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An Effective Numerical Algorithm for Coupled Systems of Emden–Fowler Equations via Shifted Airfoil Functions of the First Kind

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Abstract. The present paper deals with the designation of a new efficient numerical scheme for numerical solution of a class of coupled systems of Emden-Fowler equations describing several processes in applied sciences and technology. This method combines the shifted version of airfoil functions of the first kind (AFFK) and quasilinearization technique. More specifically, the quasilinearization technique is first applied to the original problem and the shifted AFFK (SAFFK) collocation matrix technique is then constructed for obtaining the solution of resulting family of submodels. We derive the error bound and analyze the convergence properties of the SAFFK. Computational and experimental simulations are carried out to describe the applicability and accuracy of the new technique. To show the benefit of the new approach, the computed numerical outcomes are compared with those results obtained via the Bernoulli and Haar wavelets collocation methods. It is evident from the numerical illustrations that the new developed scheme is superior to the existing available ones. The elapsed CPU time of the proposed method is provided.

Keywords: collocation points, convergent analysis, coupled ODEs, Emden–Fowler equations, shifted airfoil functions.

AMS Subject Classification: 65L60; 41A10; 65L05; 65L20.

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1 Introduction

The main goal of this manuscript is to devise an effective approximation algorithm to treat a class of nonlinear second-order coupled system of equations in the form of Emden-Fowler as follows

$$\begin{cases} x''(z) + \dfrac{\alpha}{z} x'(z) + f(x,y) = h(z), \\ y''(z) + \dfrac{\beta}{z} y'(z) + g(x,y) = k(z), \end{cases} \qquad z \in [0,1], \tag{1.1}$$

where α, β are two given positive constants, two functions h, k are known, and we assume that the functions f, g are sufficiently smooth functions with respect to their arguments. We supplement the following initial conditions

$$x(0) = x_0, \quad x'(0) = x_1, \quad y(0) = y_0, \quad y'(0) = y_1. \tag{1.2}$$

The Emden-Fowler equation arises in the study of fluid mechanic, gas dynamics in astrophysics, relativistic mechanics, electrohydrodynamic flow in a cylinder, nuclear physics, thermal explosions, oxygen diffusion in a spherical shell and chemically reacting systems. For more information on various applications of Emden-Fowler type equations, we refer the reader to [30]. In [32], Zou discussed the existence of a solution to the general type of the coupled system (1.1) with homogeneous Dirichlet boundary condition. The nonlinear differential equation (1.1) has a singularity at the initial point $z = 0$. Such equation arises in the study of diverse physical and natural events in engineering and science, such as population evolution, chemical reactions, pattern formation, and so on, see the papers [9,18,32] and references therein. Due to the singularity and the existence of nonlinear functions $f(x,y)$ and $g(x,y)$, in some cases, it may be difficult or impossible in obtaining the exact solution of coupled systems of Emden-Fowler model equations. Therefore, the design of efficient approximation techniques for tackling the singularity at $z = 0$ and nonlinearity is an interesting task.

It is important to mention that extensive studies have been done on the numerical solutions of scalar nonlinear singular differential equations, for example, see [17, 20, 21, 22, 25]. On the other hand, several and various approximation techniques have been developed for solving the coupled system of nonlinear and singular differential equations. For instance, Kumbinarasaiah et al. [14] developed Bernoulli wavelets collocation technique (BWCT) for solving the coupled system of nonlinear Lane-Emden equations. Madduri and Roul [15] used an optimal homotopy analysis method to obtain the series solution of a system of Lane-Emden equations appearing in catalytic diffusion reactions. The authors of [19] and [29] employed Adomian decomposition method for obtaining the series solution of the problem considered in [15]. The applications of the modified form of the standard Adomian decomposition and optimal homotopy analysis methods for obtaining the series solution of the coupled system of Emden-Fowler-type equations considered in [5, 6, 26]. The variational iteration procedure was used by Wazwaz [28] for solving the systems of Emden-Fowler type equations. Moreover, Sabir et al. [24] studied neuro-swarm computational heuristic approach for coupled Emden-Fowler system.

It should be noticed that there are few papers on numerical solutions of the coupled systems of Emden-Fowler equations (1.1). Furthermore, various series solution methods, as stated above, were developed for solving the coupled systems of singular differential equations. However, these methods are not computationally efficient as large number of components in the series solution are needed to obtain the results subjected to a high degree of accuracy. The aim of this paper is to design a robust computational technique to solve a class of coupled systems of Emden-Fowler equations defined by equation (1.1). We consider four test problems to indicate the efficiency and applicability of the proposed algorithm. In order to justify the benefit and advantage of our presented method, the computed outcomes are compared with those reported via the Bernoulli wavelets collocation method (BWCM) and Haar wavelets collocation method (HWCM). The elapsed numerical time (in seconds) for the suggested technique is provided. To these author's best knowledge, for the first time in the literature, we design the proposed computational technique for approximating the solutions of the considered problem. The spectral matrix collocation strategies based on various (orthogonal) bases have been successfully applied and tested to a wide variety of important model problems by the researchers in [1, 2, 11, 12, 13, 23].

This content of this research paper is planned as follows. In the next Section 2, we present some properties and perform the convergent and error analysis of the shifted AFFK. Section 3 is devoted to the construction of our proposed QLM-SAFFK technique for the approximate solution of the underlying model problem under consideration. Simulation experiments are provided and demonstrated in Section 4. Finally, the conclusions and a summary of the work are given in Section 5.

2 The shifted airfoil functions of the first kind: A convergence study

2.1 The shifted version of airfoil functions

The airfoil functions or polynomials have been utilized as expansion functions to calculate the pressure on an airfoil in both unsteady and steady subsonic flow in aerodynamics. Monograph [7] provided a detailed descriptions of properties and formulae related to these functions. Note that this set of polynomials has been addressed as the Chebyshev polynomials of the third kind. They also are a special case of the Jacobi polynomials $P_j^{(\mu,\lambda)}$ with $\mu + \lambda = 0$, see [3, 8].

The airfoil functions of the first kind are defined by

$$\mathbb{A}_s(\chi) = \frac{\cos[(s+1/2)\phi]}{\cos\phi/2}, \quad \chi = \cos\phi, \tag{2.1}$$

for $-1 \leqslant \chi \leqslant 1$ and $s \geqslant 0$ as an integer. We obviously see that $\mathbb{A}_0(\chi) = 1$. It is not difficult to show that $\mathbb{A}_1(\chi) = 2\chi - 1$. A recursion formula will be utilized to get the remaining airfoil functions for $s \geqslant 2$. Thus, we have

$$\mathbb{A}_{s+1}(\chi) = 2\chi\,\mathbb{A}_s(\chi) - \mathbb{A}_{s-1}(\chi), \quad s = 1, 2, \ldots . \tag{2.2}$$

By using (2.2), the next two terms are obtained as

$$\mathbb{A}_2(\chi) = -1 - 2\chi + 4\chi^2, \quad \mathbb{A}_3(\chi) = 8\chi^3 - 4\chi^2 - 4\chi + 1.$$

They also are solutions of the following second-order linear differential equation. In the Sturm-Liouville representation, we have

$$\frac{d}{d\chi}\Big[(1-\chi^2)\,\omega(\chi)\frac{d}{d\chi}\mathbb{A}(\chi)\Big] + s(s+1)\,\omega(\chi)\,\mathbb{A}_s(\chi) = 0, \quad s \in \mathbb{N},$$

where the related weight function is $\omega(\chi) := \sqrt{\frac{\chi+1}{1-\chi}}$. This implies that they are orthogonal with regard to $\omega(\chi)$ on $(-1,1)$ in the sense that we have [7]

$$\int_{-1}^{+1} \mathbb{A}_s(\chi)\,\mathbb{A}_{s'}(\chi)\,\omega(\chi)\,d\chi = \pi\,\delta_{ss'}. \tag{2.3}$$

Here, $\delta_{ss'}$ presents the well-known Kronecker delta function. The airfoil functions of the first kind (AFFK) can be expressed explicitly as

$$\mathbb{A}_s(\chi) = 2^{-s}\sum_{k=0}^{s}(-1)^k\binom{2s+1}{2k}(1-\chi)^k(1+\chi)^{s-k}, \quad s \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}. \tag{2.4}$$

Moreover, the zeros of $\mathbb{A}_s(\chi)$ of degree s are located on the real interval $(-1,1)$. These are given by [7]

$$\chi_i = -\cos\left[2\pi\, i/(2s+1)\right], \quad i = 1,2,\ldots,s. \tag{2.5}$$

In this study, we are aimed to utilize the AFFK on the unit interval $[0,1]$. To this end, we employ the shifted airfoil functions by using the change of variable $\chi = 2z-1$. Next, we define the shifted form of AFFK.

