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Two-Grid Method for Burgers' Equation by a New Mixed Finite Element Scheme*

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Abstract. In this article, we present two-grid stable mixed finite element method for the 2D Burgers' equation approximated by the $P_0^2 - P_1$ pair which satisfies the inf–sup condition. This method consists in dealing with the nonlinear system on a coarse mesh with width H and the linear system on a fine mesh with width $h \ll H$ by using Crank–Nicolson time-discretization scheme. Our results show that if we choose $H^2 = h$ this method can achieve asymptotically optimal approximation. Error estimates are derived in detail. Finally, numerical experiments show the efficiency of our proposed method and justify the theoretical results.

Keywords: Burgers' equation, two-grid method, stable conforming finite element, Crank–Nicolson scheme, inf–sup condition.

AMS Subject Classification: 65N30; 65N12; 65B05; 35Q30.

1 Introduction

In this paper, we consider the following 2D Burgers' equation with homogeneous boundary condition:

$$u_t - \nu(u_{xx} + u_{yy}) + u(u_x + u_y) = f, \quad \text{in } \Omega \times J, \tag{1.1}$$
$$u(x, y, 0) = u_0(x, y), \quad \text{in } \Omega \times \{0\}, \tag{1.2}$$
$$u = 0, \quad \text{on } \partial\Omega \times J, \tag{1.3}$$

where Ω is a bounded convex domain in the plane and $\partial\Omega$ is the Lipschitz continuous boundary of Ω, $J = (0, T]$. $u_0(x, y)$ is the initial value. $T > 0$ represents the given final time. $f = f(x, y, t)$ is the prescribed force. The positive number ν is the coefficient of viscosity.

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Burgers' equation can be regarded as a qualitative approximation of the Navier–Stokes equations. This equation incorporates both convection and diffusion, preserves the hybrid characteristic of the Navier–Stokes equations, and can be solved using similar numerical methods. It retains the nonlinear aspects of the governing equation in many practical transport problems such as aggregation interface growth, shock wave theory, transport and dispersion of pollutants in rivers and sediment transport. Thus, the numerical method has practical significance, and has drawn the attention of many researchers. Burgers' equation is so important that many numerical methods were developed in the past decades, for example, the spectral method, the finite difference method, the finite element method, the local discontinuous Galerkin method, see [1,3,10,11,13,14,15,18] and the references therein.

Mixed finite element methods have been found to be very important for solving the problems of groundwater through porous media. For example, there are many applications of mixed finite element methods to miscible displacement problems that describe two-phase flow in a petroleum reservoir [8]. In a mixed finite element formulation, both the pressure and the flux, or displacements and stresses, are approximated simultaneously, e.g. see References [2,8,9,17]. In [13], a new mixed finite element method is used to approximate the solution as well as the flux of Burgers' equation. So in this paper, we still use the new mixed finite element method to discretize system (1.1)–(1.3) in space.

To linearize the resulting discrete equations, we use the two-grid method, which was first introduced by Xu [26,27] as a discretization technique for nonlinear and nonsymmetric indefinite partial differential equations. It is based on the fact that the nonlinearity, nonsymmetry and indefiniteness behaving like low frequencies are governed by coarse grid and the related high frequencies are governed by some linear or symmetric positive definite operators. The basic idea of the two-grid method is to solve a complicated problem (nonlinear, nonsymmetric indefinite) on a coarse grid (mesh size H) and then solve an easier problem (linear, symmetric positive) on a fine grid (mesh size h and $h \ll H$) as correction.

The two grid method is widely used in solving nonlinear problem, for example, Wu et al. [24] used two-grid method to solve the nonlinear reaction–diffusion equations by mixed finite element methods. Dawson et al. [7] used two-grid method for mixed finite element methods approximations of nonlinear parabolic equations. Chen et al. [4,5] used two-grid method for nonlinear parabolic equations and semi-linear reaction–diffusion equations by expanded mixed finite element methods. Weng et al. [21,22,23] also used two-grid method for the semi-linear elliptic equations and the elliptic eigenvalue problem by a new mixed finite element method and so on. So we will apply two-grid scheme to the new mixed finite element methods for Burgers' equations.

