ISSN: 1392-6292/ eISSN: 1648-3510

MATHEMATICAL MODELLING and ANALYSIS

2025
Volume 30
Issue 1
Pages 74–96
https://doi.org/10.3846/mma.2025.19339

Numerical differentiation of fractional order derivatives based on inverse source problems of hyperbolic equations

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Article History:

- received June 16, 2023
- revised December 7, 2023
- accepted January 8, 2024

Abstract. In this paper, we mainly study the numerical differentiation problem of computing the fractional order derivatives from noise data of a single variable function. Firstly, the numerical differentiation problem is reformulated into an inverse source problem of first order hyperbolic equation, and the corresponding solvability and the conditional stability are provided under suitable conditions. Then, four regularization methods are proposed to reconstruct the unknown source of hyperbolic equation which is the numerical derivative, and they are implemented by utilizing the finite dimensional expansion of source function and the superposition principle of hyperbolic equation. Finally, Numerical experiments are presented to show effectiveness of the proposed methods. It can be conclude that the proposed methods are very effective for small noise levels, and they are simpler and easier to be implemented than the previous PDEs-based numerical differentiation method based on direct and inverse problems of parabolic equations.

Keywords: numerical differentiation, fractional derivative, source inversion, hyperbolic equation, ill-posed problem.

AMS Subject Classification: 65D25; 65J22; 47A52.

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1 Introduction

The problem of numerically computing derivatives of a function, called numerical differentiation for short, is one of classical ill-posed problems, where the function to be differentiated is generally contaminated by random noise. In practical applications, the main ill-posedness of numerical differentiation is instability, which means that small errors in the contaminated function will lead to dramatically changes in numerical derivatives. Hence, many methods have been developed to overcome this special instability of numerical differentiation, such as finite difference methods [26, 27], mollification methods [9, 24], differentiation by integration [31, 33], Tikhonov regularization [6, 30] and variation method [15], spline methods [11, 20, 36] and radial basis functions approximation [35], polynomial approximation with regularization [39], regularization methods with total variation and L^1 penalty term [29] and references therein.

Nowadays, fractional calculus plays an increasingly important role in science and engineering, where the fractional order derivatives are applied to model many phenomena such as control [23], viscoelastic materials [10], heterogeneous porous media [2,12], physics [7], epidemiology [3], biology [8,13], mass spectrum signal process [18], computer vision [1,37]. Especially, the fractional derivatives are used to detect R wave in electrocardiogram [8]. Although there are already some numerical methods to approximate the fractional order derivatives, such as methods based on Lagrange polynomials [4], backward difference schemes [4], spline-based methods [22, 23] and Hermite interpolation method [21], most of the existing methods are sensitive to the noise. Thus, data smoothing is usually performed before calculating the fractional order derivatives of the data, which will increase the difficulty of the fractional derivatives' applications.

Computing fractional order derivatives from noisy data is significant due to the nature of real-world data, which is inherently noisy. Therefore, to develop a direct and stable method to compute the fractional derivatives from noise data is very important for understanding and analyzing real-world systems, enhancing the robustness and reliability of data-driven models. However, there are few studies on numerical fractional derivatives of noise data [17,28]. In [17], the authors proposed a regularization method based on radial basis functions to approximate fractional derivatives numerically from one-dimensional noise data; while in [28] the authors constructed a dynamical algorithm based on the methods of control theory to approximately compute the Caputo-type fractional derivatives. In this work, we propose four regularization methods based on inverse source problems of first order hyperbolic equations for computing the fractional derivatives, which is motivated by the PDEs-based numerical differentiation methods proposed firstly in our previous work [25, 32]. Compared with the results in [25,32], this work has at least the following novelties: firstly, the idea of PDEs-based numerical differentiation is extended to the case of computing fractional derivatives by using direct and inverse problems of hyperbolic equations; secondly, the solvability and conditional stability of numerically fractional derivatives are proved under some suitable assumptions; finally, the methods proposed in this paper are simpler and easier to be implemented than the method based on the direct and inverse problems of parabolic equations. Now, we state the problem considered in this paper as follows.

Given the approximation $\varphi^\delta(x)$ of a smooth function $\varphi(x)$ on $\Omega = [a, b]$ such that

$$\left\|\varphi^\delta(x) - \varphi(x)\right\|_\infty := \max_{x\in\Omega} \left|\varphi^\delta(x) - \varphi(x)\right| \leq \delta,$$

the considered numerical differentiation problem is to compute numerically the Riemann-Liouville and Caputo fractional derivatives of order $\beta \in (0, 1)$ and the first order derivative ($\beta = 1$) of $\varphi(x)$ from $\varphi^\delta(x)$. Here, δ is the noise level.

This paper is organized as follows. The problem is formulated in Section 1. The definitions and properties of fractional integrals and derivatives are introduced in Section 2. The regularization methods based on direct and inverse problems of hyperbolic equations are proposed together with the solvability and conditional stability in Section 3. numerical examples are given in Section 4 to illustrate the stability and accuracy of the proposed methods. Finally, some conclusions are drawn in Section 5.

2 Fractional integrals and fractional derivatives

In this section, we give some necessary definitions and properties of fractional integrals and derivatives for convenience of reading. At the same time, a basically numerical method for computing the fractional integrals is introduced.

2.1 Definitions and properties

Definition 1. (Riemann-Liouville fractional integrals). Let $\Omega = [a, b]$ $(-\infty < a < b < +\infty)$ be a finite interval on $\mathbb{R}$, and $\varphi(x)$ be an arbitrary function in $L_1(\Omega)$. The Riemann-Liouville fractional integrals $I^\beta_{a+}\varphi$ and $I^\beta_{b-}\varphi$ of order $\beta \in (0, 1)$ are defined by the formulas

$$I^\beta_{a+}\varphi(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\beta}} dt, \quad x \in \Omega$$

and

$$I^\beta_{b-}\varphi(x) = \frac{1}{\Gamma(\beta)} \int_x^b \frac{\varphi(t)}{(t-x)^{1-\beta}} dt, \quad x \in \Omega$$

respectively. Here $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt.$$

The integrals $I^\beta_{a+}\varphi$ and $I^\beta_{b-}\varphi$ are called the left-sided and the right-sided fractional integrals of order β, respectively.

Definition 2. (Riemann-Liouville fractional derivatives). Let $\Omega = [a, b]$ and $\varphi(x) \in L_1(\Omega)$. The Riemann-Liouville fractional derivatives ${}^R_aD^\beta_x\varphi$ and ${}^R_xD^\beta_b\varphi$ of order $\beta \in (0, 1)$ are defined by

$${}^R_aD^\beta_x\varphi(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{\varphi(t)}{(x-t)^\beta} dt, \quad x \in \Omega$$

and

$${}^R_xD^\beta_b\varphi(x) = \frac{-1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^b \frac{\varphi(t)}{(t-x)^\beta} dt, \quad x \in \Omega.$$

${}^R_aD^\beta_x\varphi$ and ${}^R_xD^\beta_b\varphi$ are called the left-sided and the right-sided Riemann-Liouville fractional derivatives of order β, respectively.