Definition 1. The shifted form of the AFFK on $[0,1]$ of degree s is shown by $\mathbb{A}^{\dagger}_s(z)$ and is defined by

$$\mathbb{A}^{\dagger}_s(z) = \mathbb{A}_s(2z-1), \quad z \in [0,1].$$

Based on the aforementioned transformation and using (2.4), the following explicit representation form is obtained

$$\mathbb{A}^{\dagger}_s(z) = \sum_{k=0}^{s}(-1)^k\binom{2s+1}{2k}(1-z)^k\, z^{s-k}, \quad s \in \mathbb{N}_0. \tag{2.6}$$

A similar orthogonality relation for the shifted AFFK $\{\mathbb{A}^{\dagger}_s(z)\}_{s=0}^{\infty}$ is obtained if one applies the given change of variable to (2.3). An easy calculation indicates that the corresponding weight function is given by $\omega_{\dagger}(z) = \sqrt{\frac{z}{1-z}}$ for $z \in (0,1)$. Therefore, the orthogonality condition reads

$$\int_0^1 \mathbb{A}^{\dagger}_s(z)\,\mathbb{A}^{\dagger}_{s'}(z)\,\omega_{\dagger}(z)\,dz = \frac{\pi}{2}\,\delta_{ss'}. \tag{2.7}$$

Our next goal is to determine the zeros locations of the shifted AFFK. By using the relation (2.5) and Definition 1, the next result is obtained

Lemma 1. *The roots of the shifted airfoil functions* $\mathbb{A}_s^{\dagger}(z)$ *are within* $(0,1)$ *and are given by*

$$z_i = (1+\chi_i)/2, \quad i = 1,2,\ldots,s, \tag{2.8}$$

where χ_i *are given in* (2.5)*. Later, we will utilize these discrete points as the collocation nodes in our algorithm.*

Below, we show the explicit formula of shifted AFFK as a powers of z.

Lemma 2. *The shifted AFFK of degree* s *is expressed for* $s \in \mathbb{N}_0$ *as*

$$\mathbb{A}_s^{\dagger}(z) = \sum_{k=0}^{s} c_{s,k}\, z^k, \quad c_{s,k} := \sum_{i=s-k}^{s} (-1)^{2i-s+k} \binom{2s+1}{2i} \binom{i}{s-k}. \tag{2.9}$$

Proof. In view of (2.6), we first employ the following binomial expansion

$$(1-z)^k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} z^{k-i}.$$

We put this relation into (2.6) to get the result after some manipulations. □

2.2 A rigorous error analysis of the shifted AFFK

We now intend to study the convergence analysis of the shifted AFFK in detail. We note that a given function $\ell(z) \in L_2([0,1])$ can be represented in terms of SAFFK. It follows that we have

$$\ell(z) = \sum_{s=0}^{\infty} \mu_s\, \mathbb{A}_s^{\dagger}(z), \quad z \in [0,1]. \tag{2.10}$$

To get the unknown coefficients μ_s, one uses the orthogonality relation (2.7). Thus, we conclude that

$$\mu_s := \frac{2}{\pi} \int_0^1 \mathbb{A}_s^{\dagger}(z)\, \ell(z)\, \omega_{\dagger}(z)\, dz, \quad s = 1,2,\ldots. \tag{2.11}$$

Below, our fundamental goal is to assert that the infinite series form (2.10) is convergent by using the Weierstrass M test. In this respect, we now derive an estimation for the coefficients μ_s in (2.10).

Theorem 1. *Suppose that* $\ell \in C^{(2)}([0,1]) \cap L^2_{\omega_{\dagger}}([0,1])$ *is represented as* (2.10)*. Then, the coefficients* μ_s *in* (2.11) *are bounded for* $s > 1$ *as follows*

$$|\mu_s| < C_1\, s^{-4}, \quad C_1 = \frac{4}{\pi}\, \|\ell\|_{\infty,2}, \quad \|\ell\|_{\infty,2} := \max_{z\in[0,1]} |\ell''(z)|. \tag{2.12}$$

Proof. We use the change of variable $z = \frac{1}{2}(1+\cos\phi) =: t(\phi)$ in (2.11) to render

$$\mu_s = \frac{2}{\pi} \int_0^{\pi} \ell\,(t(\phi)) \cos[(s+\frac{1}{2})\phi] \cos\frac{\phi}{2} d\phi = \int_0^{\pi} \frac{\ell\,(t(\phi))}{\pi} \Big\{ \cos[s\phi] + \cos[(s+1)\phi] \Big\} d\phi. \tag{2.13}$$

We now apply two times integration by parts on (2.13) to obtain

$$\mu_s = \frac{1}{8\pi}\int_0^\pi \ell''\left(t(\phi)\right)\, \Delta_s(\phi)\, \sin(\phi)\, d\phi,$$

where

$$\Delta_s(\phi) := \frac{1}{s}\left(\frac{\sin((s-1)\phi)}{s-1} - \frac{\sin((s+1)\phi)}{s+1}\right) + \frac{1}{s+1}\left(\frac{\sin(s\phi)}{s} - \frac{\sin((s+2)\phi)}{s+2}\right).$$

We know that $|\sin(\phi)| \leqslant 1$ is valid. Owing to the upper bound of the second-order derivative, one arrives at

$$|\mu_s| \leqslant \frac{\|\ell\|_{\infty,2}}{8\pi}\left|\int_0^\pi \Delta_s(\phi)\, d\phi\right|. \tag{2.14}$$

We now attempt to compute the integral term in (2.14) exactly. By employing the new variables $n = r\phi$, for $r = s \pm 1, s, s+2$ to obtain

$$\int_0^\pi \Delta_s(\phi)\, d\phi = \frac{1-(-1)^{s-1}}{(s-1)^2\, s} + \frac{(-1)^{s+1}-1}{s(s+1)^2} + \frac{1-(-1)^s}{s^2(s+1)} + \frac{(-1)^{s+2}-1}{(s+1)(s+2)^2}.$$

One can readily observe that if $s > 1$ and depending on whether s is odd or even, two terms become zero. Therefore, for $s > 1$ we have

$$\int_0^\pi \Delta_s(\phi)\, d\phi = \begin{cases} \frac{2}{(s-1)^2\, s} - \frac{2}{s(s+1)^2} = \frac{8}{(s-1)^2(s+1)^2}, & \text{if } s \text{ even},\\ \frac{2}{s^2\,(s+1)} - \frac{2}{(s+1)(s+2)^2} = \frac{8}{s^2(s+2)^2}, & \text{if } s \text{ odd}. \end{cases}$$

In either cases, we have the following inequality $\left|\int_0^\pi \Delta_s(\phi)\, d\phi\right| \leqslant \frac{8}{(s-1)^2(s+1)^2}$. Now, by utilizing the simple inequality $s-1 \geqslant \frac{s}{2}$, which is valid for all $s \geqslant 2$ we have

$$\left|\int_0^\pi \Delta_s(\phi)\, d\phi\right| < \frac{32}{s^4}. \tag{2.15}$$

By placing (2.15) into (2.14), we have finished the proof of (2.12). □

In practice, we mainly consider a series with finite terms to approximate $\ell(z)$ in (2.10). If (2.10) truncated up to its first $(S+1)$ terms, we get

$$\ell(z) \approx \ell_S(z) = \sum_{s=0}^{S} \mu_s\, \mathbb{A}_s^\dagger(z). \tag{2.16}$$

