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Some Considerations on Numerical Methods for Cauchy Singular Integral Equations on the Real Line

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Abstract. Two different direct methods are proposed to solve Cauchy singular integral equations on the real line. The aforementioned methods differ in order to be able to prove their convergence which depends on the smoothness of the known term function in the integral equation.

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1 Introduction

We consider the following Cauchy singular integral equation on the real line:

$$\varphi(x) - \lambda \int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x} dt = f(x), \qquad -\infty < x < +\infty, \tag{1.1}$$

where the Hilbert transform of a real-valued measurable function φ on $\mathbb{R}$ is defined by

$$\mathcal{H}(\varphi; x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x} dt, \qquad x \in \mathbb{R} .$$

The divergence at $x = t$ is allowed for by taking the Cauchy principal value of the integral, i.e.,

$$\mathcal{H}(\varphi; x) = \lim_{\epsilon \to 0} \mathcal{H}^{\epsilon}(\varphi; x),$$

where

$$\mathcal{H}^{\epsilon}(\varphi; x) = \frac{1}{\pi} \int_{|x-t|>\epsilon} \frac{\varphi(t)}{t-x} dt, \qquad \epsilon > 0, \qquad x \in \mathbb{R} \ .$$

In operator form we can rewrite Equation (1.1) as

$$(I - \pi\lambda\mathcal{H})\varphi = f. \tag{1.2}$$

Several papers and books have dealt with the numerical approximation of such kind of singular integral equations in the case of bounded domain of integration (see for example [7, 8, 9, 10, 11] and the references given there). On the other hand, the literature is very poor if we consider Cauchy singular integral equations on the real line. This is because quadrature formulas for an infinite interval are associated with orthogonal Hermite polynomials whose weight function is specific exponential function. This is a serious obstacle to their use in solving the governing integral equations of some applied problems, since the considered functions of these equations, as a rule, do not have the indicated behaviour at infinity. For example in [1], a quadrature formula for Hilbert transform on an infinite interval was obtained but it was not possible to use it for solving the contact problem of elasticity theory. Even if the fate awaits the first method in the present paper, the suggested alternative method could be used. Therefore it will be possible to use it for solving a mixed boundary value problem in the theory of elasticity.

Now, we recall some basic properties of this type of integral equation (see for example [13]). Let L_p, $1 < p < +\infty$ be the usual Banach space with respect to the norm $\|u\|_p = \left[\int_{-\infty}^{\infty} [u(x)]^p dx \right]^{1/p}$. In particular L_2 is also a Hilbert space with respect to the scalar product $< u, v >= \int_{-\infty}^{\infty} u(x)v(x)dx$. Recalling the reciprocity property of $\mathcal{H}$, i.e., $\mathcal{H}(\mathcal{H}(\varphi))$ coincides almost everywhere with $-\varphi$ for any $\varphi \in L_p(\mathbb{R})$, $1 < p < +\infty$, the solution $\varphi \in L_p$ of Equation (1.2) in case of $f \in L_p$ of such integral equation can be represented by the Hilbert transform of a known function f. For this, applying the Hilbert transform to both sides of Equation (1.2), we have

$$\mathcal{H}(\varphi) + \pi\lambda\varphi = \mathcal{H}f.$$

Consequently,

$$(1 + \lambda^2\pi^2)\varphi = f + \lambda\pi\mathcal{H}(f),$$

or written explicitly

$$\varphi(x) = \frac{1}{1 + \lambda^2\pi^2} \left[f(x) + \lambda \int_{-\infty}^{+\infty} \frac{f(t)}{t-x} dt \right]. \tag{1.3}$$

Thus, we obtain the explicit inversion formula (1.3) for the singular integral equation (1.1).

The paper is organized as follows: in Sections 2 and 3, we present two different methods (one of these was proposed firstly in [2]) for the numerical solution of the Equation (1.1) whose convergence is proved under different regularity hypotheses on the smoothness of the function f in (1.1). Both methods make use of a suitable interpolation process based on the zeros of the orthogonal Hermite polynomials.