Definition 3. (Caputo fractional derivatives). Let $\Omega = [a,b]$ and $\varphi(x) \in C^1(\Omega)$. The Caputo fractional derivatives ${}_a^C D_x^\beta \varphi$ and ${}_x^C D_b^\beta \varphi$ of order $\beta \in (0,1)$ are defined by

$$ {}_a^C D_x^\beta \varphi(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{\varphi'(t)}{(x-t)^\beta} dt $$

and

$$ {}_x^C D_b^\beta \varphi(x) = \frac{-1}{\Gamma(1-\beta)} \int_x^b \frac{\varphi'(t)}{(t-x)^\beta} dt. $$

${}_a^C D_x^\beta \varphi$ and ${}_x^C D_b^\beta \varphi$ are called the left-sided and the right-sided Caputo fractional derivatives of order β, respectively.

Proposition 1. *Let $\Omega = [a,b]$ and $\beta \in (0,1)$. For a function $\varphi(x) \in L_1(\Omega)$, the Riemann-Liouville fractional derivatives hold*

$$ \frac{d}{dx} I_{a+}^{1-\beta} \varphi(x) = {}_a^R D_x^\beta \varphi(x), \quad \frac{d}{dx} I_{b-}^{1-\beta} \varphi(x) = -{}_x^R D_b^\beta \varphi(x). \tag{2.1} $$

For a function $\varphi(x) \in C^1(\Omega)$, the Caputo fractional derivatives hold

$$ \frac{d}{dx} I_{a+}^{1-\beta} \varphi(x) = {}_a^C D_x^\beta \varphi(x) + \frac{\varphi(a)}{\Gamma(1-\beta)} (x-a)^{-\beta} \tag{2.2} $$

and

$$ \frac{d}{dx} I_{b-}^{1-\beta} \varphi(x) = {}_x^C D_b^\beta \varphi(x) - \frac{\varphi(b)}{\Gamma(1-\beta)} (b-x)^{-\beta}. \tag{2.3} $$

Proof. The Equalities (2.1) are obviously true from the definitions of Riemann-Liouville fractional integrals and derivatives. The proof of Equation (2.3) is similar to that of (2.2). So, we only give the proof for Equation (2.2) in the following.

For $0 < \beta < 1$, it follows that

$$ \begin{aligned} \frac{d}{dx} I_{a+}^{1-\beta} \varphi(x) &= \frac{d}{dx} \left[\frac{1}{\Gamma(1-\beta)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-(1-\beta)}} dt \right] \\ &= \frac{-1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \varphi(t) d \frac{(x-t)^{1-\beta}}{1-\beta} \\ &= \frac{-1}{(1-\beta)\Gamma(1-\beta)} \frac{d}{dx} \left[\varphi(t)(x-t)^{1-\beta} \Big|_a^x - \int_a^x (x-t)^{1-\beta} \varphi'(t) dt \right] \\ &= \frac{\varphi(a)}{\Gamma(1-\beta)} (x-a)^{-\beta} + \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{\varphi'(t)}{(x-a)^\beta} dt \\ &= {}_a^C D_x^\beta \varphi(x) + \frac{\varphi(a)}{\Gamma(1-\beta)} (x-a)^{-\beta}. \end{aligned} $$

The proof is completed. □

2.2 Numerical method for fractional integrals

Let $\varphi(x) \in L_1(\Omega)$ and it has enough smoothness in $\Omega = [a,b]$ for numerical calculation. Without specific clarification, the method introduced here for numerically computing the fractional integrals is in the uniformed mesh with $x_k = a + kh$, $k = 0, 1, \ldots, N$ and the step size $h = \frac{b-a}{N}$, $N \in \mathbb{Z}^+$. Then, the left-sided and the right-sided Riemann-Liouville fractional derivatives of order β at x_k can be expressed respectively as

$$\begin{aligned} I_{a+}^{\beta}\varphi(x_k) &= \frac{1}{\Gamma(\beta)}\int_a^{x_k}\frac{\varphi(t)}{(x_k-t)^{1-\beta}}dt \\ &= \frac{1}{\Gamma(\beta)}\sum_{j=0}^{k-1}\int_{x_j}^{x_{j+1}}\frac{\varphi(t)}{(x_k-t)^{1-\beta}}dt, \quad 1 \le k \le N \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} I_{b-}^{\beta}\varphi(x_k) &= \frac{1}{\Gamma(\beta)}\int_{x_k}^{b}\frac{\varphi(t)}{(t-x_k)^{1-\beta}}dt \\ &= \frac{1}{\Gamma(\beta)}\sum_{j=k}^{N-1}\int_{x_j}^{x_{j+1}}\frac{\varphi(t)}{(t-x_k)^{1-\beta}}dt, \quad 0 \le k \le N-1. \end{aligned}$$

Next, we give a numerical method to approximate integrals (2.4). To this end, we approximate $\varphi(x)$ with a linear interpolation function on $[x_j, x_{j+1}]$, that is,

$$\varphi(x) \approx \varphi(x_j)\frac{x-x_{j+1}}{x_j-x_{j+1}} + \varphi(x_{j+1})\frac{x-x_j}{x_{j+1}-x_j}, \quad x \in [x_j, x_{j+1}].$$

Substitute the above interpolation into (2.4), and through integration by parts we can get

$$I_{a+}^{\beta}\varphi\left(x_k\right) \approx \frac{1}{\Gamma\left(\beta+2\right)}\sum_{j=0}^{k} c_{kj}\varphi(x_j), \tag{2.5}$$

where

$$\begin{aligned} c_{k0} &= \frac{(\beta+1)(x_k-x_0)^{\beta}(x_0-x_1)-(x_k-x_1)^{\beta+1}+(x_k-x_0)^{\beta+1}}{x_0-x_1}, \\ c_{kj} &= \frac{(x_k-x_{j-1})^{\beta+1}-(x_k-x_j)^{\beta+1}}{x_j-x_{j-1}} + \frac{(x_k-x_j)^{\beta+1}-(x_k-x_{j+1})^{\beta+1}}{x_j-x_{j+1}}, 1 \le j \le k-1, \\ c_{kk} &= (x_k-x_{k-1})^{\beta}. \end{aligned}$$

Similarly, we have

$$I_{b-}^{\beta}\varphi\left(x_k\right) \approx \frac{1}{\Gamma\left(\beta+2\right)}\sum_{j=k}^{N} d_{kj}\varphi(x_j),$$

where

$$
\begin{aligned}
d_{kN} =&\frac{(\beta+1)(x_N-x_k)^\beta(x_N-x_{N-1})-(x_N-x_k)^{\beta+1}+(x_{N-1}-x_k)^{\beta+1}}{x_N-x_{N-1}},\\
d_{kj} =&\frac{(x_{j-1}-x_k)^{\beta+1}-(x_j-x_k)^{\beta+1}}{x_j-x_{j-1}}+\frac{(x_j-x_k)^{\beta+1}-(x_{j+1}-x_k)^{\beta+1}}{x_j-x_{j+1}},\\
&k+1\le j\le N-1,\\
d_{kk} =&(x_{k+1}-x_k)^\beta.
\end{aligned}
$$

3 Regularization methods for fractional derivatives

3.1 Reformulation of numerical differentiation

From properties (2.1)–(2.3), we only need to consider the differentiation problem for numerically computing the Riemann-Liouville fractional derivatives, and transform it into an inverse source problem of a first order hyperbolic equation by the following three steps.

Step 1. Computing the fractional integral $I_{a+}^{1-\beta}\varphi(x)$ of $\varphi(x)$, and denoting it by $\Phi(x)$. That is,

$$
\Phi(x)=\begin{cases} I_{a+}^{1-\beta}\varphi(x)=\frac{1}{\Gamma(1-\beta)}\int_a^x\frac{\varphi(t)}{(x-t)^\beta}dt,\ x\in\Omega,\ \beta\in(0,1),\\ \varphi(x),\ x\in\Omega,\ \beta=1\end{cases}
$$

and $\Phi(x)$ is continuously extended with $\Phi(x)=0$ for $x<a$.

