\partial^k s \partial^l x}\right) \equiv 0$$

is not satisfied with any polynomial $p \not\equiv 0$.

The next period of investigation of the function independence is connected to a discovery by Voronin [34] of a very deep universality phenomenon for functions defined by Dirichlet series, and its applications. Roughly speaking, the universality, for example, of the function $\zeta(s)$ means that a wide class of analytic functions can be approximated by shifts $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$. Using the property of universality, Voronin obtained [32] the functional independence of the function $\zeta(s)$. He proved that if $F_0, F_1, \ldots, F_m : \mathbb{C}^N \to \mathbb{C}$ are continuous functions and the equality

$$\sum_{l=0}^{m} s^l F_l\left(\zeta(s), \zeta'(s), \ldots, \zeta^{(N-1)}(s)\right) = 0$$

is satisfied identically for $s \in \mathbb{C}$, then $F_l \equiv 0$ for all $l = 0, 1, \dots, m$. In [33], Voronin obtained the joint universality for Dirichlet L-functions

$$L(s, \chi_l) = \sum_{m=1}^{\infty} \frac{\chi_l(m)}{m^s}, \quad \sigma > 1,$$

with pairwise non-equivalent Dirichlet characters $\chi_1, \dots, \chi_r$ (two Dirichlet characters are equivalent if they are generated by the same primitive character), and derived from that if the equality

$$\sum_{l=0}^{m} s^l F_l \Big(L(s, \chi_1), L'(s, \chi_1), \dots, L^{(N-1)}(s, \chi_1), \dots, \\ L(s, \chi_r), L'(s, \chi_r), \dots, L^{(N-1)}(s, \chi_r) \Big) = 0$$

holds identically for $s \in \mathbb{C}$ with continuous functions $F_0, F_1, \dots, F_m : \mathbb{C}^{rN} \to \mathbb{C}$, then $F_l \equiv 0$ for all $l = 0, 1, \dots, m$.

The functional independence of more general zeta-functions was considered in [3,5,6,7,8,11,12,13,15,16,17,18,19,21,22,23,25,26,27]. At the moment, it is known that if a function is universal in the above sense, then it is functionally independent, see also the informative paper [20].

In the paper, we introduce the joint functional independence of the Riemann zeta-function. More precisely, we prove the following theorem.

Theorem 1. *Let $F_0, F_1, \dots, F_m : \mathbb{C}^{N_1+\dots+N_r} \to \mathbb{C}$ be continuous functions, and the equality*

$$\sum_{l=0}^{m} (s_1 \cdots s_r)^l F_l \Big(\zeta(s_1), \zeta'(s_1), \dots, \zeta^{(N_1-1)}(s_1), \dots, \\ \zeta(s_r), \zeta'(s_r), \dots, \zeta^{(N_r-1)}(s_r) \Big) = 0 \tag{1.1}$$

is satisfied identically for $s_1, \dots, s_r$. Then $F_l \equiv 0$ for all $l = 0, 1, \dots, m$.

2 Universality of the Riemann zeta-function

Voronin in [34], see also [35], proved the following statement which is called the initial version of universality for $\zeta(s)$.

Theorem 2. *Let $0 < r < 1/4$. Suppose that the function $f(s)$ is continuous and non-vanishing in the disc $|s| \leqslant r$, and analytic in the disc $|s| < r$. Then, for every $\varepsilon > 0$, there exists $\tau = \tau(\varepsilon) \in \mathbb{R}$ such that*

$$\max_{|s| \leqslant r} \left| \zeta \left(s + \frac{3}{4} + i\tau \right) - f(s) \right| < \varepsilon.$$

For the modern version of the Voronin theorem, we use the following notation. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K, and by $H_0(K)$ the subclass of the class $H(K)$ of non-vanishing on K functions. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then we have the following universality theorem, see, for example, [1, 10, 30].