Let us define the error between the infinite series defined in (2.10) and its cutted series solution $\ell_S(z)$ in (2.16) as

$$E_S(z) = \ell(z) - \ell_S(z), \tag{2.17}$$

which is referred to as the global error on $[0,1]$. In addition, by $\|h\|_{2,\dagger}$ we show the weighted $L_{2,\dagger}$ norm on $[0,1]$ with regard to $\omega_\dagger(z)$. Estimating the global error $E_S(z)$ is our next aim in both L_2 and infinity norms. Firstly, in the weighted $L_{2,\dagger}([0,1])$ norm we have

Theorem 2. *Assume that the hypotheses of Theorem 1 be fulfilled. An estimate for the upper bound of $E_S(z)$ in the $L_{2,\dagger}([0,1])$ norm satisfies*

$$\|E_S\|_{2,\dagger} < C_3 \frac{1}{\sqrt{S^7}}, \quad C_3 := \sqrt{\frac{\pi}{14}} C_1 = \sqrt{\frac{8}{7\pi}} \|\ell\|_{\infty,2}.$$

Proof. According to relations (2.10) and (2.16) we have

$$\|E_S\|_{2,\dagger}^2 = \left\| \sum_{s=0}^{\infty} \mu_s \, \mathbb{A}_s^\dagger(z) - \sum_{s=0}^{S} \mu_s \, \mathbb{A}_s^\dagger(z) \right\|_{2,\dagger}^2 = \left\| \sum_{s=S+1}^{\infty} \mu_s \, \mathbb{A}_s^\dagger(z) \right\|_{2,\dagger}^2 .$$

The orthogonality condition (2.7) follows that $\|E_S\|_{2,\dagger}^2 = \frac{\pi}{2} \sum_{s=S+1}^{\infty} \mu_s^2$. The next job is to utilize the obtained upper bound (2.12) derived in Theorem 2 to the former equality. This implies that

$$\|E_S\|_{2,\dagger}^2 \leqslant \frac{\pi}{2} C_1^2 \sum_{s=S+1}^{\infty} \frac{1}{s^8}. \tag{2.18}$$

By utilizing the Integral Test from calculus we consequently get [27]

$$\sum_{s=S+1}^{\infty} \frac{1}{s^8} \leqslant \int_S^{\infty} \frac{dw}{w^8} = \frac{1}{7S^7}.$$

In order to deduce the claimed estimate, the former inequality will be placed into (2.18). By doing the square-root operation, the proof is accomplished. □

The next result is given to pave the way for the derivation of an upper bound for (2.17) in the L_∞ norm.

Lemma 3. *For all $s \geqslant 0$, the shifted AFFK satisfying*

$$|\mathbb{A}_s^\dagger(z)| \leqslant 1 + 2s, \quad \forall z \in [0,1]. \tag{2.19}$$

Proof. We first recall that airfoil functions of the first kind (2.1) satisfy [7]

$$\mathbb{A}_s(\chi) = U_s(\chi) - U_{s-1}(\chi), \quad \chi \in [-1,1].$$

Here, by $U_s(\chi)$ we denote the Chebyshev functions of second kind. The following relation is valid for this class of classical polynomials [16]

$$|U_s(\chi)| \leqslant s+1, \quad \forall \, |\chi| \leqslant 1.$$

Based upon the change of variable $\chi = 2z-1$ followed by employing the triangle inequality, we render

$$|\mathbb{A}_s^\dagger(z)| \leqslant |U_s(2z-1)| + |U_{s-1}(2z-1)| \leqslant s+1+s = 2s+1, \quad \forall z \in [0,1].$$

Now, the proof is completed. □

Theorem 3. *Let assume that the hypotheses of Theorem 1 be satisfied. Then, an estimate for the upper bound of error* $E_S(z) = \sum_{s=S+1}^{\infty} \mu_s \mathbb{A}_s^{\dagger}(z)$ *in the* $L_\infty([0,1])$ *given by*

$$\|E_S\|_\infty < C_4 \frac{1}{S^2}, \quad C_4 := \frac{3}{2} C_1 = \frac{6}{\pi} \|\ell\|_{\infty,2}.$$

Proof. We continue by utilizing the inequality (2.19) given in Lemma (3) to arrive at

$$|E_S(z)| \leqslant \sum_{s=S+1}^{\infty} |\mu_s| \, |\mathbb{A}_s^{\dagger}(z)| \leqslant \sum_{s=S+1}^{\infty} (2s+1)|\mu_s| \leqslant \sum_{s=S+1}^{\infty} 3s|\mu_s|.$$

Owing to the inequality (2.12) in Theorem 2 we have $|E_S(z)| < 3C_1 \sum_{s=S+1}^{\infty} \frac{1}{s^3}$. In the foregoing inequality, we use the Integral Test [27] to obtain

$$\sum_{s=S+1}^{\infty} \frac{1}{s^3} \leqslant \int_S^{\infty} \frac{dw}{w^3} = \frac{1}{2S^2}.$$

By taking the supermom over all values of $z \in [0,1]$, the desired result is immediately deduced. □

3 The QLM-SAFFK matrix technique

3.1 QLM: The main idea

The methodology of QLM enables us to get rid of the intrinsic nonlinearity of the given model problem. Through various research studies have been proved that this technique is very effective in several areas of investigations and applications, see cf. [4, 10, 31]. As a starting point for discussion, the nonlinear coupled system (1.1) is reformulated as

$$\ddot{\boldsymbol{Z}}(z) = \boldsymbol{E}(z, \boldsymbol{Z}(z), \dot{\boldsymbol{Z}}(z)), \tag{3.1}$$

where

$$\boldsymbol{Z}(z) = \begin{bmatrix} x(z) \\ y(z) \end{bmatrix}, \dot{\boldsymbol{Z}}(z) = \begin{bmatrix} x'(z) \\ y'(z) \end{bmatrix}, \boldsymbol{E}(z, \boldsymbol{Z}(z), \dot{\boldsymbol{Z}}(z)) = \begin{bmatrix} h(z) - f(x,y) - \alpha\, x'(z)/z \\ k(z) - g(x,y) - \beta\, y'(z)/z \end{bmatrix}.$$

If $\boldsymbol{Z}_0(z)$ be a rough first approximation to $\boldsymbol{Z}(z)$, the QLM for (3.1) can be expressed for $n = 0, 1, \ldots$ as

$$\begin{aligned} \ddot{\boldsymbol{Z}}_{n+1}(z) \approx & \boldsymbol{E}(z, \boldsymbol{Z}_n(z), \dot{\boldsymbol{Z}}_n(z)) + \boldsymbol{E}_Z(z, \boldsymbol{Z}_n(z), \dot{\boldsymbol{Z}}_n(z)) \Big(\boldsymbol{Z}_{n+1}(z) - \boldsymbol{Z}_n(z) \Big) \\ & + \boldsymbol{E}_{\dot{\boldsymbol{Z}}}(z, \boldsymbol{Z}_n(z), \dot{\boldsymbol{Z}}_n(z)) \Big(\dot{\boldsymbol{Z}}_{n+1}(z) - \dot{\boldsymbol{Z}}_n(z) \Big). \end{aligned}$$

Note, $\boldsymbol{E}_Z = \frac{d}{d\boldsymbol{Z}} \boldsymbol{E}$ and $\boldsymbol{E}_{\dot{\boldsymbol{Z}}} = \frac{d}{d\dot{\boldsymbol{Z}}} \boldsymbol{E}$ are the corresponding Jacobian matrices. The same initial conditions as (1.2) are also supplemented with the former equations. We now apply the aforesaid QLM to the rewritten model (3.1).