2 The first numerical method and its convergence

Let $w(x) = e^{-x^2}$ be the Hermite weight function. Let $g(x) = \frac{f(x)}{w(x)}$, then $f(x) = w(x)g(x)$ and in place of relation (1.3) we consider:

$$\varphi(x) = \frac{1}{1+\lambda^2\pi^2}\left[w(x)g(x) + \lambda\pi\mathcal{H}(wg;x) \right]. \tag{2.1}$$

The methods described below are addressed to give an approximation of (2.1). It is obvious that for this to approximate the solution of (1.1), one has to guarantee that $wg \in L_p$ and the proposed approximations converge also in L_p.

Before proceeding further we recall some results concerning the Lagrange operator considered on the Hermite zeros and some results concerning some product rule used to approximate the Hilbert transform.

In the following, the symbol "C" stands for a positive constant taking different values in different occurrences. If A and B are two expressions depending on some variables, then we write $A \sim B$ if and only if $|AB^{-1}| \le C$ and $|A^{-1}B| \le C$, uniformly for the variables under consideration.

At first, we recall the definition of the best weighted uniform approximation error, i.e.,

$$E_m(g)_{\sqrt{w},\infty} := \inf_{P\in\mathbb{P}_m} \|(g-P)\sqrt{w}\|_\infty,$$

for any function

$$g \in C^0_{\sqrt{w}} := \{g \text{ continuous on } \mathbb{R} \text{ and } \lim_{|x|\to\infty} g(x)\sqrt{w(x)} = 0\},$$

and where $\mathbb{P}_m$ denotes the set of the polynomials of degree at most m.

Let $\{p_m(w)\}$ be the sequence of the orthonormal Hermite polynomials associated with the weight function $w(x) = e^{-x^2}$, so that

$$p_m(w;x) = \gamma_m x^m + ..., \quad \gamma_m > 0,$$
$$\int_{-\infty}^{\infty} p_m(w;x)p_n(w;x)w(x)dx = \delta_{m,n}.$$

The zeros of $p_m(w)$ are indexed in decreasing size, as

$$-\infty < x_{m,m} < x_{m,m-1} < ... < x_{m,2} < x_{m,1} < +\infty.$$

It is well known that $x_{m,1} = -x_{m,m} < \sqrt{2m+1}$. For a given function $g \in C^0_{\sqrt{w}}$ we denote by $\mathcal{L}_m(w;g)$ the Lagrange interpolating polynomial of g on the Hermite zeros, i.e.,

$$\mathcal{L}_m(w;g;x_{m,k}) = g(x_{m,k}), \quad k = 1,...,m.$$

Now, if we denote by $\mathcal{H}_m(wg)$ the product quadrature rule used to approximate the Hilbert transform, based on the interpolation process $\mathcal{L}_m(w)$, i.e.,

$$\mathcal{H}_m(wg;x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathcal{L}_m(w;g;t)w(t)}{t-x}dt,$$

we can approximate Equation (2.1) in the following way

$$\varphi_m(x) = \frac{1}{1+\lambda^2\pi^2}\left[\, w(x)g(x) + \lambda\mathcal{H}_m(wg;x)\,\right]. \tag{2.2}$$

Regarding the computation of (2.2) we recall the results in [2].
To prove the convergence of the proposed method at first we need to give necessary condition for the boundedness of the Hilbert transform in suitable spaces of functions. We recall that even if $\mathcal{H}$ is a bounded operator in the $L_p(\mathbb{R})$ spaces, $1 < p < \infty$, it is an unbounded operator in the space of the continuous functions on $\mathbb{R}$ equipped with the uniform norm.
For any function $g \in C^0_{\sqrt{w}}$ we define the norm

$$\|g\|_{C^0_{\sqrt{w}}} := \|g\sqrt{w}\|_\infty = \max_{x\in\mathbb{R}} |g(x)\sqrt{w(x)}|.$$

An important parameter associated with a weight w belonging to a suitable class of Freud weights is the so-called Mhaskar-Rahmanov-Saff number a_n. When w is the Hermite weight this number a_n is equal to $\sqrt{2n}$. Then, we define the following weighted modulus of continuity

$$\Omega(g,t)_{\sqrt{w},\infty} := \sup_{0<h\le t}\max_{|x|\le\sigma(h)} |\Delta_h g(x)|\sqrt{w(x)},$$

where

$$\Delta_h g(x) = g\big(x+h/2\big) - g\big(x-h/2\big), \quad w(x) = \exp\big(-x^2\big),$$

and $\sigma(u) := \inf\{a_n : a_n/n \le u\}, u > 0$.