As a continued work of Hu et al. [13], in this paper, the method we study is to combine the stable mixed finite element method with the two-grid discretization for solving the 2D Burgers' equation based on the less regularity of flux. And the time is discreted by the Crank–Nicolson scheme. The key feature of the two-grid method is that it allows one to execute all the nonlinear iterations on a system associated with a coarse spatial grid. This procedure is basically to use the coarse grid to produce a rough approximation of the solution and then use it as the initial guess on a fine grid.

This paper is divided into five sections. Notations and the new mixed formulation are stated in Section 2. In Section 3, the two-grid algorithm and its error estimates will be discussed. In Section 4, numerical experiments are given to illustrate the theoretical results and the efficiency of the proposed method. And the conclusions are given in the end of the paper.

2 Some Notations and Mixed Finite Element Approximation

Suppose that $f \in L^2(\Omega)$. By introducing the flux $p = -\nabla u$, the mixed formulation of (1.1)–(1.3) is to find $(p, u) \in V \times W$, such that

$$\begin{aligned} (p, q) + (q, \nabla u) &= 0, \qquad \forall q \in V, \\ (u_t, v) - \nu(p, \nabla v) - \big([u, u]p, v\big) &= (f, v), \quad \forall v \in W. \end{aligned} \tag{2.1}$$

Here we denote by $V = L^2(\Omega)^2$, $W = H_0^1(\Omega)$.

Throughout the paper, we employ the standard notations $H^s(\Omega)$, $\|\cdot\|_s$, $(\cdot,\cdot)_s$, $s \geq 0$ for the Sobolev spaces of all functions having square integrable derivatives up to order s on Ω, the standard Sobolev norm, and inner product, respectively. When $s = 0$, we will write $L^2(\Omega)$ instead of $H^0(\Omega)$, the L^2-inner product and L^2- norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. As usual, $H_0^1(\Omega)$ will denote the closure of C_0^∞ with respect to the norm $\|\cdot\|$. The norm and semi-norm in $H^s(\Omega)^d$ are denoted by $\|\cdot\|_s$ and $|\cdot|_s$, respectively. We use C to denote a generic positive constant whose value may change from place to place but remains independent of the mesh parameter h.

For any $t \in J$, we define the following bilinear forms

$$\begin{aligned} a(p, q) &= (p, q), \qquad \forall p, q \in V, \\ b(p, v) &= -(p, \nabla v), \quad \forall p \in V,\ \forall v \in W. \end{aligned}$$

From (2.1), for any $t \in J$, a new variational formulation to Burgers' equation (1.1)–(1.3) is to find $(p, u) \in V \times W$, such that

$$\begin{aligned} a(p, q) - b(q, u) &= 0, \qquad \forall q \in V, \\ (u_t, v) + \nu b(p, v) - \big([u, u]p, v\big) &= (f, v), \quad \forall v \in W. \end{aligned} \tag{2.2}$$

Concerning this system, we give some properties.

Lemma 1. [19] *Bilinear form $b(\cdot,\cdot)$ satisfies the so called inf–sup condition, i.e., there exists a constant $\beta_1 > 0$, such that*

$$\inf_{v \in W} \sup_{q \in V} \frac{-(q, \nabla v)}{\|q\|_V \|v\|_W} \geq \beta_1.$$

Theorem 1. [13] *Suppose that $u_0(x, y) \in L^2(\Omega)$, then there exists a unique solution $(p, u) \in V \times W$ to variational formulation (2.2). Moreover, there exists a constant $M_0 > 0$, such that $\|u\|_{0,\infty} \leq M_0$.*