After doing some simple calculations and manipulations, one gets the following set of linear system of equations

$$\ddot{\boldsymbol{Z}}_{n+1}(z)+\boldsymbol{m}_{1,n}(z)\,\dot{\boldsymbol{Z}}_{n+1}(z)+\boldsymbol{m}_{0,n}(z)\,\boldsymbol{Z}_{n+1}(z)=\boldsymbol{r}_n(z),\qquad n=0,1,\dots, \tag{3.2}$$

where $\boldsymbol{Z}_{n+1}(z)=\begin{bmatrix} x_{n+1}(z)\\ y_{n+1}(z)\end{bmatrix}$, $\boldsymbol{m}_{1,n}(z)=\begin{bmatrix} \frac{\alpha}{z} & 0\\ 0 & \frac{\beta}{z}\end{bmatrix}$,

$$\boldsymbol{m}_{0,n}(z)=\begin{bmatrix} f_x(x_n(z),y_n(z)) & f_y(x_n(z),y_n(z))\\ g_x(x_n(z),y_n(z)) & g_y(x_n(z),y_n(z))\end{bmatrix},$$
$$\boldsymbol{r}_n(z)=\begin{bmatrix} h(z)-f(x_n(z),y_n(z))+x_n(z)f_x(x_n(z),y_n(z))+y_n(z)f_y(x_n(z),y_n(z))\\ k(z)-g(x_n(z),y_n(z))+x_n(z)g_x(x_n(z),y_n(z))+y_n(z)g_y(x_n(z),y_n(z))\end{bmatrix}.$$

By virtue of (1.2), the initial conditions are given as

$$\boldsymbol{Z}_{n+1}(0)=\begin{bmatrix} x_{n+1}(0)\\ y_{n+1}(0)\end{bmatrix}=\begin{bmatrix} x_0\\ y_0\end{bmatrix},\quad \dot{\boldsymbol{Z}}_{n+1}(0)=\begin{bmatrix} x'_{n+1}(0)\\ y'_{n+1}(0)\end{bmatrix}=\begin{bmatrix} x_1\\ y_1\end{bmatrix}. \tag{3.3}$$

Below, we devise a matrix collocation approach based on the SAFFK to solve the aforementioned systems (3.2)–(3.3).

3.2 The QLM-SAFFK method

We will approximate the solution of linearized model (3.2) in the form of the truncated series solutions (2.16) consisting of $(S+1)$-terms. Let assume that the approximate solutions $\mathcal{X}_S^{(n)}(z)$ and $\mathcal{Y}_S^{(n)}(z)$ to $x_n(z)$ and $y_n(z)$ are already obtained in the iteration number n for $n=0,1,\dots$. Note that for $n=0$, we will use the given function $\boldsymbol{Z}_0(z)$ as the initial (rough) guess. The next step is to consider the forms of approximations in the subsequent iteration $n+1$. Thus, the approximate solutions are presumed as

$$x_{n+1}(z)\approx\mathcal{X}_S^{(n+1)}(z)=\sum_{s=0}^{S}\mu_{s,1}^{(n)}\,\mathbb{A}_s^{\dagger}(z),\quad y_{n+1}(z)\approx\mathcal{Y}_S^{(n+1)}(z)=\sum_{s=0}^{S}\mu_{s,2}^{(n)}\,\mathbb{A}_s^{\dagger}(z), \tag{3.4}$$

for $z\in[0,1]$. Thus, our goal will reduce to find the unknown coefficients $\{\mu_{s,j}^{(n)}\}_{s=0}^{S}$ for $n=1,2,\dots$ and $j=1,2$ by developing a spectral collocation strategy relying on SAFFK. To continue, we mention the following result. A proof of which is a simple task.

Lemma 4. *The truncated expansion solutions in* (3.4) *for* $j=1,2$ *can be expressed as*

$$\sum_{s=0}^{S}\mu_{s,j}^{(n)}\,\mathbb{A}_s^{\dagger}(z)=\boldsymbol{A}_S(z)\,\boldsymbol{F}_{S,j}^{(n)}, \tag{3.5}$$

where $\boldsymbol{F}_{S,j}^{(n)}=\begin{bmatrix}\mu_{0,j}^{(n)} & \mu_{1,j}^{(n)} & \dots & \mu_{S,j}^{(n)}\end{bmatrix}^T$ *represents the unknown vectors and* $\boldsymbol{A}_S(z)=\begin{bmatrix}\mathbb{A}_0^{\dagger}(z) & \mathbb{A}_1^{\dagger}(z) & \dots & \mathbb{A}_S^{\dagger}(z)\end{bmatrix}$ *is the vector of SAFFK.*

We may further decompose the vector $\boldsymbol{A}_S(z)$ as shown in the next Lemma

Lemma 5. *The vector of SAFFK can be decomposed as*

$$\boldsymbol{A}_S(z) = \boldsymbol{T}_S(z)\,\boldsymbol{C}_S, \tag{3.6}$$

where $\boldsymbol{T}_S(z) = \begin{bmatrix}1 & z & z^2 & \dots & z^S\end{bmatrix}$ *and* $\boldsymbol{C}_S = (c_{s,k})_{s,k=0}^{S}$ *denotes a matrix with upper-triangular structure and its elements are* $c_{s,k}$ *given in* (2.9). *Note that* $c_{s,k} = 0$ *for* $s > k$. *It can be easily seen that the matrix* $\boldsymbol{C}_S$ *is non-singular.*

Proof. The proof is easily concluded on account of relation (2.9). To do so, we perform left-hand multiplication of the matrix $\boldsymbol{C}_S$ by $\boldsymbol{T}_S$. □

We now combine the outcomes of two forgoing Lemmas (i.e., relations (3.5) and (3.6)) to get

$$\begin{cases} \mathcal{X}_S^{(n+1)}(z) = \boldsymbol{A}_S(z)\,\boldsymbol{F}_{S,1}^{(n)} = \boldsymbol{T}_S(z)\,\boldsymbol{C}_S\,\boldsymbol{F}_{S,1}^{(n)}, \\ \mathcal{Y}_S^{(n+1)}(z) = \boldsymbol{A}_S(z)\,\boldsymbol{F}_{S,2}^{(n)} = \boldsymbol{T}_S(z)\,\boldsymbol{C}_S\,\boldsymbol{F}_{S,2}^{(n)}, \end{cases} \quad z \in [0,1]. \tag{3.7}$$

By a straightforward calculation we can show that

$$\frac{d}{dz}\boldsymbol{T}_S(z) = \boldsymbol{T}_S(z)\,\boldsymbol{D}_S. \tag{3.8}$$

Here, the matrix $\boldsymbol{D}_S = (d_{i,j})_{i,j}^{S}$ is a zero matrix except for entries $d_{i,i+1} = i+1$ for $i = 0,1,\dots,S-1$. From this matrix the first-order derivatives of truncated series forms in (3.7) are represented as

$$\begin{cases} \frac{d}{dz}\mathcal{X}_S^{(n+1)}(z) = \boldsymbol{T}_S(z)\,\boldsymbol{D}_S\,\boldsymbol{C}_S\,\boldsymbol{F}_{S,1}^{(n)}, \\ \frac{d}{dz}\mathcal{Y}_S^{(n+1)}(z) = \boldsymbol{T}_S(z)\,\boldsymbol{D}_S\,\boldsymbol{C}_S\,\boldsymbol{F}_{S,2}^{(n)}, \end{cases} \quad z \in [0,1]. \tag{3.9}$$

Similarly, we can approximate the second-order derivatives of approximate solutions by combining the relation (3.8) and (3.9) to gain

$$\begin{cases} \frac{d^2}{dz^2}\mathcal{X}_S^{(n+1)}(z) = \boldsymbol{T}_S(z)\,\boldsymbol{D}_S^2\,\boldsymbol{C}_S\,\boldsymbol{F}_{S,1}^{(n)}, \\ \frac{d^2}{dz^2}\mathcal{Y}_S^{(n+1)}(z) = \boldsymbol{T}_S(z)\,\boldsymbol{D}_S^2\,\boldsymbol{C}_S\,\boldsymbol{F}_{S,2}^{(n)}, \end{cases} \quad z \in [0,1]. \tag{3.10}$$

