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A priori estimate and existence of solutions with symmetric derivatives for a third-order boundary value problem

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Abstract. We study a priori estimate, existence, and uniqueness of solutions with symmetric derivatives for a third-order boundary value problem. The main tool in the proof of our existence result is Leray-Schauder continuation principle. Two examples are included to illustrate the applicability of the results.

Keywords: nonlinear boundary value problems; a priori estimate of solutions; existence of solutions; uniqueness of solution; Leray-Schauder continuation principle.

AMS Subject Classification: 34B15; 34B27.

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1 Introduction

We study the third-order nonlinear differential equation

$$x''' = f(t, x, x', x''), \; t \in (0, 1), \tag{1.1}$$

subject to the boundary conditions

$$x(0) = 0, \; x(1) = 0, \; x'(t) = x'(1 - t). \tag{1.2}$$

Throughout the paper we assume that $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $f(t, 0, 0, 0) \not\equiv 0$ for $t \in [0, 1]$ to exclude the existence of the trivial solution, and

$$f(1 - t, -x, x', -x'') = f(t, x, x', x'') \;\text{ for }\; (t, x, x', x'') \in [0, 1] \times \mathbb{R}^3.$$

By a solution of (1.1)–(1.2) we mean $C^3[0, 1]$ function that satisfies differential equation (1.1) for $0 < t < 1$ and boundary conditions (1.2).

The purpose of the paper is to obtain existence and uniqueness theorems for (1.1)–(1.2). For the existence result we get a priory estimate for solutions

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and apply the Leray-Schauder continuation principle [17]. For the uniqueness result we use estimates for a solution and its first and second derivatives. Let us recall the Leray-Schauder continuation principle here.

Theorem 1. *Let X be a Banach space and $T : X \to X$ be a completely continuous operator (T is continuous and maps bounded sets into relatively compact sets). Suppose that there exists an $r > 0$ such that if $x = \lambda Tx$ for $\lambda \in (0,1)$, then $\|x\| \leq r$. Then T has a fixed point.*

Actually, our main results state that under certain growth conditions on the function f, problem (1.1)–(1.2) has at least one nontrivial solution or has exactly one nontrivial solution.

In order to obtain the existence result, we first rewrite problem (1.1)–(1.2) as an equivalent integral equation. Then, we define an operator in a suitable set of functions, and hence the problem turns to prove that the operator has a fixed point. In the end, we show the existence of a fixed point by establishing an a priori estimate for a solution and its first and second derivatives.

The study of the existence of solutions to boundary value problems often involves rewriting the problem as an equivalent integral equation by the construction of the corresponding Green's functions. A survey of results on the Green's functions for stationary problems with nonlocal boundary conditions is presented in [15]. Green's functions for third-order boundary value problems with different additional conditions were studied in [13].

The nonlocal nature of boundary conditions (1.2) is attributed to the fact that the boundary conditions specify solution values not just at the ends of the interval, but also inside the interval. Papers [2, 3, 12, 16] contain some recent achievements in the field of nonlocal problems.

The Leray-Schauder continuation principle is a very effective and widely applied tool of nonlinear functional analysis, also in view of its applicability to boundary value problems for ordinary differential equations. For instance, the Leray-Schauder continuation principle was used in the papers [6, 7, 8], which motivated the present investigation.

There are several reasons why problem (1.1)–(1.2) should be studied. First, various fields of physics encounter third-order boundary value problems. Heat power transmission theory, deflection of a curved beam are just some of the topics covered. Second, the importance of boundary value problems with symmetric solutions has made them more popular in recent years due to their essential role in different branches of applied mathematics. In [4], the existence of symmetric positive solutions for a 2n-order nonlinear ordinary differential equation with integral boundary conditions by applying the theory of fixed point index in cones is studied. In [5], the existence of symmetric positive solutions to higher-order problems is discussed. Authors obtain sufficient conditions for the problem to have one, any finite number, and a countably infinite number of such solutions. In [9], by applying an iterative technique, a necessary and sufficient condition is obtained for the existence of symmetric positive solutions of second-order nonlinear singular boundary value problems. In [10], authors prove the existence, multiplicity, and nonexistence of symmetric positive solutions to nonlinear boundary value problems with the Laplacian operator.