This modulus of continuity is related to the weighted Ditzian-Lubinsky modulus of smoothness $\omega(g,t)_{\sqrt{w},\infty}$ defined by (see [5, p.102])

$$\omega(f,t)_{\sqrt{w},\infty} = \Omega(f,t)_{\sqrt{w},\infty} + \inf_{P\in\mathbb{P}_0}\max_{|x|\ge\sigma(t)} |[f(x)-P(x)]\sqrt{w(x)}|, \tag{2.3}$$

where $\mathbb{P}_0$ denotes the set of the polynomials of degree zero. Under the assumption on $g \in C^0_{\sqrt{w}}$, we can bound $\mathcal{H}(wg)$ as follows

$$\max_{x\in\mathbb{R}} |\mathcal{H}(wg;x)| \le C\left[\, \|g\sqrt{w}\|_\infty + \int_0^1 \frac{\omega(g,t)_{\sqrt{w},\infty}}{t}dt\,\right],$$

with some constant C independent of g, (see [3]). Then, if $\Omega(g,t)_{\sqrt{w},\infty}$ has a prescribed behaviour, we can deduce the boundedness of $\mathcal{H}(wg)$ on $\mathbb{R}$. Moreover, we can state [2]

Theorem 1. *Assume the $g \in C^0_{\sqrt{w}}$ satisfies the condition*

$$\int_0^1 \frac{\omega(g,t)_{\sqrt{w},\infty}}{t} dt < \infty,$$

we have

$$\max_{x\in\mathbb{R}} |\mathcal{H}(wg;x) - \mathcal{H}_m(wg;x)| \le C\left\{ E_{m-1}(g)_{\sqrt{w},\infty} \log m + \int_0^{\frac{1}{\sqrt{m}}} \frac{\omega(g,t)_{\sqrt{w},\infty}}{t} dt \right\},$$

for some constant C independent of g and m.

Consequentely, we can state the following result.

Theorem 2. *For all $g \in C^0_{\sqrt{w}}$ satisfying the condition*

$$\int_0^1 \frac{\omega(g,t)_{\sqrt{w},\infty}}{t} dt < \infty,$$

we have

$$\max_{x\in\mathbb{R}} |\varphi(x) - \varphi_m(x)| \le C \left\{ E_{m-1}(g)_{\sqrt{w},\infty} \log m + \int_0^{\frac{1}{\sqrt{m}}} \frac{\omega(g,t)_{\sqrt{w},\infty}}{t} dt \right\},$$

for some constant C independent of g and m.

Proof. The result is obtained immediately by applying Theorem 1, observing that, subtracting (2.1) and (2.2), in this case we have

$$|\varphi(x) - \varphi_m(x)| = |\mathcal{H}(wg;x) - \mathcal{H}_m(wg;x)|.$$

$\square$

3 The second method and its convergence

Now, we want to consider another suitable space of functions and to examine the boundedness of the function $\mathcal{H}(wg;x)$ when g belongs to it, (see [3] for more details).
If g belongs to the set

$$W_0^\infty := \{\, g \in C^0_{LOC}(\mathbb{R}) : \lim_{|x|\to\infty} g(x) e^{-x^2/2} = 0 \,\} \tag{3.1}$$

and satisfies a Dini type condition by the Ditzian-Totik modulus of continuity, then $\mathcal{H}(wf)$ is bounded on $\mathbb{R}$. To be more precise, let $g \in W_0^\infty$, where W_0^∞ is the set defined in (3.1), equipped with the norm

$$\|g\|_{W_0^\infty} := \|g\sqrt{w}\|_\infty = \max_{x\in\mathbb{R}} |g(x)\sqrt{w(x)}|.$$