Theorem 3. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 3 has its joint generalization. For real numbers $a_1, \ldots, a_r$, denote $a = \max_{1\leqslant j\leqslant r}|a_j|^{-1}$ and $\widehat{a} = \max_{1\leqslant j\leqslant r}|a_j|$. Then, in [14], the following statement has been obtained.

Theorem 4. *Suppose that $a_1, \ldots, a_r$ are real algebraic numbers linearly independent over the field of rational numbers $\mathbb{Q}$, and $\widehat{a}(Ta)^{1/3}(\log Ta)^{26/15} \leqslant H \leqslant T$. Let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K)$, $j = 1, \ldots, r$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H}\text{meas}\left\{\tau \in [T, T+H] : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|\zeta(s+ia_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

Let $H(D)$ be the space of analytic on D functions endowed with the topology of uniform convergence on compacta. Then the Voronin universality theorem shows the denseness of the set $\{\zeta(s+i\tau)\}$ in $H(D)$.

3 Denseness theorem

By the Bohr-Courant theorem [2], the set $\{\zeta(\sigma+it) : t \in \mathbb{R}\}$ with every fixed $1/2 < \sigma < 1$ is dense in $\mathbb{C}$. Theorem 2 implies a more general result [31] that the set

$$\left\{\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(N-1)}(\sigma+it) : t \in \mathbb{R}\right\}$$

with every fixed $1/2 < \sigma < 1$ is dense in $\mathbb{C}^N$, $N \in \mathbb{N}$. In this section, we will prove the following statement.

Theorem 5. *Suppose that $a_1, \ldots, a_r$ are real algebraic numbers linearly independent over $\mathbb{Q}$. Then the set*

$$\Big\{\Big(\zeta(\sigma+ia_1t), \zeta'(\sigma+ia_1t), \ldots, \zeta^{(N_1-1)}(\sigma+ia_1t), \ldots,$$
$$\zeta(\sigma+ia_rt), \zeta'(\sigma+ia_rt), \ldots, \zeta^{(N_r-1)}(\sigma+ia_rt)\Big) : t \in \mathbb{R}\Big\}$$

with every fixed $1/2 < \sigma < 1$ is dense in $\mathbb{C}^{N_1+\cdots+N_r}$, $N_j \in \mathbb{N}$, $j = 1, \ldots, r$.

For the proof of Theorem 5, we will use one property of polynomials [9].

Lemma 1. *Suppose that $s_0 \neq 0$, $s_1, \dots, s_m$ are complex numbers. Then there exists a polynomial $p(s)$ such that*

$$s_k = \left(\mathrm{e}^{p(s)}\right)^{(k)}\Big|_{s=0}.$$

Proof. (Proof of Theorem 5) In view of the metric of $\mathbb{C}^{N_1+\cdots+N_r}$, it suffices to show that, for all s_{jk}, $j = 0, \dots, N_k - 1$, $s_{0k} \neq 0$, $k = 1, \dots, r$, and $\varepsilon > 0$, there exists $\tau \in \mathbb{R}$ such that

$$\left|\zeta^{(j)}(\sigma + ia_k\tau) - s_{jk}\right| < \varepsilon. \tag{3.1}$$

Let the set $K \in \mathcal{K}$ contain σ. By Lemma 1, there exist polynomials $p_k(s)$ such that

$$s_{jk} = \left(\mathrm{e}^{p_k(s-\sigma)}\right)^{(j)}\Big|_{s=\sigma}.$$

Then Theorem 4 implies the existence of $\tau \in \mathbb{R}$ (actually, there exists a sequence $\{\tau_n\} \subset \mathbb{R}$, $\tau_n \to \infty$ as $n \to \infty$) such that, for every $\varepsilon_1 > 0$,

$$\sup_{s \in K}\left|\zeta(s + ia_k\tau) - \mathrm{e}^{p_k(s-\sigma)}\right| < \varepsilon_1.$$