The analysis mainly relies on the fixed point theorem of cone expansion and compression of norm type. In [11], a necessary and sufficient condition for the existence of symmetric positive solutions to higher-order nonlinear boundary value problems is obtained. The analysis relies on the monotone iterative technique. In [14], authors are concerned with the existence and multiplicity of symmetric positive solutions for a second-order three-point boundary value problem. In [18], authors establish various results on the existence and nonexistence of symmetric positive solutions to fourth-order boundary value problems with integral boundary conditions. The arguments are based upon a specially constructed cone and the fixed point theory in a cone. In [1], authors study symmetric positive solutions of a nonlinear fourth-order four-point boundary value problem. It is important to note that solutions to boundary value problems with symmetric derivatives are not enough investigated. The present paper is an attempt to decrease this gap. The next reason is that the nonlinear term in our problem has first and second derivative dependence. When f depends explicitly on x' and x'' there is no unified theory to study such problems.

The paper contains four sections besides the Introduction. In Section 2, we rewrite the main problem as an equivalent integral equation and prepare some technical details for application to our main results. In Section 3, we prove our existence theorem and give an illustrative example. In Section 4, we prove our uniqueness theorem and provide an example to illustrate the result. In Section 5, we give the conclusion.

2 Equivalent integral equation

Proposition 1. *A function $x = x(t)$ is a solution of boundary value problem (1.1)–(1.2) if and only if x is a solution of the integral equation*

$$
\begin{aligned}
x(t) = &\int_0^t \frac{1}{4}\left(2s^2 - t\right)(1-t)\, f(s, x(s), x'(s), x''(s))ds \\
&+ \int_t^1 \frac{1}{4}\, t\,(2(2-s)s - 1 - t)\, f(s, x(s), x'(s), x''(s))ds. \qquad (2.1)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
x'(t) = &\int_0^t \frac{1}{4}\left(2t - 2s^2 - 1\right) f(s, x(s), x'(s), x''(s))ds \\
&+ \int_t^1 \frac{1}{4}\left(1 - 2t - 2(1-s)^2\right) f(s, x(s), x'(s), x''(s))ds, \\
x''(t) = &\int_0^t \frac{1}{2}\, f(s, x(s), x'(s), x''(s))ds - \int_t^1 \frac{1}{2}\, f(s, x(s), x'(s), x''(s))ds.
\end{aligned}
$$

By a solution of (2.1) we understand $C^2[0,1]$ function that satisfies integral equation (2.1) for $0 \le t \le 1$.

Proof. Suppose that $x(t)$ is a solution to problem (1.1)–(1.2), then $x'''(t) = f(t, x(t), x'(t), x''(t))$ or $x'''(t) = h(t)$, where $h(t) \equiv f(t, x(t), x'(t), x''(t))$. Integrating the equation $x'''(t) = h(t)$ thrice, we get

$$\begin{aligned}
x''(t) =& x''(0) + \int_0^t h(s)ds, \\
x'(t) =& x'(0) + t\,x''(0) + \int_0^t (t-s)h(s)ds, \\
x(t) =& x(0) + t\,x'(0) + \frac{1}{2}\,t^2 x''(0) + \frac{1}{2}\int_0^t (t-s)^2 h(s)ds.
\end{aligned}$$

Since $x'(t) = x'(1-t)$, we get $x''(t) = -x''(1-t)$, $x(t) = -x(1-t)$, $h(t) = h(1-t)$ for $t \in [0,1]$, and

$$x''(0) + \int_0^t h(s)ds = -x''(0) - \int_0^{1-t} h(s)ds.$$

Therefore,

$$\begin{aligned}
x''(0) =& -\frac{1}{2}\int_0^t h(s)ds - \frac{1}{2}\int_0^{1-t} h(s)ds \\
=& -\frac{1}{2}\int_0^t h(s)ds - \frac{1}{2}\int_t^1 h(s)ds = -\frac{1}{2}\int_0^1 h(s)ds.
\end{aligned}$$