Then, let us introduce the following weighted modulus of continuity

$$\Omega^r(f,t)_{\sqrt{w},\infty} = \sup_{0<h\le t} \max_{|x|\le \frac{1}{h}} |\Delta_h^r f(x)| \sqrt{w(x)}, \quad r \ge 1,$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^i \begin{pmatrix} r \\ i \end{pmatrix} f\left(x + \frac{h}{2}[r-2i] \right).$$

By $\Omega^r(f,t)_{\sqrt{w},\infty}$, we can define the $r-$th Ditzian-Totik weighted modulus of smoothness $\omega^r(f,t)_{\sqrt{w},\infty}$ [6, p.182]

$$\begin{aligned}\omega^r(f,t)_{\sqrt{w},\infty} =& \Omega^r(f,t)_{\sqrt{w},\infty} + \|(f-P_{r-1})\sqrt{w}\|_{L_\infty(-\infty,-\frac{1}{t})} \\ &+ \|(f-Q_{r-1})\sqrt{w}\|_{L_\infty(\frac{1}{t},\infty)},\end{aligned} \tag{3.2}$$

where P_{r-1} and Q_{r-1} are the orthonormal projections of f onto $\mathbb{P}_{r-1}$ in $L_\infty(-\infty,-\frac{1}{t})$ and $L_\infty(\frac{1}{t},\infty)$, respectively. We remark that the differences between the modulus defined in (2.3) and the one defined in (3.2), in addition to being in the class to which the function for which they can be belongs, lies in the motivations of the authors of [5] and [6] who induced to introduce them. Now, if we suppose that $g \in W_0^\infty$ and

$$\int_0^1 \frac{\Omega^1(g,t)_{\sqrt{w},\infty}}{t} dt < \infty,$$

under these assumptions on g, we can bound $\mathcal{H}(wg)$ as follows

$$\max_{x\in\mathbb{R}} |\mathcal{H}(wg;x)| \le C \left[\|g\sqrt{w}\|_\infty + \int_0^1 \frac{\Omega^1(g,t)_{\sqrt{w},\infty}}{t} dt \right], \tag{3.3}$$

with some constant C independent of g (see [3]). So that if $\Omega^1(g,t)_{\sqrt{w},\infty}$ has a prescribed behaviour, then we can deduce the boundedness of $\mathcal{H}(wg)$ on $\mathbb{R}$. Now, for a given function $g \in W_0^\infty$ we consider again the Lagrange interpolating polynomial of g on the Hermite zeros, i.e.,

$$\mathcal{L}_m(w;g;x_{m,k}) = g(x_{m,k}), \qquad k = 1, ..., m.$$

Unfortunately, this interpolation process is not efficient in W_0^∞. Indeed, the corresponding Lebesgue constants

$$\|\mathcal{L}_m(w)\|_{W_0^\infty} = \sup_{\|g\sqrt{w}\|_\infty=1} \|\mathcal{L}_m(w;g)\sqrt{w}\|_\infty = \max_{\mathbb{R}} \sqrt{w(x)} \sum_{k=1}^{m} \frac{|l_{m,k}(x)|}{\sqrt{w(x_{m,k})}}$$

satisfy the following relation

$$\|\mathcal{L}_m(w)\|_{W_0^\infty} \sim m^{\frac{1}{6}},$$

(see [12]). Denoting with $x_{m,0} = \sqrt{2m}$, $x_{m,m+1} = -\sqrt{2m}$, we consider the matrix of interpolation nodes $T := \{x_{m,k} = 0, 1, ..., m+1\}$. Therefore, for a given function $g \in W_0^\infty$ we denote by $\tilde{\mathcal{L}}_{m+2}(w;g)$ the Lagrange interpolating polynomial of g on the zeros $x_{m,k}$ of $p_m(w)$, plus the additional knots $x_{m,0} = -x_{m,m+1} = \sqrt{2m}$, i.e.,

$$\tilde{\mathcal{L}}_{m+2}(w;g;x_{m,k}) = g(x_{m,k}), \qquad k = 0, 1, ..., m+1.$$

The theorem stated below shows the optimal convergence of this interpolation process (see [12] for more details).