Let us back to the matrix differential equation (3.2). The vector $\boldsymbol{Z}_{n+1}(z)$ and its derivatives $\frac{d^l}{dz^l}\boldsymbol{Z}_{n+1}(z)$ can be approximated by $\boldsymbol{Z}_S^{(n+1)}(z)$ and $\frac{d^l}{dz^l}\boldsymbol{Z}_S^{(n+1)}(z)$ respectively for $l = 1,2$ as

$$\boldsymbol{Z}_S^{(n+1)}(z) := \begin{bmatrix} \mathcal{X}_S^{(n+1)}(z) \\ \mathcal{Y}_Q^{(n+1)}(z) \end{bmatrix}, \quad \frac{d^l}{dz^l}\boldsymbol{Z}_S^{(n+1)}(z) := \begin{bmatrix} \frac{d^l}{dz^l}\mathcal{X}_S^{(n+1)}(z) \\ \frac{d^l}{dz^l}\mathcal{Y}_S^{(n+1)}(z) \end{bmatrix}. \tag{3.11}$$

Lemma 6. *In the matrix formats, the approximated solutions* $\frac{d^l}{dz^l}\boldsymbol{Z}_S^{(n+1)}(z)$, $l = 0,1,2$ *in* (3.11) *can be written as*

$$\boldsymbol{Z}_S^{(n+1)}(z) = \bar{\boldsymbol{T}}(z)\,\bar{\boldsymbol{C}}\,\bar{\boldsymbol{F}}^n, \quad \frac{d^l}{dz^l}\boldsymbol{Z}_S^{(n+1)}(z) = \bar{\boldsymbol{T}}(z)\,\bar{\boldsymbol{D}}_l\,\bar{\boldsymbol{C}}\,\bar{\boldsymbol{F}}^n, \quad l = 1,2, \tag{3.12}$$

where we have $\bar{\boldsymbol{F}}^n = \begin{bmatrix}\boldsymbol{F}_{S,1}^{(n)} & \boldsymbol{F}_{S,2}^{(n)}\end{bmatrix}^T$ *and*

$$\bar{\boldsymbol{T}}(z) = \begin{bmatrix} \boldsymbol{T}_S(z) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{T}_S(z) \end{bmatrix}, \quad \bar{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C}_S & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_S \end{bmatrix}, \quad \bar{\boldsymbol{D}}_l = \begin{bmatrix} (\boldsymbol{D}_S)^l & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{D}_S)^l \end{bmatrix}.$$

Proof. The desired results can be simply obtainable by just putting relations (3.7), (3.9) and (3.10) into the related vector form in (3.11). □

We now employ the zeros of SAFFK given in (2.8) as the collocation nodes. To this end, the zeros of $\mathbb{A}^{\dagger}_{S+1}(z)$ on $[0,1]$ are labeled as $z_0, z_1, \ldots, z_S$. Now, we insert these zeros into the linear matrix differential equation (3.2) to get

$$\ddot{\boldsymbol{Z}}_{n+1}(z_j) + \boldsymbol{m}_{1,n}(z_j)\,\dot{\boldsymbol{Z}}_{n+1}(z_j) + \boldsymbol{m}_{0,n}(z_j)\,\boldsymbol{Z}_{n+1}(z_j) = \boldsymbol{r}_n(z_j),\ j = 0,1,\ldots,S, \tag{3.13}$$

for $n = 0, 1, \ldots$. Next task is to define the vectors and matrices as

$$\ddot{\boldsymbol{W}}_n = \begin{bmatrix} \ddot{\boldsymbol{Z}}_{n+1}(z_0) \\ \ddot{\boldsymbol{Z}}_{n+1}(z_1) \\ \vdots \\ \ddot{\boldsymbol{Z}}_{n+1}(z_S) \end{bmatrix}, \quad \dot{\boldsymbol{W}}_n = \begin{bmatrix} \dot{\boldsymbol{Z}}_{n+1}(z_0) \\ \dot{\boldsymbol{Z}}_{n+1}(z_1) \\ \vdots \\ \dot{\boldsymbol{Z}}_{n+1}(z_S) \end{bmatrix}, \quad \boldsymbol{W}_n = \begin{bmatrix} \boldsymbol{Z}_{n+1}(z_0) \\ \boldsymbol{Z}_{n+1}(z_1) \\ \vdots \\ \boldsymbol{Z}_{n+1}(z_S) \end{bmatrix},$$

$$\boldsymbol{R}_n = \begin{bmatrix} \boldsymbol{r}_n(z_0) \\ \boldsymbol{r}_n(z_1) \\ \vdots \\ \boldsymbol{r}_n(z_S) \end{bmatrix}, \quad \boldsymbol{M}_{i,n} = \begin{bmatrix} \boldsymbol{m}_{i,n}(z_0) & \boldsymbol{0} & \ldots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m}_{i,n}(z_1) & \ldots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{m}_{i,n}(z_S) \end{bmatrix}, \quad i = 0, 1.$$

Therefore, the vector representation of relations (3.13) can be written in a compact format as

$$\ddot{\boldsymbol{W}}_n + \boldsymbol{M}_{1,n}\,\dot{\boldsymbol{W}}_n + \boldsymbol{M}_{0,n}\,\boldsymbol{W}_n = \boldsymbol{R}_n, \quad n = 0, 1, \ldots. \tag{3.14}$$

The next Lemma provides the matrix forms of $\boldsymbol{W}_n$, $\dot{\boldsymbol{W}}_n$, and $\ddot{\boldsymbol{W}}_n$. It is suffices to substitute the SAFFK nodes into relations (3.12).

Lemma 7. *The matrix forms of relations* (3.12) *at the collocation points* (2.8) *are given by*

$$\boldsymbol{W}_n = \bar{\bar{\boldsymbol{T}}}\,\bar{\boldsymbol{C}}\,\bar{\boldsymbol{F}}^n, \qquad \dot{\boldsymbol{W}}_n = \bar{\bar{\boldsymbol{T}}}\,\bar{\boldsymbol{D}}_1\,\bar{\boldsymbol{C}}\,\bar{\boldsymbol{F}}^n, \quad \ddot{\boldsymbol{W}}_n = \bar{\bar{\boldsymbol{T}}}\,\bar{\boldsymbol{D}}_2\,\bar{\boldsymbol{C}}\,\bar{\boldsymbol{F}}^n. \tag{3.15}$$

Here, the matrix $\bar{\bar{\boldsymbol{T}}}$ *is defined by* $\bar{\bar{\boldsymbol{T}}} = [\bar{\boldsymbol{T}}(z_0)\bar{\boldsymbol{T}}(z_1)\ldots\bar{\boldsymbol{T}}(z_S)]^T$. *Moreover, the matrices* $\bar{\boldsymbol{T}}, \bar{\boldsymbol{C}}, \bar{\boldsymbol{D}}_l$, $l = 1, 2$ *the vector* $\bar{\boldsymbol{F}}^n$ *are already defined in* (3.12).

Now, by inserting the former relations (3.15) into (3.14) we arrive at the fundamental matrix equation (FME). It follows that

$$\boldsymbol{A}_n\,\bar{\boldsymbol{F}}^n = \boldsymbol{R}_n, \ \text{ or } \ [\boldsymbol{A}_n; \boldsymbol{R}_n], \ \ n = 0, 1, \ldots, \tag{3.16}$$

where $\boldsymbol{A}_n := \left\{\bar{\bar{\boldsymbol{T}}}\,\bar{\boldsymbol{D}}_2 + \boldsymbol{M}_{1,n}\,\bar{\bar{\boldsymbol{T}}}\,\bar{\boldsymbol{D}}_1 + \boldsymbol{M}_{0,n}\,\bar{\bar{\boldsymbol{T}}}\right\}\bar{\boldsymbol{C}}$.

One should note that the foregoing FME (3.16) is a (linear) algebraic system comprises $2(S+1)$ unknowns $\mu^{(n)}_{s,j}$ for $s = 0, 1, \ldots, S$ and $j = 1, 2$ to be determined as the SAFFK coefficients. However, this system is not yet completed due to missing of initial conditions (1.2).