Step 2. Taking $\Phi(x)$ as the initial distribution of the concentration $u_1(x,t)$, and solving the following direct problem of a first order hyperbolic equation

$$
\begin{cases} \frac{\partial u_1}{\partial t}=-\frac{\partial u_1}{\partial x},\ \ (x,t)\in\Omega\times(0,T],\\ u_1\left(x,0\right)=\Phi\left(x\right),\ \ x\in\Omega,\\ u_1\left(0,t\right)=0,\ \ t\in[0,T].\end{cases} \tag{3.1}
$$

Step 3. Let $u_1\left(x,t\right)=w_1\left(x,t\right)+\Phi\left(x\right)$. Then, $w_1(x,t)$ satisfies

$$
\begin{cases} \frac{\partial w_1}{\partial t}=-\frac{\partial w_1}{\partial x}-f(x),\ \ x\in\Omega\times(0,T],\\ w_1\left(x,0\right)=0,\ \ x\in\Omega,\\ w_1\left(0,t\right)=0,\ \ t\in[0,T],\end{cases} \tag{3.2}
$$

where $f\left(x\right)=\frac{d}{dx}\Phi\left(x\right)$, that is $f(x)={}_0^RD_x^\beta\varphi\left(x\right)$ for $\beta\in(0,1)$ and $f(x)=\varphi'(x)$ for $\beta=1$. Then the differentiation problem for numerically computing the fractional derivative and the first order derivative is reduced to reconstructing the source term $f(x)$ from the overspecified condition

$$
w_1(x,T)=u_1(x,T)-\Phi(x), \tag{3.3}
$$

where $u_1(x,t)$ is the solution to direct problem (3.1) which can be obtained by the characteristic method that $u_1(x,t)=\Phi(x-t)$. In the practical application, the overspecified condition (3.3) is replaced by its noise-contained form with

$$
w_1^\delta(x,T)=u_1^\delta(x,T)-\Phi^\delta(x),
$$

where $u_1^\delta(x,T) = \Phi^\delta(x-T)$, and $\Phi^\delta(x-T)$ is obtained from **Step 1** after replacing φ with φ^δ.

In the subsequent theoretical analysis and numerical algorithm construction, we also need the solution to direct problem (3.2) with respect to the source term $f(x)$, that is,

$$w_1(x,t) = -\int_0^t f(x-t+\tau)d\tau, \quad (x,t) \in \Omega \times (0,T], \tag{3.4}$$

where we specify $f(x-t+\tau) = 0$ if $x-t+\tau < 0$.

3.2 Existence, uniqueness and conditional stability

As we know, inverse source problems of partial differential equations are always ill-posed, which means that the solutions of inverse problems may not exist, may not be unique, or may be unstable. In this subsection, we discuss the conditional well-posedness of the inverse source problem occurred in **Step 3**.

Theorem 1. (*Existence, uniqueness*) *Let $f(x)$ have a derivative of first order. If T is not a period of $f(x)$ together with $f(x)=0, x<a$, then the inverse source problem for reconstructing $f(x)$ from problem (3.2) and the overspecified condition (3.3) has a unique solution.*

Proof. From the solution (3.4) of problem (3.2), we know that

$$w_1(x,T) = -\int_0^T f(x-T+\tau)d\tau.$$

Taking the derivative with respect to x on both sides of the above equation, we have

$$\frac{dw_1(x,T)}{dx} = -\int_0^T f'(x-T+\tau)d\tau = -f(x) + f(x-T).$$

It follows that

$$f(x) = f(x-T) - \frac{dw_1(x,T)}{dx}. \tag{3.5}$$

The existence of $f(x)$ is attributed to $f(x)=0, x<a$.

The uniqueness of the inverse solution can be summarized as $f(x)=0$ when $w_1(x,T)=0$. In this case, we have

$$0 = -\int_0^T f'(x-T+\tau)d\tau = -f(x) + f(x-T), \quad \forall x \in \Omega.$$

Therefore, $f(x)=0$ since T is not a period of $f(x)$, which implies that the inverse solution is unique. The proof is completed. □

Theorem 2. (*Conditional stability of source inversion*) *Let $f(x)$ have a derivative of first order and $f(x)=0$ for $x<a$. Let T be not a period of $f(x)$ with*

$T \geq b-a$, and $\frac{\partial^2 w_1(x,T)}{\partial x^2} \in L_2(\Omega)$. If $w_1(b,T)=0$ or $\frac{\partial w_1(b,T)}{\partial x}=0$, then the following estimation is valid:

$$\left\|f(x)\right\|_{L_2(\Omega)} \leq \sqrt{\left\|\frac{\partial^2 w_1(x,T)}{\partial x^2}\right\|_{L_2(\Omega)}} \sqrt{\left\|w_1(x,T)\right\|_{L_2(\Omega)}}. \tag{3.6}$$

Proof. Noting that $f(x-T)=0$ when $T \geq b-a$ and $w_1(a,T)=0$, from (3.5) we have

$$\begin{aligned}\int_a^b \left|f(x)\right|^2 dx &= \int_a^b \left|f(x-T) - \frac{\partial w_1(x,T)}{\partial x}\right|^2 dx = \int_a^b \left(\frac{\partial w_1(x,T)}{\partial x}\right)^2 dx \\ &= \left(w_1(x,T)\frac{\partial w_1(x,T)}{\partial x}\right)\Bigg|_a^b - \int_a^b w_1(x,T)\frac{\partial^2 w_1(x,T)}{\partial x^2} dx \\ &\leq \left\|\frac{\partial^2 w_1}{\partial x^2}(x,T)\right\|_{L_2(\Omega)} \left\|w_1(x,T)\right\|_{L_2(\Omega)}.\end{aligned}$$

The above last inequality is obtained by using the Hölder inequality. Thus,

$$\left\|f(x)\right\|_{L_2(\Omega)} \leq \sqrt{\left\|\frac{\partial^2 w_1}{\partial x^2}(x,T)\right\|_{L_2(\Omega)}} \sqrt{\left\|w_1(x,T)\right\|_{L_2(\Omega)}}.$$

The proof is completed. □

Remark 1. The condition $w_1(b,T)=0$ in Theorem 2 is trivial in the proposed method for numerical fractional derivative. Making the following transformation for $\Phi(x)$ as

$$\Phi(x) := \Phi(x) - \Phi(b)\frac{x-a}{b-a},$$

we have $\Phi(a)=\Phi(b)=0$. Thus, $w_1(b,T)=u_1(b,T)-\Phi(b)=0$ for $T \geq b-a$.

Theorem 3. *(Conditional stability of numerical differentiation) Let $\varphi(x)$ be smooth enough such that $\Phi''(x) \in L_2(\Omega)$ and $\left\|\Phi''(x)\right\|_{L_2(\Omega)} \leq M$ for a constant $M>0$. Under the conditions of Theorem 2, then the following conditional stabilities hold:*

$$\left\|{}_0^R D_x^\beta \varphi(x)\right\|_{L_2(\Omega)} \leq C \max_{x\in\Omega} \sqrt{|\varphi(x)|}$$

for numerically computing the fractional derivative, and

$$\left\|\varphi'(x)\right\|_{L_2(\Omega)} \leq C \max_{x\in\Omega} \sqrt{|\varphi(x)|}$$

for numerically computing the first order derivative with a positive constant C.