Since $x(0) = 0$, we have

$$x(t) = t\,x'(0) - \frac{1}{4}\,t^2 \int_0^1 h(s)ds + \frac{1}{2}\int_0^t (t-s)^2 h(s)ds.$$

Since $x(1) = 0$, we get

$$x'(0) = \frac{1}{4}\int_0^1 h(s)ds - \frac{1}{2}\int_0^1 (1-s)^2 h(s)ds$$

and hence

$$\begin{aligned}
x(t) =& \frac{1}{4}\,t\int_0^1 h(s)ds - \frac{1}{2}\,t\int_0^1 (1-s)^2 h(s)ds - \frac{1}{4}\,t^2\int_0^1 h(s)ds + \frac{1}{2}\int_0^t (t-s)^2 h(s)ds \\
=& \int_0^1 \left(-\frac{1}{4}\,t\,(1 + 2(-2+s)s + t)\right) h(s)ds + \frac{1}{2}\int_0^t (t-s)^2 h(s)ds \\
=& \int_0^t \left(-\frac{1}{4}\,t\,(1 + 2(-2+s)s + t) + \frac{1}{2}(t-s)^2\right) h(s)ds \\
&+ \int_t^1 \left(-\frac{1}{4}\,t\,(1 + 2(-2+s)s + t)\right) h(s)ds \\
=& \int_0^t \frac{1}{4}\left(2s^2 - t\right)(1-t)h(s)ds + \int_t^1 \frac{1}{4}\,t\,(2(2-s)s - 1 - t)h(s)ds
\end{aligned}$$

or $x(t)$ is a solution of (2.1).

Now let $x(t)$ be a solution of integral equation (2.1). To show that $x(t)$ is a solution to problem (1.1)–(1.2), one can differentiate thrice equation (2.1) and verify the continuity. □

Proposition 2. *If $x(t)$ is a solution of boundary value problem* (1.1)–(1.2), *then,*

$$\begin{aligned} |x(t)| \leq & \frac{1}{16}\int_0^1 |f(s,x(s),x'(s),x''(s))|ds, \\ |x'(t)| \leq & \frac{1}{4}\int_0^1 |f(s,x(s),x'(s),x''(s))|ds, \\ |x''(t)| \leq & \frac{1}{2}\int_0^1 |f(s,x(s),x'(s),x''(s))|ds, \quad t\in[0,1]. \end{aligned}$$

Proof. Let $h(t) \equiv f(t,x(t),x'(t),x''(t))$. For all $t\in[0,1]$, we have

$$\begin{aligned} |x(t)| \leq & \frac{1}{4}\int_0^t |2s^2-t|(1-t)\,|h(s)|ds + \frac{1}{4}\int_t^1 t\,|2(2-s)s-1-t|\,|h(s)|ds \\ \leq & \frac{1}{4}\max_{0\leq s\leq t\leq 1}\left\{|2s^2-t|(1-t)\right\}\int_0^t |h(s)|ds \\ & + \frac{1}{4}\max_{0\leq t\leq s\leq 1}\left\{t\,|2(2-s)s-1-t|\right\}\int_t^1 |h(s)|ds \\ = & \frac{1}{16}\int_0^t |h(s)|ds + \frac{1}{16}\int_t^1 |h(s)|ds = \frac{1}{16}\int_0^1 |h(s)|ds, \\ |x'(t)| \leq & \frac{1}{4}\int_0^t |2t-2s^2-1||h(s)|ds + \frac{1}{4}\int_t^1 |1-2t-2(1-s)^2||h(s)|ds \\ \leq & \frac{1}{4}\max_{0\leq s\leq t\leq 1}\left\{|2t-2s^2-1|\right\}\int_0^t |h(s)|ds \\ & + \frac{1}{4}\max_{0\leq t\leq s\leq 1}\left\{|1-2t-2(1-s)^2|\right\}\int_t^1 |h(s)|ds \\ = & \frac{1}{4}\int_0^t |h(s)|ds + \frac{1}{4}\int_t^1 |h(s)|ds = \frac{1}{4}\int_0^1 |h(s)|ds, \\ |x''(t)| \leq & \frac{1}{2}\int_0^t |h(s)|ds + \frac{1}{2}\int_t^1 |h(s)|ds = \frac{1}{2}\int_0^1 |h(s)|ds. \end{aligned}$$