Let L be a simple closed contour lying in K and enclosing σ. Then, by the Cauchy integral formula,

$$\begin{aligned}&\left|\zeta^{(j)}(\sigma + ia_k\tau) - \left(\mathrm{e}^{p_k(s-\sigma)}\right)^{(j)}\right| \\ &\qquad \leqslant \frac{j!}{2\pi}\int_L \frac{\sup_{s\in K}|\zeta(z + ia_k\tau) - \mathrm{e}^{p_k(z-\sigma)}||\mathrm{d}z|}{|z-\sigma|^{j+1}} \ll_{j,L} \varepsilon_1.\end{aligned}$$

This and the corresponding choice of ε_1 prove (3.1). □

4 Proof of Theorem 1

For brevity, let $N = N_1 + \cdots + N_r$. Denote the distance in the space $\mathbb{C}^N$ by d_N.

Proof. (Proof of Theorem 1) On the contrary, suppose that $F_m \not\equiv 0$. Then there exists $\underline{s}_0 \in \mathbb{C}^N$ such that $F_m(\underline{s}_0) \neq 0$. Since the function F_m is continuous, for a certain fixed $\delta > 0$, we find $\varepsilon > 0$ such that

$$|F_m(\underline{s})| > \delta \tag{4.1}$$

for all $\underline{s} \in G_\varepsilon = \{\underline{s} \in \mathbb{C}^N : d_N(\underline{s}, \underline{s}_0) < \varepsilon\}$. Let $0 < \varepsilon_1 < \varepsilon$ and $\underline{G}_{\varepsilon_1} = \{\underline{s} \in \mathbb{C}^N : d_N(\underline{s}, \underline{s}_0) \leqslant \varepsilon_1\}$. Then the functions $F_0, F_1, \dots, F_{m-1}$ are bounded in $\underline{G}_{\varepsilon_1}$. Define one more disc $G_{\varepsilon_2} = \{\underline{s} \in \mathbb{C}^N : d_N(\underline{s}, \underline{s}_0) < \varepsilon_2\}$, where $0 < \varepsilon_2 < \varepsilon_1$. Denote $\underline{a} = (a_1, \dots, a_r)$,

$$\begin{aligned}\underline{\zeta}(\sigma, \underline{a}, t) = \Big(&\zeta(\sigma + ia_1t), \zeta'(\sigma + ia_1t), \dots, \zeta^{(N_1)}(\sigma + ia_1t), \\ &\dots, \zeta(\sigma + ia_rt), \zeta'(\sigma + ia_rt), \dots, \zeta^{(N_r)}(\sigma + ia_rt)\Big),\end{aligned}$$

and fix $1/2 < \sigma < 1$. Then, in view of Theorem 5, there exists a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to \infty$ as $n \to \infty$, such that $\underline{\zeta}(\sigma, \underline{a}, t_n) \in G_{\varepsilon_2}$. Thus, by (4.1),

$$\left|F_m\left(\underline{\zeta}(\sigma, \underline{a}, t_n)\right)\right| > \delta.$$

Hence, putting $\tau_l(\sigma, \underline{a}, t_n) = ((\sigma + ia_1 t_n) \cdots (\sigma + ia_r t_n))^l$, we obtain

$$\begin{aligned}\Big|F_0\left(\underline{\zeta}(\sigma, \underline{a}, t_n)\right) &+ \tau_1(\sigma, \underline{a}, t_n) F_1\left(\underline{\zeta}(\sigma, \underline{a}, t_n)\right) + \cdots \\ &+ \tau_m(\sigma, \underline{a}, t_n) F_m\left(\underline{\zeta}(\sigma, \underline{a}, t_n)\right)\Big| \xrightarrow[n\to\infty]{} \infty,\end{aligned}$$

and this contradicts (1.1). This shows that $F_m \equiv 0$. Therefore, the summing in (1.1) runs over $l = 0, 1, \ldots, m-1$. Similarly, we obtain that $F_{m-1}, \ldots, F_1 \equiv 0$. This and (1.1) prove that $F_0 \equiv 0$ as well. The theorem is proved. □

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