Proof. Noting that $w_1(x,T)=\Phi(x-T)-\Phi(x)$ and the definition of $\Phi(x)$, we have

$$\left\|\frac{\partial^2 w_1}{\partial x^2}(x,T)\right\|_{L_2(\Omega)} \leq 2\left\|\Phi''(x)\right\|_{L_2(\Omega)},$$

and for $\beta \in (0,1)$ have

$$\begin{aligned}\left\|w_1(x,T)\right\|^2_{L_2(\Omega)} \le 4\left\|\Phi(x)\right\|^2_{L_2(\Omega)} &= \frac{4}{(\Gamma(1-\beta))^2}\int_a^b\left(\int_a^x \frac{\varphi(t)}{(x-t)^\beta}dt\right)^2 dx\\ &\le \frac{4}{(\Gamma(1-\beta))^2}\left(\max_{x\in\Omega}|\varphi(x)|\right)^2\int_a^b\left(\int_a^x \frac{1}{(x-t)^\beta}dt\right)^2 dx\\ &= \frac{4}{(\Gamma(1-\beta))^2}\frac{(b-a)^{3-2\beta}}{(1-\beta)^2(3-2\beta)}\left(\max_{x\in\Omega}|\varphi(x)|\right)^2.\end{aligned}$$

Meanwhile, noticing that $f(x) = {}^R_0D^\beta_x\varphi(x)$, it can be derived directly from (3.6) that

$$\left\|{}^R_0D^\beta_x\varphi(x)\right\|_{L_2(\Omega)} \le \sqrt{\left\|\frac{\partial^2 w_1}{\partial x^2}(x,T)\right\|_{L_2(\Omega)}}\sqrt{\left\|w_1(x,T)\right\|_{L_2(\Omega)}} \le C\max_{x\in\Omega}\sqrt{|\varphi(x)|},$$

where $C = 2\sqrt{\frac{M(b-a)^{3/2-\beta}}{\Gamma(2-\beta)\sqrt{3-2\beta}}}$. On the other hand, for $\beta = 1$ we can easily obtain that

$$\left\|\varphi'(x)\right\|_{L_2(\Omega)} \le \sqrt{\left\|\frac{\partial^2 w_1}{\partial x^2}(x,T)\right\|_{L_2(\Omega)}}\sqrt{\left\|w_1(x,T)\right\|_{L_2(\Omega)}} \le C\max_{x\in\Omega}\sqrt{|\varphi(x)|},$$

where $C = 2\sqrt{M}(b-a)^{\frac{1}{4}}$.

The proof is completed from the above two inequalities. □

3.3 Regularization method for source inversion

It is easy to see from (3.5) that reconstruction of the source term $f(x)$ is not stable, and its instability is equivalent to that for numerically computing the derivative of $w_1(x,T)$. Hence, small errors in the noise data $w_1^\delta(x,T)$ of $w_1(x,T)$ will be enlarged dramatically in the solution of source inversion. To overcome this instability, we adopt the Tikhonov regularization method to reconstruct the source term, that is, minimizing the Tikhonov functional

$$J^\delta_\alpha(f) = \left\|w_1(x,T;f) - w_1^\delta(x,T)\right\|^2_{L_2(\Omega)} + \alpha\|f\|^2_{L_2(\Omega)} \tag{3.7}$$

in a reasonably admissible set $S = \left\{f\,\middle|\,f\in L_2(\Omega) \text{ and } |f|\le M\right\}$, where M is a positive constant. Here, α is a regularization parameter, and $w_1(x,t;f)$ represents the solution to problem (3.2) with respect to $f(x)$.

To numerically solve the minimizing problem of Tikhonov functional (3.7), we expand $f(x)$ in a finite dimensional space with

$$f(x) \approx \sum_{i=1}^K f_i\xi_i(x),$$

where $\{\xi_i(x)\}_{i=1}^K$ is linearly independent on Ω and is a set of basis functions. Obviously, problem (3.2) satisfies the superposition principle due to the linearity of the hyperbolic equation and the homogeneity of the initial-boundary conditions. Therefore, we have

$$w_1(x,T;f) \approx \sum_{i=1}^{K} f_i w_1(x,T;\xi_i(x)),$$

and $w_1(x,T;\xi_i(x)) = -\int_0^T \xi_i(x-T+\tau)d\tau = -\int_{x-T}^{x} \xi_i(\tau)d\tau$.

Let $F = (f_1, f_2, \ldots, f_{K-1}, f_K)^T$. Then we obtain the approximation of Tikhonov functional (3.7) as follows

$$J_\alpha^\delta(F) = \left\| \sum_{i=1}^{K} f_i w_1(x,T;\xi_i(x)) - w_1^\delta(x,T) \right\|_{L_2(\Omega)}^2 + \alpha \left\| \sum_{i=1}^{K} f_i \xi_i(x) \right\|_{L_2(\Omega)}^2 . \quad (3.8)$$

According to the necessary conditions for minimizing functional (3.8), a system of linear algebraic equations with respect to F can be obtained:

$$(A + \alpha G)F = b, \quad (3.9)$$

where $A = (a_{ij})_{K\times K}, G = (g_{ij})_{K\times K}, b = (b_i)_{K\times 1}$, and

$$a_{ij} = \int_a^b w_1(x,T;\xi_i(x)) w_1(x,T;\xi_j(x))dx,$$

$$g_{ij} = \int_a^b \xi_i(x)\xi_j(x)dx, \quad b_i = \int_a^b w_1(x,T;\xi_i(x)) w_1^\delta(x,T)dx.$$

For a given regularization parameter α, we solve system (3.9) and obtain the discrete solution $F^* = (f_1^*, f_2^*, \ldots, f_{K-1}^*, f_K^*)^T$. It follows that the regularization solution of source term is

$$f_\alpha^\delta(x) = \sum_{i=1}^{K} f_i^* \xi_i(x),$$

which is the numerical approximation of fractional derivative ${}_0^R D_x^\beta \varphi(x)$ or first order derivative $\varphi'(x)$ for the noise data $\varphi^\delta(x)$.

3.4 Joint regularization methods for source inversion

Numerical experiments show that the above approach performs poorly at the right endpoint when numerically computing derivatives. The reason for this phenomenon may be that the first-order hyperbolic equations in (3.1) and (3.2) propagate in the characteristic direction (the right-hand direction), and accumulate errors toward the right. Therefore, we construct another approach in which the hyperbolic equations propagate along the opposite direction of those in (3.1) and (3.2), and establish two joint regularization methods for numerically computing the fractional and the first order derivatives.