□

3 A priori estimate and existence of solutions

Theorem 2. *If there exist continuous functions α, β, γ, $\delta:[0,1]\to[0,+\infty)$, such that*

$$|f(t,x,x',x'')| \leq \alpha(t)\,|x|+\beta(t)\,|x'|+\gamma(t)\,|x''|+\delta(t) \ \textit{for}\ (t,x,x',x'')\in[0,1]\times\mathbb{R}^3,$$

and

$$\frac{1}{16}\int_0^1 \alpha(s)ds + \frac{1}{4}\int_0^1 \beta(s)ds + \frac{1}{2}\int_0^1 \gamma(s)ds < 1,$$

then, boundary value problem (1.1)–(1.2) *has at least one nontrivial solution.*

Proof. Let $X = C^2[0,1]$ with the norm $\|x\| = \max\left\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\right\}$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$ and define the operator $T : X \to X$ by

$$\begin{aligned}(Tx)(t) &= \int_0^t \frac{1}{4}\left(2s^2 - t\right)(1-t)\, f(s, x(s), x'(s), x''(s))ds \\ &\quad + \int_t^1 \frac{1}{4} t\,(2(2-s)s - 1 - t)\, f(s, x(s), x'(s), x''(s))ds, \ t \in [0,1].\end{aligned}$$

T is completely continuous by application of Arzelà-Ascoli theorem. It is necessary to show that operator T has a fixed point. Let $\lambda \in (0,1]$ and consider

$$x''' = \lambda\, f(t, x, x', x''), \ t \in (0,1) \tag{3.1}$$

subject to boundary conditions (1.2). Our goal is to demonstrate that the set of all possible solutions of (3.1),(1.2) is a priori bounded in X by a constant that does not depend on λ. We include the case $\lambda = 1$, to get an estimate for solutions to problem (1.1)–(1.2). Let $x(t)$ be a solution of (3.1),(1.2) and consider

$$\begin{aligned}&\int_0^1 |x'''(s)|ds = \lambda \int_0^1 |f(s, x(s), x'(s), x''(s))|ds \le \int_0^1 |f(s, x(s), x'(s), x''(s))|ds \\ &\le \int_0^1 \big(\alpha(s)\,|x(s)| + \beta(s)\,|x'(s)| + \gamma(s)\,|x''(s)| + \delta(s)\big)\, ds \\ &= \int_0^1 \alpha(s)\,|x(s)|ds + \int_0^1 \beta(s)\,|x'(s)|ds + \int_0^1 \gamma(s)\,|x''(s)|ds + \int_0^1 \delta(s)ds \\ &\le \frac{1}{16}\int_0^1 |x'''(s)|ds \int_0^1 \alpha(s)ds + \frac{1}{4}\int_0^1 |x'''(s)|ds \int_0^1 \beta(s)ds \\ &\quad + \frac{1}{2}\int_0^1 |x'''(s)|ds \int_0^1 \gamma(s)ds + \int_0^1 \delta(s)ds.\end{aligned}$$

Hence,

$$\int_0^1 |x'''(s)|ds \le \frac{\displaystyle\int_0^1 \delta(s)ds}{1 - \dfrac{1}{16}\displaystyle\int_0^1 \alpha(s)ds - \dfrac{1}{4}\displaystyle\int_0^1 \beta(s)ds - \dfrac{1}{2}\displaystyle\int_0^1 \gamma(s)ds} = r.$$