Based on new variational formulation (2.2), we address the stable conforming finite element approximation for $P_0^2 - P_1$ pair. Let K_h be a uniformly regular family of triangulation of Ω. Now choose (V_h, W_h) as the $P_0^2 - P_1$ finite-element pair as follows:

$$
\begin{aligned}
V_h &= \{q_h = (q_1, q_2) \in V : q_i \in P_0(K),\ \forall K \in K_h,\ i = 1, 2\}, \\
W_h &= \{v \in C^0(\Omega) \cap W : v \in P_1(K),\ \forall K \in K_h\}.
\end{aligned} \tag{2.3}
$$

Lemma 2. [19] *The $P_0^2 - P_1$ finite element pair defined by the spaces* (2.3) *satisfies the discrete inf–sup condition as follows*:

$$\inf_{v_h \in W_h} \sup_{q_h \in V_h} \frac{-(q_h, \nabla v_h)}{\|q_h\|_V \|v_h\|_W} \geq \beta_2 > 0.$$

Lemma 3. [20] *There exists a standard L^2 projection operator $\Pi : L^2(\Omega) \to V_h$, which satisfies the following properties:*

$$(p - \Pi p, q) = 0, \qquad \forall q \in V_h, \tag{2.4}$$

$$\|\Pi p\|_0 \leq C\|p\|_0, \qquad \forall p \in V, \tag{2.5}$$

$$\|p - \Pi p\|_0 \leq Ch\|p\|_1, \quad \forall p \in H^1(\Omega) \cap V. \tag{2.6}$$

Lemma 4. [20] *There exists a projection $\Lambda : W \to W_h$, such that*

$$\|\Lambda u\|_0 \leq C\|u\|_1, \qquad \forall u \in W, \tag{2.7}$$

$$\|u - \Lambda u\|_0 + h\|u - \Lambda u\|_1 \leq Ch^2\|u\|_2, \quad \forall u \in H^2(\Omega) \cap W, \tag{2.8}$$

and if $u \in H_0^1(\Omega)$, then we have

$$\big(\nabla(u - \Lambda u), q\big) = 0, \quad \forall q \in V_h. \tag{2.9}$$

Theorem 2. [13] *If $u_0(x, y) \in L^2(\Omega)$, then there exists a unique finite element solution $(p_h, u_h) \in V_h \times W_h$ to the following equations for $P_0^2 - P_1$ finite element pair*

$$
\begin{aligned}
a(p_h, q) - b(q, u_h) &= 0, && \forall q \in V_h, \\
(u_{ht}, v) + \nu b(p_h, v) - \big([u_h, u_h]p_h, v\big) &= (f, v), && \forall v \in W_h.
\end{aligned}
$$

Moreover, there exists a positive constant M_1 independent of h, such that $\|u_h\|_0 \leq M_1$.

3 The Two-Grid Algorithm Based on Crank–Nicolson Scheme

From now on, H and $h \ll H$ will be two real positive parameters tending to zero. Also, a coarse mesh triangulation of $K_H(\Omega)$ of Ω is made like in Section 2 and a fine mesh triangulation $K_h(\Omega)$ is generated by a mesh refinement process to $K_H(\Omega)$. The conforming finite element space pairs (V_h, W_h) and

$(V_H, W_H) \subset (V_h, W_h)$ based on the triangulations $K_h(\Omega)$ and $K_H(\Omega)$, respectively, are constructed as in Section 2. Let $\tau = \frac{T}{N}$ be the time step and u_h^n be the approximation of $u(t)$ at $t = t_n = n\tau (n = 1, 2, \dots, N)$ in W_h. Applying the Crank–Nicolson scheme to time derivative $\frac{\partial u}{\partial t}$ around the point $t_{n-\frac{1}{2}} = (n - \frac{1}{2})\tau$, the two-grid stable finite element approximations are defined as follows:

Step 1: On the coarse grid K_H, for given $(p_H^0, u_H^0) = (\nabla u_0, u_0)$, solve the following nonlinear system for $(p_H^n, u_H^n) \in V_H \times W_H$:

$$\begin{aligned} \left(\frac{u_H^n - u_H^{n-1}}{\tau}, v\right) - \nu\big(\overline{p}_H^n, \nabla v\big) - \big(\overline{\phi}_H^n, v\big) &= (\overline{f}, v), \\ \big(\overline{p}_H^n, q\big) + \big(q, \nabla \overline{u}_H^n\big) &= 0, \end{aligned} \tag{3.1}$$

where $\overline{p}_H^n = \frac{p_H^n + p_H^{n-1}}{2}$, $\overline{u}_H^n = \frac{u_H^n + u_H^{n-1}}{2}$, $\overline{\phi}_H^n = \frac{[u_H^n, u_H^n] p_H^n + [u_H^{n-1}, u_H^{n-1}] p_H^{n-1}}{2}$, $\overline{f} = \frac{f^n + f^{n-1}}{2}$, $v \in W_H$, $q \in V_H$.

Step 2: On the fine grid K_h, for given $(p_h^0, u_h^0) = (\nabla u_0, u_0)$, compute $(p_h^n, u_h^n) \in V_h \times W_h$ to satisfy the following linear system:

$$\begin{aligned} \left(\frac{u_h^n - u_h^{n-1}}{\tau}, v\right) - \nu\big(\overline{p}_h^n, \nabla v\big) - \big(\overline{\phi}_H^n, v\big) &= (\overline{f}, v), \\ \big(\overline{p}_h^n, q\big) + \big(q, \nabla \overline{u}_h^n\big) &= 0, \end{aligned} \tag{3.2}$$

where $\overline{p}_h^n = \frac{p_h^n + p_h^{n-1}}{2}$, $\overline{u}_h^n = \frac{u_h^n + u_h^{n-1}}{2}$, $v \in W_h$, $q \in V_h$.

In order to obtain error estimate, we introduce some useful lemmas as follows:

Lemma 5. [6] *Let C and a_k, c_k, d_k, for integer $k \geq 0$, be non-negative numbers such that*

$$a_n \leq \tau \sum_{k=0}^{n-1} d_k a_k + \tau \sum_{k=0}^{n-1} c_k + C, \quad \forall n \geq 1.$$

Then

$$a_n \leq \exp\left(\tau \sum_{k=0}^{n-1} d_k\right)\left(\tau \sum_{k=0}^{n-1} c_k + C\right), \quad \forall n \geq 1.$$

Lemma 6. [20] *For each $n \geq 1$, if $u_{tt}, u_{ttt} \in L^2(0, T; L^2(\Omega))$, then we have*

$$\begin{aligned} \left\| \frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}} \right\|^2 &\leq C\tau^3 \int_{t_{n-1}}^{t_n} \|u_{tt}\|^2 \, \mathrm{d}t, \\ \left\| \frac{u^n - u^{n-1}}{\tau} - u_t^{n-\frac{1}{2}} \right\|^2 &\leq C\tau^3 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|^2 \, \mathrm{d}t. \end{aligned}$$

First, we can give the error estimates of Step 1.

Theorem 3. *Let $(p_H, u_H) \in V_H \times W_H$ be the solution of Eqs. (3.1), for the $P_0^2 - P_1$ finite element pair, there exists a positive constant C such that*

$$\begin{aligned}\left\|u^n - u_H^n\right\|_1 + \left\|p^n - p_H^n\right\|_0 \le CH\bigg(\left\|u^0\right\|_2 + \left\|p^0\right\|_1 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t \\ + \int_0^{t_n} \|p_t\|_1 \,\mathrm{d}t\bigg) + C\tau^2\bigg(\int_0^{t_n} \|u_{ttt}\|_0^2 \,\mathrm{d}t\bigg)^{\frac{1}{2}}. \qquad (3.3)\end{aligned}$$