From results of Remark 1, we can suppose that $\Phi(b) = 0$ and $\Phi(x) = 0$ for $x > b$. Then, take $\Phi(x)$ as the initial value of concentration $u_2(x,t)$, and solve the following initial-boundary problem:

$$\begin{cases} \frac{\partial u_2}{\partial t} = \frac{\partial u_2}{\partial x}, & (x,t) \in \Omega \times (0,T], \\ u_2(x,0) = \Phi(x), & x \in \Omega, \\ u_2(b,t) = 0, & t \in [0,T]. \end{cases} \tag{3.10}$$

Subsequently, reconstruct the source term $f(x)$ from problem

$$\begin{cases} \frac{\partial w_2}{\partial t} = \frac{\partial w_2}{\partial x} + f(x), & (x,t) \in \Omega \times (0,T], \\ w_2(x,0) = 0, & x \in \Omega, \\ w_2(b,t) = 0, & t \in [0,T], \end{cases}$$

and the overspecified condition

$$w_2(x,T) = u_2(x,T) - \Phi(x), \quad x \in \Omega. \tag{3.11}$$

Obviously, $f(x) = \Phi'(x) = {}_0^R D_x^\beta \varphi(x)$ for $\beta \in (0,1)$ and $f(x) = \varphi'(x)$ for $\beta = 1$, $u_2(x,t) = \Phi(x+t)$ and $w_2(x,t) = \int_0^t f(x+t-\tau)d\tau$, where $f(x) = 0$ for $x > b$. Thus, we can stably reconstruct the source $f(x)$ by minimizing the Tikhonov functional

$$J_\alpha^\delta(f) = \left\| w_2(x,T;f) - w_2^\delta(x,T) \right\|_{L_2(\Omega)}^2 + \alpha \|f\|_{L_2(\Omega)}^2 \tag{3.12}$$

from the noise data $w_2^\delta(x,T) = u_2^\delta(x,T) - \Phi^\delta(x)$ in the admissible set S.

In order to obtain better inversion of the source term $f(x)$, we construct two joint regularization methods, which combine the overspecified conditions (3.3) and (3.11) to minimize respectively the Tikhonov regularization functionals

$$J_\alpha^\delta(f) = \left\| w_1(x,T;f) - w_1^\delta(x,T) \right\|_{L_2(\Omega)}^2 + \left\| w_2(x,T;f) - w_2^\delta(x,T) \right\|_{L_2(\Omega)}^2 + \alpha \|f\|_{L_2(\Omega)}^2 \tag{3.13}$$

and

$$J_\alpha^\delta(f) = \left\| w_1(x,T;f) + w_2(x,T;f) - w_1^\delta(x,T) - w_2^\delta(x,T) \right\|_{L_2(\Omega)}^2 + \alpha \|f\|_{L_2(\Omega)}^2 \tag{3.14}$$

in the reasonably admissible set S, where α is a regularization parameter. The implementation of minimizing regularized functionals (3.12)–(3.14) is same to that of minimizing (3.7) stated in subsection 3.3. So, we do not repeat it.

3.5 Strategy for choosing regularization parameters

As is known to all, the effectiveness of minimizing Tikhonov regularization functionals (3.7) and (3.12)–(3.14) depends strongly on the value of regularization parameter α [5, 16, 19]. Therefore, how to choose an appropriate value of

α is very important. Here, we adopt the Morozov's discrepancy principle [14] for choosing regularization parameters, that is, choose the value of α satisfying respectively the following discrepancy equations

$$\left\|w_j(x,T;f_\alpha^\delta) - w_j^\delta(x,T)\right\|^2_{L_2(\Omega)} = C_1\delta^2,\ j=1,2, \tag{3.15}$$

$$\left\|w_1(x,T;f_\alpha^\delta)-w_1^\delta(x,T)\right\|^2_{L_2(\Omega)}+\left\|w_2(x,T;f_\alpha^\delta)-w_2^\delta(x,T)\right\|^2_{L_2(\Omega)}=2C_1\delta^2$$

and

$$\left\|w_1(x,T;f_\alpha^\delta)+w_2(x,T;f_\alpha^\delta)-w_1^\delta(x,T)-w_2^\delta(x,T)\right\|^2_{L_2(\Omega)}=C_2\delta^2, \tag{3.16}$$

where C_1 and C_2 are positive constants.

4 Numerical experiments

In this section, we give some numerical examples to verify effectiveness of the proposed methods for numerically computing the derivatives of fractional and first order. In all examples, we always take $\Omega = [0,1]$ and $T = 1.0$. Then, we partition $\Omega = [0,1]$ into N equal subintervals by using equidistant nodes $0 = x_0 < x_1 < \ldots < x_N = 1$ with stepsize $h = \frac{1}{200}$. The noise is added to $\varphi(x)$ in a discrete form

$$\varphi^\delta(x_i) = \varphi(x_i) + \delta\, r_i,$$

where δ is a noise level and, $R = [r_0, r_1, \ldots, r_N]$ is a random vector subject to standard normal distribution. R is used repeatedly in all examples for comparisons. To show the practicability and applications, we choose the piecewise linear functions as the basis set to implement the proposed methods, that is, approximate the source function $f(x)$ by an expansion of functions $\xi_1(x), \xi_2(x), \ldots, \xi_{N+1}(x)$ which defined by

$$\xi_1(x) = \begin{cases} \frac{x-x_1}{x_0-x_1}, & x\in[x_0,x_1], \\ 0, & x\in(x_1,x_N], \end{cases}, \quad \xi_{N+1}(x) = \begin{cases} 0, & x\in[x_0,x_{N-1}). \\ \frac{x-x_{N-1}}{x_N-x_{N-1}}, & x\in[x_{N-1},x_N], \end{cases}$$

$$\xi_{i+1}(x) = \begin{cases} 0, & x\in[x_0,x_{i-1}), \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}, & x\in[x_{i-1},x_i], \\ \frac{x-x_{i+1}}{x_i-x_{i+1}}, & x\in[x_i,x_{i+1}], \\ 0, & x\in[x_{i+1},x_N), \end{cases} ,i=1,2,\ldots,N-1.$$

When using strategies (3.15)–(3.16) to choose regularization parameters, we always take $C_2 = 2C_1$, and $C_1 = 0.00005$ for $\beta = 0.3$, $C_1 = 0.005$ for $\beta = 0.7$, $C_1 = 0.5$ for $\beta = 1.0$, which shows a certain regularity. Now, we only present the algorithm based on the regularization method of minimizing (3.13) as an example in the following Algorithm 1.

The first three numerical examples are to calculate the fractional derivatives with respect to different functions, and the last example is to compare the proposed method with some existing methods for computing the fractional and first order derivatives. In the first three numerical examples, the relative errors of regularization solutions are presented in Tables 1–3 for orders $\beta = 0.3, 0.7, 1.0$, and noise levels $\delta = 0.001, 0.01, 0.03$, respectively. From results in Tables 1–3, we find that the regularization method of minimizing functional (3.13) is better than those of minimizing functionals (3.7), (3.12) and (3.14). Hence, we only give the regularization solutions of minimizing functional (3.13) in Figures 1–9.

Algorithm 1: Regularization method of minimizing (3.13).