It follows that

$$|x(t)| \leq \frac{1}{16}\int_0^1 |\lambda f(s,x(s),x'(s),x''(s))|ds = \frac{1}{16}\int_0^1 |x'''(s)|ds \leq \frac{1}{16} r,$$
$$|x'(t)| \leq \frac{1}{4}\int_0^1 |\lambda f(s,x(s),x'(s),x''(s))|ds = \frac{1}{4}\int_0^1 |x'''(s)|ds \leq \frac{1}{4} r,$$
$$|x''(t)| \leq \frac{1}{2}\int_0^1 |\lambda f(s,x(s),x'(s),x''(s))|ds = \frac{1}{2}\int_0^1 |x'''(s)|ds \leq \frac{1}{2} r, \quad t\in[0,1].$$

According to these estimates, the set of all possible solutions of (3.1),(1.2) is a priori bounded in X or $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\} \leq 1/2\, r$ and therefore, by Theorem 1, T has a fixed point. □

Remark 1. An estimate in the norm for the nontrivial solution of (1.1)–(1.2) was obtained during the proof of Theorem 2.

Example 1. Consider boundary value problem for the differential equation

$$x''' = t(1-t)\frac{x^3 \sin x}{1+x^2} + (2t-1)^2\left(x' - 2\arctan x'\right) + \left(\sqrt{1+(x'')^2} - 1\right) + 23,$$
$$t\in(0,1), \tag{3.2}$$

with boundary conditions (1.2). The function $f(t,x,x',x'') = t(1-t)\frac{x^3\sin x}{1+x^2}$ $+(2t-1)^2\left(x'-2\arctan x'\right) + \left(\sqrt{1+(x'')^2}-1\right) + 23$ is continuous for $(t,x,x',x'')\in[0,1]\times\mathbb{R}^3$, and $f(t,0,0,0)\not\equiv 0$ for $t\in[0,1]$. We have $f(1-t,-x,x',-x'') = f(t,x,x',x'')$ for all $(t,x,x',x'')\in[0,1]\times\mathbb{R}^3$. Consider

$$|f(t,x,x',x'')| \leq t(1-t)\frac{|x|\,x^2\,|\sin x|}{1+x^2} + (2t-1)^2|x'|\left|\frac{x'-2\arctan x'}{x'}\right|$$
$$+|x''|\left|\frac{\sqrt{1+(x'')^2}-1}{x''}\right| + 23 \leq t(1-t)\,|x| + (2t-1)^2\,|x'| + |x''| + 23.$$

We have

$$\alpha(t) = t(1-t), \quad \beta(t) = (2t-1)^2, \quad \gamma(t) = 1, \quad \delta(t) = 23,$$
$$\frac{1}{16}\int_0^1 \alpha(s)ds + \frac{1}{4}\int_0^1 \beta(s)ds + \frac{1}{2}\int_0^1 \gamma(s)ds = \frac{19}{32} < 1.$$

Thus, by Theorem 2, problem (3.2),(1.2) has at least one nontrivial solution $x(t)$, which together with its derivatives $x'(t)$ and $x''(t)$ is depicted in Figures 1, 2, 3. These figures were obtained by using the program Wolfram Mathematica 11.1. The initial conditions for this solution are $x(0) = 0$, $x'(0) \approx 2.24838$, $x''(0) \approx -14.344$. We have the following estimates

$$|x(t)| \leq \frac{1}{16} r = \frac{46}{13}, \quad |x'(t)| \leq \frac{1}{4} r = \frac{184}{13}, \quad |x''(t)| \leq \frac{1}{2} r = \frac{368}{13}.$$

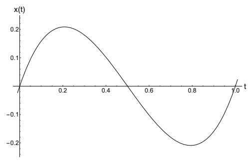

Figure 1. Solution $x(t)$.

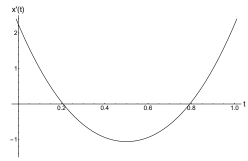

Figure 2. Derivative $x'(t)$.