We are aimed to enter the given initial conditions (3.3) into the FME (3.16). To begin, let us pay attention to the matrix forms (3.12) for the approximation $\boldsymbol{Z}_S^{(n+1)}(z)$. In this respect, we let $z \to 0$ to arrive at

$$\widehat{\boldsymbol{A}_0}\,\bar{\boldsymbol{F}}^n = \widehat{\boldsymbol{R}}_0, \quad \widehat{\boldsymbol{A}_0} := \bar{\boldsymbol{T}}(0)\,\bar{\boldsymbol{C}}, \quad \widehat{\boldsymbol{R}}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad \text{or} \quad \left[\widehat{\boldsymbol{A}_0}; \widehat{\boldsymbol{R}}_0\right].$$

For the second initial condition in (3.3), we consider again (3.12) for $l = 1$ followed by tending $z \to 0$ to obtain the following matrix representation

$$\widehat{\boldsymbol{A}_1}\,\bar{\boldsymbol{F}}^n = \widehat{\boldsymbol{R}}_1, \quad \widehat{\boldsymbol{A}_1} := \bar{\boldsymbol{T}}(0)\,\bar{\boldsymbol{D}}_1\,\bar{\boldsymbol{C}}, \quad \widehat{\boldsymbol{R}}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{or} \quad \left[\widehat{\boldsymbol{A}_1}; \widehat{\boldsymbol{R}}_1\right].$$

Here, the constants x_0, y_0, and x_1, y_1 are known in (1.2). The replacement of four rows of the matrix $[\boldsymbol{A}_n; \boldsymbol{R}_n]$ in (3.16) will be performed next by the row matrices $\left[\widehat{\boldsymbol{A}_0}; \widehat{\boldsymbol{R}}_0\right]$ and $\left[\widehat{\boldsymbol{A}_1}; \widehat{\boldsymbol{R}}_1\right]$. Let us denote the modified FME by

$$\widehat{\boldsymbol{A}_n}\,\bar{\boldsymbol{F}}^n = \widehat{\boldsymbol{R}_n} \quad \text{or} \quad \left[\widehat{\boldsymbol{A}_n}; \widehat{\boldsymbol{R}_n}\right]. \tag{3.17}$$

Thus, after solving the modified FME (3.17), the unknown SAFFK coefficients will be gotten. Any linear solver or algorithm can be taken into consideration to get the solution of this system. Once the vector $\bar{\boldsymbol{F}}^n$ is determined, all unknowns $\mu_{s,j}^{(n)}$, for $j = 1, 2$, and $s = 0, 1, \ldots, S$ as the coefficients in the expansion series (3.4) are determined in the iteration n. Therefore, an approximation for the solution of model (1.1) will be at hand.

4 Computational results

To carry out experimental computations, our platform of choice is Matlab software version 2021a on a personal laptop with the following capabilities: 16 GB RAM, 1 TB memory, and CPU Intel Core-i7-10870H.

Let $n = 5$ is taken as the quasilinearization parameter in the experimental results. During the running of QLM-SAFFK algorithm, we set $\boldsymbol{Z}_0(z)$ as the zero functions or we take it as the initial conditions (1.2). The errors are also defined as follows

$$\mathcal{E}_{x,S}^{(n)}(z) := |x(z) - \mathcal{X}_S^{(n)}(z)|, \quad \mathcal{E}_{y,S}^{(n)}(z) := |y(z) - \mathcal{Y}_S^{(n)}(z)|, \quad z \in [0,1], \tag{4.1}$$

in the iteration $n = 1, 2, \ldots$. We also calculate the norms in the L_∞ norm for a fixed value of n in accordance to the relations

$$L_\infty^x \equiv L_\infty^x(S) := \max_{z\in[0,1]} \mathcal{E}_{x,S}^{(n)}(z), \quad L_\infty^y \equiv L_\infty^y(S) := \max_{z\in[0,1]} \mathcal{E}_{y,S}^{(n)}(z).$$

In what follows, we further compute the numerical order of convergence (NOC) as

$$\text{NOC}_\infty^x := \log_2\left(\frac{L_\infty^x(S)}{L_\infty^x(2S)}\right), \quad \text{NOC}_\infty^y := \log_2\left(\frac{L_\infty^y(S)}{L_\infty^y(2S)}\right). \tag{4.2}$$

Problem 1. Let us set $\alpha = 1$ and $\beta = 2$ in the first test example. Here, the following coupled system will be studied [5, 6, 29]

$$\begin{cases} x''(z) + \frac{1}{z}x'(z) + x^2(z)\,y(z) - (4z^2+5)x(z) = 0, \\ y''(z) + \frac{2}{z}y'(z) + x(z)\,y^2(z) - (4z^2-5)y(z) = 0, \end{cases} \qquad \begin{cases} x(0) = y(0) = 1, \\ x'(0) = y'(0) = 0. \end{cases}$$

One can easily verify that the exact solutions are $(x(z), y(z)) = (e^{z^2}, e^{-z^2})$.

We first take $S = 8$ in the computations. The outputs of our QLM-SAFFK technique are the following approximations

$$\mathcal{X}_8^{(5)}(z) = 0.4262447\,z^8 - 1.455108\,z^7 + 2.651745\,z^6 - 2.398299\,z^5 + 1.887582\,z^4 - 0.46969036\,z^3 + 1.0807017\,z^2 - 8.5159197\times 10^{-108}\,z + 1.0,$$

$$\mathcal{Y}_8^{(5)}(z) = -0.02678217\,z^8 + 0.177758\,z^7 - 0.3979087\,z^6 + 0.180474\,z^5 + 0.4130984\,z^4 + 0.02458554\,z^3 - 1.0035135\,z^2 + 7.98\times 10^{-110}\,z + 1.0.$$

The former approximate solutions together with related exact solutions are visualized in Figure 1. As one can see that the approximate solutions obtained via QLM-SAFFK are in good alignment with the associated exact solutions. Precisely, we show the graphical plots of the achieved absolute errors defined in (4.1). These errors using $S = 8$ are seen in Figure 2. Note that beside $S = 8$, the plots of absolute errors with $S = 16$ and $S = 24$ are further visualized in Figure 2. It can be readily inferred that by increasing the number of bases we get the desired level of accuracy in our proposed approach.

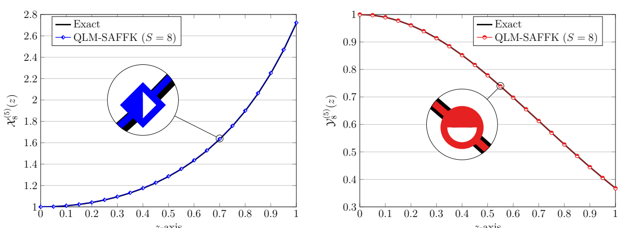


Figure 1. Graphics of approximate solutions for $x(z)$ (a) and $y(z)$ (b) using QLM-SAFFK technique in Problem 1 with $S = 8$, $n = 5$.

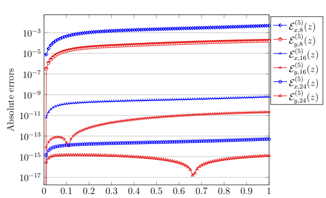


Figure 2. Comparing of absolute errors using QLM-SAFFK procedure for Problem 1 with $S = 8, 16, 24$ and $n = 5$.

A thorough comparison of numerical outcomes with the corresponding analytical true solutions is shown in Table 1. Here, we utilize $S = 15$. Obviously, eight to ten-digit agreement is found between our presented results and the exact true ones.

Table 1. Numerical results and absolute errors for $x(z), y(z)$ obtained via QLM-SAFFK procedure using $S = 15$, $n = 5$ in Problem 1.