Input: $a = 0$, $b = 1$, $\varphi^\delta(x_i)$, $x_i = ih$, $h = \frac{1}{N}$, $i = 0, 1, \ldots, N$, $T = 1$
Output: $f^\delta := \left({}_0^R D_x^\beta \varphi^\delta(x_0), {}_0^R D_x^\beta \varphi^\delta(x_1), \ldots, {}_0^R D_x^\beta \varphi^\delta(x_N)\right)$

Compute $\Phi^\delta(x_i) = I_{a+}^{1-\beta}\varphi^\delta(x_i)$ by using (2.5);
Solve (3.1) for obtaining $u_1^\delta(x_i, t) = \Phi^\delta(x_i - T)$;
Compute $W1 := \left(w_1^\delta(x_0, T), w_1^\delta(x_1, T), \ldots, w_1^\delta(x_N, T)\right)$ where
  $w_1^\delta(x_i, T) = u_1^\delta(x_i, T) - \Phi^\delta(x_i)$, $i = 0, 1, \ldots, N$;
Solve (3.10) for obtaining $u_2^\delta(x_i, T) = \Phi^\delta(x_i + T)$;
Compute $W2 := \left(w_2^\delta(x_0, T), w_2^\delta(x_1, T), \cdots, w_2^\delta(x_N, T)\right)$ where
  $w_2^\delta(x_i, T) = u_2^\delta(x_i, T) - \Phi^\delta(x_i)$, $i = 0, 1, \ldots, N$;
Choose the basis functions $\{\xi_k(x)\}_{k=1}^K$;
$V1 = \text{zeros}(K, N+1)$; $V2 = \text{zeros}(K, N+1)$; $R = \text{zeros}(K, N+1)$;
for $k = 1 : K$ **do**
  $V1(k, :) = \left(w_1(x_0, T; \xi_k(x)), w_1(x_1, T; \xi_k(x)), \ldots, w_1(x_N, T; \xi_k(x))\right)$;
  $V2(k, :) = \left(w_2(x_0, T; \xi_k(x)), w_2(x_1, T; \xi_k(x)), \ldots, w_2(x_N, T; \xi_k(x))\right)$;
  $R(k, :) = \left(\xi_k(x_0), \xi_k(x_1), \ldots, \xi_k(x_N)\right)$;
end
Generate matrices $A1$, $A2$ by $A1 = V1 * V1^T$, $A2 = V2 * V2^T$;
Generate matrices $B1$, $B2$ by $B1 = V1 * W1^T$, $B2 = V2 * W2^T$;
Generate matrices G by $G = R * R^T$;
Choose $\alpha_0 = 1$ and $q = \frac{1}{2}$;
Let $j = 0$;
while
  $h * \text{trapz}((W5 - W1)(W5 - W1)^T + (W6 - W2)(W6 - W2)^T) > 2C_1\delta^2$ **do**
  $\alpha_j = \alpha_0 q^j$;
  $F = (A1 + A2 + \alpha_j G)^{-1}(B1 + B2)$;
  $f^\delta = F^T$ R;
  $W5 = (w_1(x_0, T; f^\delta), w_1(x_1, T; f^\delta), \ldots, w_1(x_N, T; f^\delta)) = F^T V1$;
  $W6 = (w_2(x_0, T; f^\delta), w_2(x_1, T; f^\delta), \ldots, w_2(x_N, T; f^\delta)) = F^T V2$;
  $j = j + 1$;
end
return f^δ (Values of the fractional derivative)

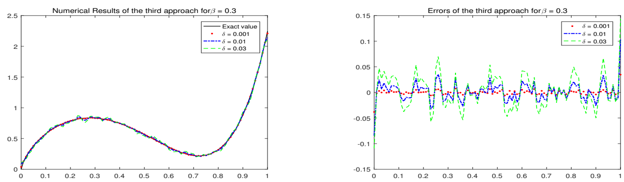
Example 1. Take $\varphi_1(x) = 3x - 3x^2 - \frac{5}{3}x^3 + 3x^6$. So, its Riemann-Liouville fractional derivative of order β is

$$ {}_0^R D_x^\beta \varphi_1(x) = \frac{3\Gamma(1+1)}{\Gamma(1-\beta+1)} x^{1-\beta} - \frac{3\Gamma(2+1)}{\Gamma(2-\beta+1)} x^{2-\beta} - \frac{5\Gamma(3+1)}{3\Gamma(3-\beta+1)} x^{3-\beta} + \frac{3\Gamma(6+1)}{\Gamma(6-\beta+1)} x^{6-\beta}, $$

and its derivative of first order is $\varphi_1'(x) = 3 - 6x - 5x^2 + 18x^5$. In this example, the relative errors of $\varphi_1^\delta(x)$ are $0.161855\%, 1.618551\%, 4.855653\%$ for $\delta = 0.001, 0.01, 0.03$, respectively. The computational results are showed in Table 1 and Figures 1–3.

Table 1. Example 1 relative errors of regularization solutions.

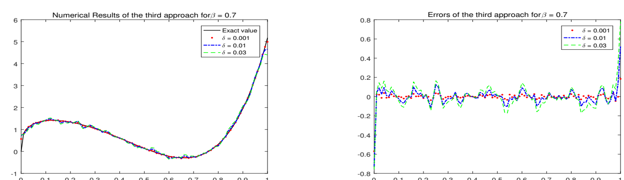
	δ	Relative error $= \left\| f_\beta^\delta - {}_0^R D_x^\beta \varphi \right\| \Big/ \left\| {}_0^R D_x^\beta \varphi \right\|$			
		min(3.7)	min(3.12)	min(3.13)	min(3.14)
$\beta = 0.3$	0.001	1.596528e-02	1.467824e-02	6.371363e-03	9.976664e-03
	0.01	8.929022e-02	5.381751e-02	2.355025e-02	4.972828e-02
	0.03	1.457824e-01	7.672907e-02	4.342223e-02	8.181446e-02
$\beta = 0.7$	0.001	7.075689e-02	4.054403e-02	3.237865e-02	4.159473e-02
	0.01	2.177177e-01	6.381559e-02	5.997846e-02	1.378819e-01
	0.03	2.939425e-01	8.624024e-02	8.615749e-02	1.958447e-01
$\beta = 1.0$	0.001	1.756364e-01	5.391811e-02	2.826946e-02	1.355291e-01
	0.01	3.796694e-01	1.335865e-01	1.148947e-01	2.932596e-01
	0.03	5.509703e-01	2.511444e-01	2.461262e-01	4.094092e-01



(a) Numerical fractional derivatives

(b) Errors of numerical fractional derivatives

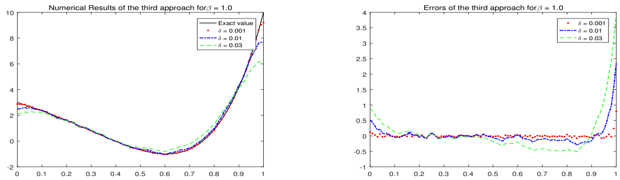
Figure 1. Results of minimizing (3.13) for $\beta = 0.3$ and $C_1 = 0.00005$ in Example 1.



(a) Numerical derivatives of first order

(b) Errors of numerical first order derivatives

Figure 2. Results of minimizing (3.13) for $\beta = 0.7$ and $C_1 = 0.5$ in Example 1.



(a) Numerical derivatives of first order

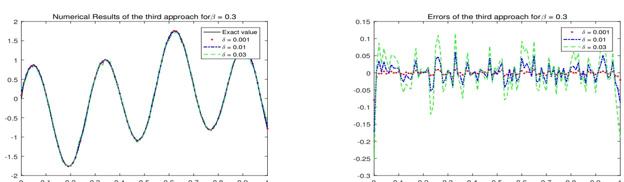
(b) Errors of numerical first order derivatives

Figure 3. Results of minimizing (3.13) for $\beta = 1.0$ and $C_1 = 0.5$ in Example 1.

Example 2. Take $\varphi_2(x) = \frac{1}{2}\sin(7\pi x) - \frac{3}{10}\sin(2\pi x)$. Obviously, its derivative of first order is $\varphi_2'(x) = \frac{7}{2}\pi\cos(7\pi x) - \frac{3}{5}\pi\cos(2\pi x)$. The exact solutions for fractional derivatives of order $\beta \in (0,1)$ are obtained from the exact value of $\varphi_2(x)$ by using the numerical method presented Subsection 2.2. In this example, the relative errors of $\varphi_2^\delta(x)$ are $0.2186717\%, 2.186717\%, 6.560151\%$ for $\delta = 0.001, 0.01, 0.03$, respectively. The computational results are showed in Table 2 and Figures 4–6.