Furthermore, we have

$$\left\|u^n - u_H^n\right\|_0 \le CH^2\bigg(\left\|u^0\right\|_2 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t\bigg) + C\tau^2\bigg(\int_0^{t_n} \|u_{ttt}\|_0^2 \,\mathrm{d}t\bigg)^{\frac{1}{2}}. \qquad (3.4)$$

Proof. Let

$$\begin{aligned}u^n - u_H^n &= u^n - \Lambda u^n + \Lambda u^n - u_H^n = \phi_1^n + \theta_1^n = \varepsilon_1^n,\\ p^n - p_H^n &= p^n - \Pi p^n + \Pi p^n - p_H^n = \rho_1^n + \xi_1^n = \eta_1^n.\end{aligned}$$

From (2.6) and (2.8), we have

$$\left\|\phi_1^n\right\|_1 = \left\|u^n - \Lambda u^n\right\|_1 \le CH\bigg(\left\|u^0\right\|_2 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t\bigg), \qquad (3.5)$$

$$\left\|\phi_1^n\right\|_0 = \left\|u^n - \Lambda u^n\right\|_0 \le CH^2\bigg(\left\|u^0\right\|_2 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t\bigg), \qquad (3.6)$$

$$\left\|\rho_1^n\right\|_0 = \left\|p^n - \Pi p^n\right\|_0 \le CH\bigg(\left\|p^0\right\|_1 + \int_0^{t_n} \|p_t\|_1 \,\mathrm{d}t\bigg). \qquad (3.7)$$

Using (2.1) and (3.1), for any $q \in V_H$ and $v \in W_H$, we obtain the error equations as follows:

$$\begin{aligned}&\left(\frac{\varepsilon_1^n - \varepsilon_1^{n-1}}{\tau}, v\right) - \nu\left(\frac{\eta_1^n + \eta_1^{n-1}}{2}, \nabla v\right) - (\varphi_H^n, v) - \left(\varphi_H^{n-1}, v\right)\\ &= \left(\frac{u^n - u^{n-1}}{\tau} - u_t^{n-\frac{1}{2}}, v\right) - \left(\frac{u_t^n + u_t^{n-1}}{2} - u_t^{n-\frac{1}{2}}, v\right), \qquad (3.8)\\ &\left(\frac{\eta_1^n + \eta_1^{n-1}}{2}, q\right) + \left(q, \nabla\frac{\varepsilon_1^n + \varepsilon_1^{n-1}}{2}\right) = 0, \qquad (3.9)\end{aligned}$$

where $\varphi_H^n = \frac{[u^n, u^n]p^n - [u_H^n, u_H^n]p_H^n}{2}$. From (2.4) and (2.9), we get

$$\begin{aligned}&\left(\frac{\theta_1^n - \theta_1^{n-1}}{\tau}, v\right) - \nu\left(\frac{\xi_1^n + \xi_1^{n-1}}{2}, \nabla v\right) - (\varphi_H^n, v) - \left(\varphi_H^{n-1}, v\right)\\ &= \left(\frac{u^n - u^{n-1}}{\tau} - u_t^{n-\frac{1}{2}}, v\right) - \left(\frac{u_t^n + u_t^{n-1}}{2} - u_t^{n-\frac{1}{2}}, v\right), \qquad (3.10)\\ &\left(\frac{\xi_1^n + \xi_1^{n-1}}{2}, q\right) + \left(q, \nabla\frac{\theta_1^n + \theta_1^{n-1}}{2}\right) = 0. \qquad (3.11)\end{aligned}$$