Theorem 3. *For a given constant C independent of m it is proved that*

$$\frac{1}{C}\log m \le \|\tilde{\mathcal{L}}_{m+2}(w)\|_{W_0^\infty} \le C\log m.$$

Equivalently, for all $g \in W_0^\infty$ the estimate

$$\|[g - \tilde{\mathcal{L}}_{m+2}(w;g)]\sqrt{w}\|_\infty \le CE_{m+1}(g)_{\sqrt{w},\infty}\log m$$

holds, where C is a positive constant independent of m and g.

Now, if we denote by $\tilde{\mathcal{H}}_{m+2}(wg)$ the product quadrature rule used to approximate the Hilbert transform, based on the interpolation process $\tilde{\mathcal{L}}_{m+2}(w)$, i.e.,

$$\tilde{\mathcal{H}}_{m+2}(wg;x) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tilde{\mathcal{L}}_{m+2}(w;g;t)w(t)}{t-x}dt, \tag{3.4}$$

we can approximate Equation (2.1) in the following way

$$\tilde{\varphi}_{m+2}(x) = \frac{1}{1+\lambda^2\pi^2}\left[w(x)\tilde{\mathcal{L}}_{m+2}(w;g;x) + \lambda\tilde{\mathcal{H}}_{m+2}(wg;x)\right]. \tag{3.5}$$

The idea of product rule (3.4) has been considered for the first time in [4]. Now, we can state the principal result of the Section.

Theorem 4. *For all $g \in W_0^\infty$, satisfying the condition*

$$\int_0^1 \frac{\Omega^r(g,t)_{\sqrt{w},\infty}}{t}dt < \infty,$$

we have

$$\max_{x\in\mathbb{R}}|\varphi(x) - \tilde{\varphi}_{m+2}(x)| \le C\log^2 m\left[\int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega^r(g,t)_{\sqrt{w},\infty}}{t}dt\right],$$

where C is a constant independent of m and g.

Proof. Substracting (2.1) and (3.5), and applying relation (3.3), we obtain

$$\begin{aligned}|\varphi(x) - \tilde{\varphi}_{m+2}(x)| \le & C[\|[g - \tilde{\mathcal{L}}_{m+2}(w;g)]\sqrt{w}\|_\infty \\ & + \int_0^1 \frac{\Omega^1([g-\tilde{\mathcal{L}}_{m+2}(w;g)],t)_{\sqrt{w},\infty}}{t}dt].\end{aligned}$$

Recalling Theorem 3, we can write (see also [4] for more details)

$$\begin{aligned}&|\varphi(x) - \tilde{\varphi}_{m+2}(x)| \le C\Bigg[E_{m+1}(g)_{\sqrt{w},\infty}\log m \\ &+ \int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega^1([g-\tilde{\mathcal{L}}_{m+2}(w;g)],t)_{\sqrt{w},\infty}}{t}dt + \int_{\frac{1}{\sqrt{m}}}^1\frac{\Omega^1([g-\tilde{\mathcal{L}}_{m+2}(w;g)],t)_{\sqrt{w},\infty}}{t}dt\Bigg] \\ &\le C\left[E_{m+1}(g)_{\sqrt{w},\infty}\log m + E_{m+1}(g)_{\sqrt{w},\infty}\log^2 m + \int_0^{\frac{1}{\sqrt{m}}}\frac{\Omega^r(g,t)_{\sqrt{w},\infty}}{t}dt\right].\end{aligned}$$

Applying the weaker version of the Jackson theorem(see [6], pp.180–195), we obtain

$$E_{m+1}(g)_{\sqrt{w},\infty} \leq C \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega^r(g,t)_{\sqrt{w},\infty}}{t} dt,$$

thus, the assertion easily follows. □

4 Conclusions

We have suggested two operable different methods for the numerical solution of the Equation (1.1) whose convergence is proved under different regularity hypotheses on the smoothness of the function f in (1.1). Both methods make use of a suitable interpolation process based on the zeros of the orthogonal Hermite polynomials. In particular we observe that even if the first formula (2.2) is more simple of formula (3.5), it converges under more strong assumptions on the smoothness of f.

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