Table 2. Example 2 relative errors of regularization solutions.

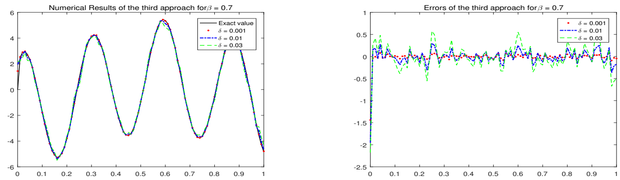
	δ	Relative error $= \left\|f_\beta^\delta - {}_0^RD_x^\beta\varphi\right\| \Big/ \left\|{}_0^RD_x^\beta\varphi\right\|$			
		min(3.7)	min(3.12)	min(3.13)	min(3.14)
	0.001	1.071734e-02	7.545431e-03	7.718370e-03	9.999962e-03
$\beta = 0.3$	0.01	4.706951e-02	2.730137e-02	2.871942e-02	4.002675e-02
	0.03	8.275197e-02	5.667192e-02	5.882381e-02	7.381842e-02
	0.001	4.121455e-02	2.777151e-02	3.493327e-02	4.184775e-02
$\beta = 0.7$	0.01	1.076430e-01	5.120607e-02	6.242095e-02	8.756341e-02
	0.03	1.585615e-01	9.018222e-02	9.466815e-02	1.262816e-01
	0.001	5.924834e-02	4.679699e-02	1.615752e-02	2.252384e-02
$\beta = 1.0$	0.01	1.513511e-01	1.187316e-01	6.404829e-02	6.215945e-02
	0.03	2.562021e-01	2.175347e-01	1.700716e-01	1.263851e-01



(a) Numerical fractional derivatives

(b) Errors of numerical fractional derivatives

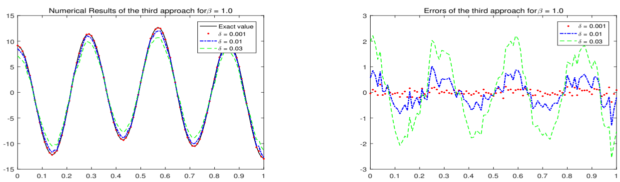
Figure 4. Results of minimizing (3.13) for $\beta = 0.3$ and $C_1 = 0.00005$ in Example 2.



(a) Numerical fractional derivatives

(b) Errors of numerical fractional derivatives

Figure 5. Results of minimizing (3.13) for $\beta = 0.7$ and $C_1 = 0.005$ in Example 2.



(a) Numerical derivatives of first order

(b) Errors of numerical first order derivatives

Figure 6. Results of minimizing (3.13) for $\beta = 1.0$ and $C_1 = 0.5$ in Example 2.

Example 3. Take

$$\varphi_3(x) = \begin{cases} -10x^2 + 5x, & x \in [0, \frac{1}{2}], \\ 10x^2 - 15x + 5, & x \in (\frac{1}{2}, 1], \end{cases} \qquad \varphi_3'(x) = \begin{cases} -20x + 5, & x \in [0, \frac{1}{2}], \\ 20x - 15, & x \in (\frac{1}{2}, 1]. \end{cases}$$

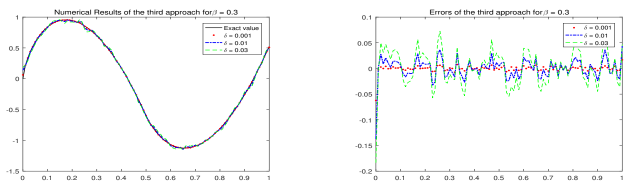
Also, the exact solutions for fractional derivatives of order $\beta \in (0, 1)$ are obtained from the exact value of $\varphi_3(x)$ by using the numerical method presented Subsection 2.2. In this example, the relative errors of $\varphi_3^\delta(x)$ are 0.1975321%, 1.975321%, 5.925963% for $\delta = 0.001, 0.01, 0.03$, respectively. The computational results are showed in Table 3 and Figures 7–9.

From numerical results of the above three examples, it can be seen that the smaller the fractional order, the better the effectiveness of the proposed methods for numerical differentiation. Furthermore, the proposed methods are very effective when the error level is small. Especially, the regularization method of minimizing (3.13) is generally better than the other three methods of minimizing (3.7), (3.12) and (3.14).

Like the PDEs-based numerical differentiation method in references [25], the proposed methods in this paper are also based on the source inversion of partial differential equations. Next, let's take the calculation of the first order derivative as an example to compare the effectiveness between the method of minimizing (3.13) and the PDEs-based one of reference [25].

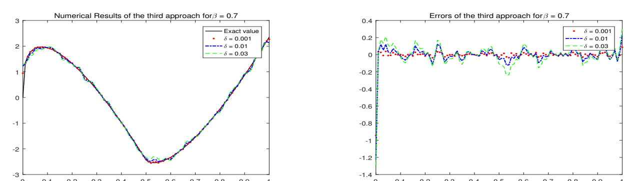
Table 3. Example 3 relative errors of regularization solutions.

	δ	Relative error $= \left\| f_\beta^\delta - {}_0^R D_x^\beta \varphi \right\| \Big/ \left\| {}_0^R D_x^\beta \varphi \right\|$			
		min(3.7)	min(3.12)	min(3.13)	min(3.14)
$\beta = 0.3$	0.001	1.350497e-02	5.176865e-03	7.350985e-03	8.447939e-03
	0.01	5.090400e-02	2.290229e-02	2.445106e-02	4.012851e-02
	0.03	7.531465e-02	4.619148e-02	4.507702e-02	6.952910e-02
$\beta = 0.7$	0.001	6.246526e-02	2.575625e-02	4.446216e-02	3.871467e-02
	0.01	1.403841e-01	6.461969e-02	6.636020e-02	1.139471e-01
	0.03	1.816194e-01	1.055957e-01	8.369378e-02	1.667994e-01
$\beta = 1.0$	0.001	1.131460e-01	1.132491e-01	1.831613e-02	1.226036e-01
	0.01	2.221883e-01	2.277898e-01	7.614857e-02	2.615364e-01
	0.03	3.249182e-01	3.257938e-01	1.723761e-01	3.621127e-01



(a) Numerical fractional derivatives (b) Errors of numerical fractional derivatives

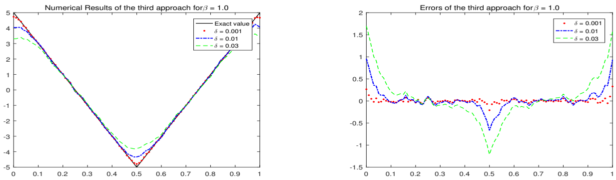
Figure 7. Results of minimizing (3.13) for $\beta = 0.3$ and $C_1 = 0.00005$ in Example 3.



(a) Numerical fractional derivatives (b) Errors of numerical fractional derivatives

Figure 8. Results of minimizing (3.13) for $\beta = 0.7$ and $C_1 = 0.005$ in Example 3.