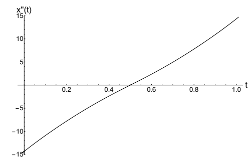

Figure 3. Derivative $x''(t)$.

4 Uniqueness of solution

Theorem 3. *If there exist continuous functions* p, q, $r : [0,1] \to [0,+\infty)$, *such that*

$$|f(t,x,x',x'') - f(t,y,y',y'')| \le p(t)\,|x-y| + q(t)\,|x'-y'| + r(t)\,|x''-y''|$$
$$\text{for}\ \ (t,x,x',x''),\ (t,y,y',y'') \in [0,1]\times\mathbb{R}^3,$$

and

$$\frac{1}{16}\int_0^1 p(s)ds + \frac{1}{4}\int_0^1 q(s)ds + \frac{1}{2}\int_0^1 r(s)ds < 1,$$

then boundary value problem (1.1)–(1.2) *has exactly one nontrivial solution.*

Proof. If $y = y' = y'' = 0$, we get

$$|f(t,x,x',x'')| \le p(t)\,|x| + q(t)\,|x'| + r(t)\,|x''| + |f(t,0,0,0)|$$

for $(t,x,x',x'') \in [0,1]\times\mathbb{R}^3$. Hence, in view of Theorem 2, boundary value problem (1.1)–(1.2) has at least one nontrivial solution. To prove the uniqueness of a solution, let $x(x)$, $y(t)$ be two solutions for (1.1)–(1.2). If $u(t) = x(t) - y(t)$, we get

$$\begin{aligned} u'''(t) =& f(t,x(t),x'(t),x''(t)) - f(t,y(t),y'(t),y''(t)), \\ u(0) =& 0,\ u(1) = 0,\ u'(t) = u'(1-t). \end{aligned}$$

Let us consider

$$\begin{aligned} &\int_0^1 |u'''(s)|ds = \int_0^1 |f(s,x(s),x'(s),x''(s)) - f(s,y(s),y'(s),y''(s))|ds \\ &\le \int_0^1 p(s)\,|u(s)|ds + \int_0^1 q(s)\,|u'(s)|ds + \int_0^1 r(s)\,|u''(s)|ds \\ &\le \frac{1}{16}\int_0^1 |u'''(s)|ds \int_0^1 p(s)ds + \frac{1}{4}\int_0^1 |u'''(s)|ds \int_0^1 q(s)ds \\ &\quad + \frac{1}{2}\int_0^1 |u'''(s)|ds \int_0^1 r(s)ds \\ &= \int_0^1 |u'''(s)|ds \left(\frac{1}{16}\int_0^1 p(s)ds + \frac{1}{4}\int_0^1 q(s)ds + \frac{1}{2}\int_0^1 r(s)ds \right). \end{aligned}$$

It follows that $\int_0^1 |u'''(s)|ds = 0$ for every $s \in [0,1]$, and hence $u(t) = 0$ for every $t \in [0,1]$. We get that $x(t) = y(t)$ for every $t \in [0,1]$. □

Example 2. Let us now consider boundary value problem for the linear differential equation

$$x''' = 1, \ t \in (0,1), \tag{4.1}$$

with boundary conditions (1.2). Problem (4.1),(1.2) meets all the requirements of Theorem 3 and therefore has exactly one nontrivial solution. Since the equation is linear, it is not difficult to verify, that the unique solution of the problem is $x(t) = \frac{1}{12}\left(t - 3t^2 + 2t^3\right)$.

5 Conclusions

We have proved the existence of solutions with symmetric derivatives for boundary value problem (1.1)–(1.2) by obtaining an a priori estimate for solutions and their first and second derivatives and applying Leray-Schauder continuation principle. The nontrivial solution of (1.1)–(1.2) was estimated in the norm during the proof of our existence theorem. Also, we have proved the existence of a unique solution to (1.1)–(1.2). Illustrative examples (3.2) and (4.1) were provided to show the applicability of the obtained results.

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