We consider

$$\left(\frac{\xi_1^n-\xi_1^{n-1}}{\tau},q\right)+\left(q,\nabla\frac{\theta_1^n-\theta_1^{n-1}}{\tau}\right)=0,\quad \forall q\in V_h,\ n=1,2,\dots,N \quad (3.12)$$

instead of (3.11). From (3.11) and taking $q=\nabla\frac{\theta_1^n-\theta_1^{n-1}}{2}$, applying the Cauchy–Schwartz and Young inequality, we obtain

$$\begin{aligned}\left\|\nabla\theta_1^n\right\|_0^2-\left\|\nabla\theta_1^{n-1}\right\|_0^2&\le\left\|\xi_1^n+\xi_1^{n-1}\right\|_0\left\|\nabla\theta_1^n+\nabla\theta_1^{n-1}\right\|_0\\&\le\frac{\|\xi_1^n+\xi_1^{n-1}\|_0^2}{2\delta}+\frac{\delta\|\nabla\theta_1^n+\nabla\theta_1^{n-1}\|_0^2}{2}.\end{aligned}$$

Choosing $\delta\ge 0$ such that $1-\delta>0$ and due to Poincaré inequality, we have

$$\left\|\theta_1^n\right\|_1^2\le C\left(\left\|\theta_1^{n-1}\right\|_1^2+\left\|\xi_1^n\right\|_0^2\right).$$

Considering $u_H^0=\Lambda u^0$ and adding all equations for each n with $1\le n\le N$ and from Lemma 5, we get

$$\left\|\theta_1^n\right\|_1\le C_1\left\|\xi_1^n\right\|_0. \quad (3.13)$$

From the sum of (3.10) with $v=\frac{\theta_1^n-\theta_1^{n-1}}{\tau}$ and (3.12) with $q=\frac{\xi_1^n+\xi_1^{n-1}}{2}$, applying the Cauchy–Schwartz and Young inequality and Lemma 6, we obtain

$$\begin{aligned}\nu\left(\left\|\xi_1^n\right\|_0^2-\left\|\xi_1^{n-1}\right\|_0^2\right)&\le\left(\left[u^{n-1},u^{n-1}\right]p^{n-1}-\left[u_H^{n-1},u_H^{n-1}\right]p_H^{n-1},\theta_1^n-\theta_1^{n-1}\right)\\&+\left(\left[u^n,u^n\right]p^n-\left[u_H^n,u_H^n\right]p_H^n,\theta_1^n-\theta_1^{n-1}\right)+C\tau^4\int_{t_{n-1}}^{t_n}\|u_{ttt}\|_0^2\,\mathrm{d}t. \quad (3.14)\end{aligned}$$

From Theorem 1 and Theorem 2, using (3.6), (3.7), (3.13) and applying the Cauchy–Schwartz and Young inequality again, we obtain

$$\begin{aligned}&\left(\left[u^{n-1},u^{n-1}\right]p^{n-1}-\left[u_H^{n-1},u_H^{n-1}\right]p_H^{n-1},\theta_1^n-\theta_1^{n-1}\right)\\&\quad=\left(\left[u^{n-1},u^{n-1}\right](\rho_1^{n-1}+\xi_1^{n-1})\right.\\&\qquad\left.+\left[\phi_1^{n-1}+\theta_1^{n-1},\phi_1^{n-1}+\theta_1^{n-1}\right]p_H^{n-1},\theta_1^n-\theta_1^{n-1}\right)\\&\quad\le\frac{\epsilon_1M_0^2}{2}\left\|\rho_1^{n-1}+\xi_1^{n-1}\right\|^2+\frac{\epsilon_2M_1^2}{2}\left\|\phi_1^{n-1}+\theta_1^{n-1}\right\|^2\\&\qquad+\left(\frac{1}{2\epsilon_1}+\frac{1}{2\epsilon_2}\right)\left\|\theta_1^n-\theta_1^{n-1}\right\|^2\\&\quad\le CH^2\left[\left(\left\|p^0\right\|_1+\int_0^{t_{n-1}}\|p_t\|_1\,\mathrm{d}t\right)^2+\left(\left\|u^0\right\|_2+\int_0^{t_{n-1}}\|u_t\|_2\,\mathrm{d}t\right)^2\right]\\&\qquad+C\left\|\xi_1^{n-1}\right\|^2+\left(\frac{1}{2\epsilon_1}+\frac{1}{2\epsilon_2}\right)\left(1+\frac{1}{\epsilon_3}\right)C_1\left\|\xi_1^n\right\|^2. \quad (3.15)\end{aligned}$$