Example 4. Comparisons with some existing methods for computing the fractional order and first order derivatives of the above three functions $\varphi_i(x), i = 1, 2, 3$ with different noise levels $\delta = 0.0001,\ 0.0005,\ 0.001,\ 0.01,\ 0.03$ and the same random noise vector R. Relative errors between the exact derivatives and the corresponding numerical solutions are listed in Tables 4–6, and numerical



(a) Numerical derivatives of first order (b) Errors of numerical first order derivatives

Figure 9. Results of minimizing (3.13) for $\beta = 1.0$ and $C_1 = 0.5$ in Example 3.

results of the first order derivatives for $\delta = 0.001$ are showed in Figure 10. Here M1 represents the method of minimizing (3.13); M2 represents the PDEs-based method proposed in reference [25]; M3 represents the Lanczos' method presented in reference [34]; M4 represents the B-spline based method given in reference [23]; M5 is the Jacobi spectral method provided in reference [38]. From results of Table 4 and Figure 10, we find that the numerical derivatives of first order of M1 are better than the ones of M2 when the noise level is smaller, but the opposite is true between the numerical derivatives of first order of M1 and the ones of M3. The reason of these phenomena may be that both the heat diffusion equation and the corresponding regularization [25] can suppress larger noise at the same time, and the Lanczos' method requires data from outside the interval Ω when calculating the derivatives near the endpoint. From Figure 10, we can also see that accuracy of M1 is better near the endpoints while accuracy of M2 is little better in the middle part of interval Ω. Meanwhile, numerical results in Tables 5 and 6 show that the proposed method of minimizing (3.13) is superior to B-spline based method (M4) and the Jacobi spectral method (M5), since the two latter methods are sensitive to larger noise so that some appropriate regularization techniques may be required.

Table 4. Comparing minimizing (3.13) with the M1 and M2 methods for $\beta = 1.0$.

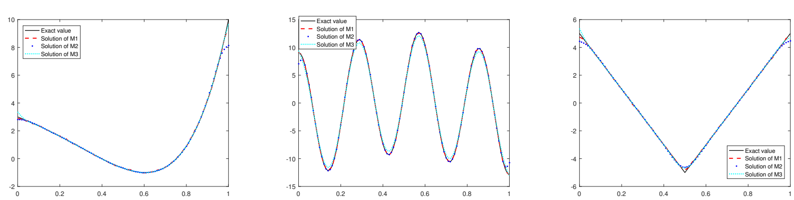
		Relative errors between the exact and regularization solutions				
		δ=0.0001	δ=0.0005	δ=0.001	δ=0.01	δ=0.03
$\varphi_1(x)$	M1	8.215536e-03	1.938692e-02	2.826946e-02	1.148947e-01	2.461262e-01
	M2	5.929323e-02	7.611632e-02	8.064482e-02	9.724040e-02	8.516548e-02
	M3	2.496119e-03	9.055309e-03	1.967900e-02	1.284523e+00	1.926497e+01
$\varphi_2(x)$	M1	4.652172e-03	1.151011e-02	1.615752e-02	6.404829e-02	1.700716e-01
	M2	2.743994e-02	3.523590e-02	3.918616e-02	6.979682e-02	1.396360e-01
	M3	1.194272e-02	3.557661e-02	5.596699e-02	2.540363e-01	2.323064e+00
$\varphi_3(x)$	M1	5.882776e-03	1.281768e-02	1.831613e-02	7.614857e-02	1.723761e-01
	M2	2.491232e-02	3.164266e-02	3.872320e-02	6.674063e-02	8.336076e-02
	M3	7.057460e-03	1.360892e-02	2.281829e-02	8.711955e-01	1.105324e+01

Table 5. Comparing minimizing (3.13) with the M4 and M5 methods for $\beta = 0.7$.

		Relative errors between the exact and regularization solutions				
		δ=0.0001	δ=0.0005	δ=0.001	δ=0.01	δ=0.03
	M1	2.218375e-02	2.804790e-02	3.237865e-02	5.997846e-02	8.615749e-02
	M4	2.668845e-02	2.810264e-02	3.148385e-02	1.565093e-01	4.599641e-01
	M5	1.090776e-02	5.453880e-02	1.090776e-01	1.090776e+00	3.272328e+00
$\varphi_2(x)$	M1	2.753688e-02	3.155913e-02	3.493327e-02	6.242095e-02	9.466815e-02
	M4	3.640491e-02	3.669049e-02	3.733784e-02	8.002699e-02	2.137145e-01
	M5	3.981085e-03	1.954361e-02	3.910807e-02	3.914811e-01	1.174556e+00
$\varphi_3(x)$	M1	3.309556e-02	4.003899e-02	4.446216e-02	6.636020e-02	8.369378e-02
	M4	3.942603e-02	4.031215e-02	4.242696e-02	1.463355e-01	4.203445e-01
	M5	1.106476e-02	5.444219e-02	1.088051e-01	1.087609e+00	3.262759e+00

Table 6. Comparing minimizing (3.13) with the M4 and M5 methods for $\beta = 0.3$.

		Relative errors between the exact and regularization solutions				
		δ=0.0001	δ=0.0005	δ=0.001	δ=0.01	δ=0.03
$\varphi_1(x)$	M1	2.383282e-03	4.435124e-03	6.371363e-03	2.355025e-02	4.342223e-02
	M4	6.354648e-03	6.588961e-03	7.199635e-03	3.290840e-02	9.664872e-02
	M5	5.698683e-04	2.849342e-03	5.698683e-03	5.698683e-02	1.709605e-01
$\varphi_2(x)$	M1	4.231292e-03	5.980453e-03	7.718370e-03	2.871942e-02	5.882381e-02
	M4	1.222714e-02	1.232346e-02	1.255847e-02	2.850274e-02	7.742369e-02
	M5	1.031295e-03	2.366986e-03	4.484589e-03	4.427122e-02	1.328839e-01
$\varphi_3(x)$	M1	3.403974e-03	5.476084e-03	7.350985e-03	2.445106e-02	4.507702e-02
	M4	7.740294e-03	7.950835e-03	8.479624e-03	3.308243e-02	9.613757e-02
	M5	7.308849e-04	3.105115e-03	6.166354e-03	6.146931e-02	1.843877e-01



(a) Example 1 (b) Example 2 (c) Example 3

Figure 10. Comparisons for $\delta = 0.001$ between minimizing (3.13) and the method in [25].

5 Conclusions

Four regularization methods based on the direct and inverse problems of first order hyperbolic equations are proposed for numerically computing the fractional order and the first order derivatives, and the corresponding existence, uniqueness and conditional stability are analyzed. Compared with the PDEs-based numerical differentiation method presented in [25, 32], the proposed methods are simpler and easier to be implemented. Results of numerical examples show that the proposed methods with the Morozov's discrepancy principle for choosing regularization parameters are very effective in the cases of small noise levels. However, the implementation of Morozov's discrepancy principle becomes impractical when the noise level is not known. Consequently, it is imperative to investigate whether the proposed methods demonstrate equal or superior efficacy under alternative strategies for selecting regularization parameters, such as the L-curve technique or the GCV approach in future studies. Additionally, investigating the potential of employing alternative regularization methods, such as the mollification method, regularization methods with total variation and L^1 penalty term, in source term inversion is also valuable to ascertain the possibility of achieving improved results of numerical derivatives. Further examination is needed to compare the methods with more other numerical differentiation approaches. Moreover, applying the proposed methods to solve multivariate numerical derivatives from noise data also requires further research.

Acknowledgements

This work is supported by National Natural Science Foundation of China (12261004), Guangdong Basic and Applied Basic Research Foundation (2025), Jiangxi Provincial Natural Science Foundation (20212ACB201001, 20232BAB201019), Research Fund of Guangzhou Maritime University.

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