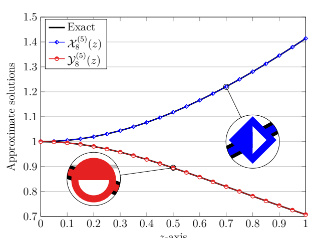
z	$\mathcal{X}_{15}^{(1)}(z)$	$\mathcal{E}_{x,15}^{(1)}(z)$	Exact	$\mathcal{Y}_{15}^{(1)}(z)$	$\mathcal{E}_{y,15}^{(1)}(z)$	Exact
0.2	1.040810773	1.68_{-9}	1.04081077419239	0.9607894392	3.99_{-11}	0.96078943915232
0.4	1.173510869	2.37_{-9}	1.17351087099181	0.8521437890	6.79_{-11}	0.85214378896621
0.6	1.433329411	3.07_{-9}	1.43332941456034	0.6976763262	1.10_{-10}	0.69767632607103
0.8	1.896480875	4.06_{-9}	1.89648087930495	0.5272924242	1.57_{-10}	0.52729242404305
1.0	2.718281823	5.69_{-9}	2.71828182845904	0.3678794414	1.99_{-10}	0.36787944117144

Problem 2. We investigate the following nonlinear singular system [5, 6, 28]

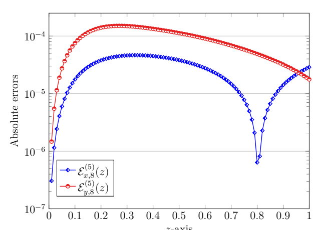
$$\begin{cases} x''(z) + \frac{1}{z}x'(z) + y^3(z)\big(x^2(z)+1\big) = 0, \\ y''(z) + \frac{3}{z}y'(z) + y^5(z)\big(x^2(z)+3\big) = 0, \end{cases} \qquad \begin{cases} x(0) = y(0) = 1, \\ x'(0) = y'(0) = 0. \end{cases}$$

The exact solutions are given by $(x(z), y(z)) = (\sqrt{1+z^2}, 1/\sqrt{1+z^2})$.

Let us first set $S = 8$ for this example. We visualize the approximate solutions $\mathcal{X}_8^{(5)}(z)$ and $\mathcal{Y}_8^{(5)}(z)$ using the QLN-SAFFK collocation approach as shown in Figure 3. Moreover, the graphical representations of the achieved absolute errors are visualized in Figure 3. Indeed, these approximations are obtained as



(a)



(b)

Figure 3. Graphics of approximate solutions for $x(z)$ (a) and $y(z)$ (b) using QLM-SAFFK technique in Problem 2 with $S = 8$, $n = 5$.

follows

$$\mathcal{X}_8^{(5)}(z) = 0.0052868077\,z^8 - 0.022458304\,z^7 + 0.01615031\,z^6 + 0.077130627\,z^5 - 0.17803247\,z^4 + 0.019410543\,z^3 + 0.49675486\,z^2 + 1.70\times 10^{-108}\,z + 1.0,$$

$$\mathcal{Y}_8^{(5)}(z) = -0.008682165\,z^8 - 0.005352826\,z^7 + 0.20901949\,z^6 - 0.63166476\,z^5 + 0.74189028\,z^4 - 0.11393863\,z^3 - 0.48414691\,z^2 + 8.30\times 10^{-110}\,z + 1.0.$$

We now check the related NOC via relations (4.2) for this example. Theoretically in Theorem 3 we showed that this order of convergence is 2 with respect to number of bases S. However, as reported in Table 2, the corresponding NOC are shown to behaved exponentially as we increase S. In this table, we used $S = 2^k$, $k = 0, 1, 2, 4, 5$. The results of L_∞ related to both approximated solutions are also presented in Table 2. Furthermore, we record the execution time of our algorithm to solve the modified augmented linear system (3.17). Thus, the required CPU times given in seconds are shown in the last column. Note that this time is needed to obtain both approximate solutions simultaneously.

Table 2. The outcomes of errors in L_∞ norms, the associated NOC, and the elapsed CPU times for Problem 2 with diverse S.

S	L^x_∞	NOC^x_∞	L^y_∞	NOC^y_∞	CPU(s)
1	4.1421×10^{-01}	–	2.9289×10^{-01}	–	0.70016
2	4.5104×10^{-02}	3.1990	8.7107×10^{-02}	1.7495	0.92313
4	8.5101×10^{-03}	2.4060	5.0466×10^{-03}	4.1094	1.35134
8	4.6489×10^{-05}	7.5161	1.5047×10^{-04}	5.0678	2.81615
16	1.0898×10^{-09}	15.381	3.6853×10^{-09}	15.317	7.22782
32	1.8502×10^{-15}	19.168	4.3897×10^{-16}	23.001	19.9206

Some comparisons are performed between the results of our algorithm and the outcomes of the Bernoulli wavelets collocation method (BWCM) with $M = 10$ as well as with those obtained via Haar wavelets collocation method (HWCM) with $J = 4$. Both methods are developed in [14]. The results are reported in Table 3. Although it seems that the results of BWCM are more accurate, but in overall our method performed better than BWCM. The reason is that the maximum absolute errors in the BWCM is 3.83×10^{-2} (at $z = 0.8$) while in the QLM-SAFFK is 3.37547×10^{-9} for the first solution $x(z)$.

Table 3. Numerical results and absolute errors for $x(z), y(z)$ obtained via QLM-SAFFK procedure using $S = 15$, $n = 5$ in Problem 2.

z	$\mathcal{X}^{(1)}_{15}(z)$	$\mathcal{E}^{(1)}_{x,15}(z)$	BWCM	HWCM	$\mathcal{Y}^{(1)}_{15}(z)$	$\mathcal{E}^{(1)}_{y,15}(z)$	BWCM	HWCM
0.1	1.004987564	1.68_{-9}	5.60_{-9}	1.19_{-6}	0.995037187	3.65_{-09}	3.04_{-9}	2.43_{-6}
0.2	1.019803906	2.37_{-9}	5.79_{-9}	3.67_{-6}	0.980580672	3.64_{-09}	3.49_{-9}	3.63_{-5}
0.3	1.044030654	3.07_{-9}	3.61_{-9}	7.28_{-6}	0.957826282	3.30_{-09}	3.57_{-9}	6.50_{-7}
0.4	1.077032965	4.06_{-9}	4.31_{-9}	1.16_{-5}	0.928476688	2.85_{-09}	3.53_{-9}	2.96_{-6}
0.5	1.118033992	1.68_{-9}	2.39_{-9}	1.18_{-5}	0.894427189	2.37_{-09}	3.31_{-9}	3.98_{-6}
0.6	1.166190382	2.37_{-9}	3.12_{-9}	2.09_{-5}	0.857492924	1.90_{-09}	3.01_{-9}	3.78_{-7}
0.7	1.220655565	3.07_{-9}	1.42_{-9}	2.69_{-5}	0.819231919	1.46_{-09}	2.69_{-9}	5.56_{-6}
0.8	1.280624851	4.06_{-9}	3.83_{-2}	3.38_{-2}	0.780868808	1.08_{-09}	2.48_{-9}	1.48_{-5}
0.9	1.345362408	4.06_{-9}	3.44_{-9}	3.21_{-5}	0.743294145	7.65_{-10}	1.78_{-9}	1.24_{-5}
1.0	1.414213565	5.69_{-9}	2.34_{-9}	3.27_{-5}	0.707106781	5.04_{-10}	1.68_{-9}	2.35_{-5}

Problem 3. The following nonlinear singular coupled system is considered [5, 26, 28, 29]

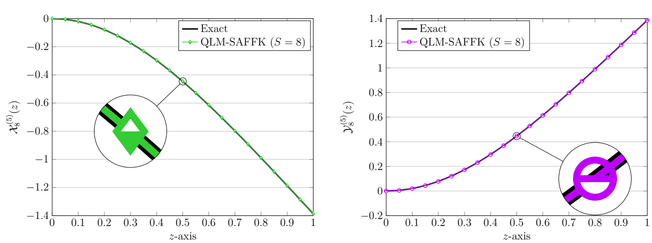
$$\begin{cases} x''(z) + \frac{5}{z}x'(z) + 8\big(e^{x(z)} + 2e^{-\frac{1}{2}\,y(z)}\big) = 0, \\ y''(z) + \frac{3}{z}y'(z) - 8\big(e^{-y(z)} + 2e^{\frac{1}{2}\,x(z)}\big) = 0, \end{cases} \qquad \begin{cases} x(0) = y(0) = 0, \\ x'(0) = y'(0) = 0. \end{cases}$$

The exact solutions are given by $(x(z), y(z)) = (-2\ln(z^2+1), 2\ln(z^2+1))$.

For this test problem we again consider $S = 8$ and run our QLM-SAFFK algorithm with $n = 5$ number of iterations. The approximated solutions are

$$\begin{aligned} \mathcal{X}_8^{(5)}(z) &= -0.053337755\,z^8 + 0.19417638\,z^7 - 0.028109583\,z^6 - 0.88580349\,z^5 \\ &\quad + 1.5259934\,z^4 - 0.16063339\,z^3 - 1.9785469\,z^2 + 4.7\times 10^{-108}\,z - 3.4\times 10^{-108}, \\ \mathcal{Y}_8^{(5)}(z) &= 0.045612319\,z^8 - 0.15278663\,z^7 - 0.065219781\,z^6 + 1.0000873\,z^5 \\ &\quad - 1.6072752\,z^4 + 0.19330416\,z^3 + 1.9724346\,z^2 - 3.3\times 10^{-108}\,z - 1.1\times 10^{-108}. \end{aligned}$$

We plot these approximate solutions in Figure 4 along with the related exact true solutions, which are shown by thick lines. We now show that the order



(a) (b)

Figure 4. Graphics of approximate solutions for $x(z)$ (a) and $y(z)$ (b) using QLM-SAFFK technique in Problem 3 with $S = 8$, $n = 5$.

of convergence of the proposed scheme behaves exponentially. The results of L_∞ norms and the estimated NOC are presented in Table 4. Finally, we present the numerical outcomes evaluated at some discrete points $z_j = 2j/10$ for $j = 1, 2, 3, 4, 5$. Using $S = 15$ and $n = 5$, the outcomes are shown in Table 5.

Problem 4. In the last test problem, let us consider the following nonlinear singular coupled system [5, 26, 29]

$$\begin{cases} x''(z) + \frac{8}{z}x'(z) + x(z)\big(18 - 4\ln y(z)\big) = 0, \\ y''(z) + \frac{4}{z}y'(z) - y(z)\big(10 - 4\ln x(z)\big) = 0, \end{cases} \qquad \begin{cases} x(0) = y(0) = 1, \\ x'(0) = y'(0) = 0. \end{cases}$$

One shows that the exact solutions are given by $(x(z), y(z)) = (e^{-z^2}, e^{z^2})$.

Table 4. The outcomes of error norms in L_∞, the associated NOC, and the elapsed CPU times for Problem 3 with diverse S.

S	L^x_∞	NOC^x_∞	L^y_∞	NOC^y_∞	CPU(s)
1	$1.3863 \times 10^{+00}$	–	$1.3863 \times 10^{+00}$	–	0.61321
2	3.0165×10^{-01}	2.2003	3.7499×10^{-01}	1.8863	0.78018
4	1.8966×10^{-02}	3.9914	2.3172×10^{-02}	4.0164	1.14082
8	1.8021×10^{-04}	6.7176	2.8375×10^{-04}	6.3516	2.07383
16	3.0304×10^{-09}	15.860	5.1466×10^{-09}	15.751	4.85578
32	7.8714×10^{-17}	25.198	2.7973×10^{-16}	24.133	16.1101

Table 5. The outcomes of absolute errors for $x(z), y(z)$ using QLM-SAFFK procedure with $S = 15$, $n = 5$ for Problem 3.

z	$\mathcal{X}^{(1)}_{15}(z)$	$\mathcal{E}^{(1)}_{x,15}(z)$	Exact	$\mathcal{Y}^{(1)}_{15}(z)$	$\mathcal{E}^{(1)}_{y,15}(z)$	Exact
0.2	−0.078441432	5.81_{-9}	−0.078441426306563	0.078441436	9.55_{-9}	0.078441426306563
0.4	−0.296840015	4.75_{-9}	−0.296840010236547	0.296840019	8.43_{-9}	0.296840010236547
0.6	−0.614969403	3.40_{-9}	−0.614969399495921	0.614969406	6.82_{-9}	0.614969399495921
0.8	−0.989392486	2.11_{-9}	−0.989392483672214	0.989392489	5.25_{-9}	0.989392483672214
1.0	−1.386294362	1.03_{-9}	−1.386294361119891	1.386294365	3.94_{-9}	1.386294361119891

By using $S = 8$, we get the following approximations for this example as

$$\mathcal{X}^{(5)}_8(z) = -0.025077205\, z^8 + 0.16860662\, z^7 - 0.37732895\, z^6 + 0.15564878\, z^5 + 0.4301283\, z^4 + 0.017987073\, z^3 - 1.0020509\, z^2 + 1.0106002 \times 10^{-106}\, z + 1.0,$$

$$\mathcal{Y}^{(5)}_8(z) = 0.39381724\, z^8 - 1.2787462\, z^7 + 2.2428373\, z^6 - 1.876918\, z^5 + 1.4924375\, z^4 - 0.29473112\, z^3 + 1.0406634\, z^2 - 3.4063679 \times 10^{-108}\, z + 1.0.$$

These solutions can be compared with the expansion series of the exact analytical solutions given by $e^{\pm z^2} = 1 \pm z^2 + \frac{1}{2} z^4 \pm \frac{1}{6} z^6 + \frac{1}{24} z^8 + \dots$.

Figure 5 shows the achieved absolute errors related to above approximate solutions using $S = 8, 16, 24$. By utilizing $S = 15$, the numerical results together with absolute errors computed at some points $z \in [0, 1]$ are presented in Table 6.

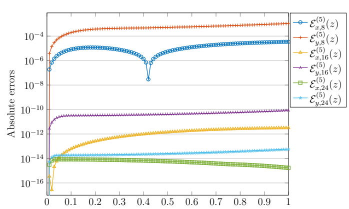


Figure 5. Comparing of absolute errors using QLM-SAFFK procedure for Problem 4 with $S = 8, 16, 24$ and $n = 5$.

Table 6. Numerical results and absolute errors for $x(z), y(z)$ obtained via QLM-SAFFK procedure using $S = 15$, $n = 5$ in Problem 4.

z	$\mathcal{X}_{15}^{(1)}(z)$	$\mathcal{E}_{x,15}^{(1)}(z)$	Exact	$\mathcal{Y}_{15}^{(1)}(z)$	$\mathcal{E}_{y,15}^{(1)}(z)$	Exact
0.2	0.96078943916	5.00_{-12}	0.960789439152323	1.040810774	3.03_{-10}	1.040810774192388
0.4	0.85214378896	2.44_{-12}	0.852143788966211	1.173510871	3.44_{-10}	1.173510870991810
0.6	0.69767632606	1.20_{-11}	0.697676326071031	1.433329414	4.19_{-10}	1.433329414560340
0.8	0.52729242402	2.07_{-11}	0.527292424043049	1.896480879	5.51_{-10}	1.896480879304951
1.0	0.36787944115	2.64_{-11}	0.367879441171442	2.718281828	7.80_{-10}	2.718281828459045

5 Conclusions

We have designed an accurate and effective spectral matrix collocation algorithm using the (novel) shifted airfoil function of the first kind (SAFFK) to determine the approximate (polynomial) solutions to a class of nonlinear coupled system of Emden-Fowler equations. The error estimation and convergence analysis for the SAFFK have been investigated. Four numerical test examples were provided to demonstrate the effectiveness and accuracy of the presented numerical matrix scheme. The computed computational results reveal that the presented QLM-SAFFK algorithm well approximates the solution of nonlinear system of coupled Emden-Fowler equations. By comparing with the Bernoulli and Haar wavelets collocation methods, computational outcomes demonstrate the high efficiency and competitiveness of our method. Moreover, the recorded computational time justified that our presented numerical matrix collocation algorithm is numerically efficient. To conclude, our method is robust, efficient (in terms of numerical accuracy) and easy to implement for solving the class of nonlinear system of coupled Emden-Fowler equations. Further, the method is capable of treating the singularity at the zero. The proposed technique may be easily extended to solve other similar classes of coupled systems of nonlinear singular differential equations given by either initial or boundary conditions.

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