The second term of the right side of (3.14) is similar to (3.15). Here the constants of the Young inequality are chosen appropriately such that the coefficient of $\|\xi_1^n\|^2$ in the right side of (3.14) is less than ν. Combining (3.14) with (3.15), adding all equations for each n with $1 \le n \le N$ and from Lemma 5, we have

$$\|\xi_1^n\|_0 \le CH\Big(\|u^0\|_2 + \|p^0\|_1 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t + \int_0^{t_n} \|p_t\|_1 \,\mathrm{d}t \Big) \\ + C\tau^2 \Big(\int_0^{t_n} \|u_{ttt}\|_0^2 \,\mathrm{d}t \Big)^{\frac{1}{2}}. \tag{3.16}$$

Consequently, using (3.5), (3.7), (3.13), (3.16) and the triangle inequality, we complete the proof of (3.3). □

Furthermore, we need to prove (3.4). Taking $v = \frac{\theta_1^n - \theta_1^{n-1}}{\tau}$, $q = \nabla \frac{\theta_1^n - \theta_1^{n-1}}{\tau}$, applying the Cauchy–Schwartz and Young inequality, we obtain from the sum of (3.10) and (3.11) such that

$$\nu\big(\|\nabla\theta_1^n\|_0^2 - \|\nabla\theta_1^{n-1}\|_0^2\big) \le \big([u^{n-1}, u^{n-1}]p^{n-1} - [u_H^{n-1}, u_H^{n-1}]p_H^{n-1}, \theta_1^n - \theta_1^{n-1}\big) \\ + \big([u^n, u^n]p^n - [u_H^n, u_H^n]p_H^n, \theta_1^n - \theta_1^{n-1}\big) + C\tau^4 \int_{t_{n-1}}^{t_n} \|u_{ttt}\|_0^2 \,\mathrm{d}t. \tag{3.17}$$

From Theorem 1 and Theorem 2, using Green's formula, applying the Cauchy–Schwartz and Young inequality, we obtain

$$\begin{aligned} &\big([u^{n-1}, u^{n-1}]p^{n-1} - [u_H^{n-1}, u_H^{n-1}]p_H^{n-1}, \theta_1^n - \theta_1^{n-1}\big) \\ &\quad = \big([u_H^{n-1}, u_H^{n-1}]\nabla u_H^{n-1}, \theta_1^n - \theta_1^{n-1}\big) - \big([u^{n-1}, u^{n-1}]\nabla u^{n-1}, \theta_1^n - \theta_1^{n-1}\big) \\ &\quad = \big(\big[(u^{n-1})^2 - (u_H^{n-1})^2, (u^{n-1})^2 - (u_H^{n-1})^2\big], \nabla(\theta_1^n - \theta_1^{n-1})\big) \\ &\quad \le C\|\phi_1^{n-1}\|_0^2 + C\|\nabla\theta_1^{n-1}\|_0^2 + \frac{1}{2\delta_1}\Big(1 + \frac{1}{\delta_2}\Big)\|\nabla\theta_1^n\|_0^2. \end{aligned} \tag{3.18}$$

Similarly,

$$\begin{aligned} &\big([u^n, u^n]p^n - [u_H^n, u_H^n]p_H^n, \theta_1^n - \theta_1^{n-1}\big) \\ &\quad = \big([u_H^n, u_H^n]\nabla u_H^n, \theta_1^n - \theta_1^{n-1}\big) - \big([u^n, u^n]\nabla u^n, \theta_1^n - \theta_1^{n-1}\big) \\ &\quad \le \Big[\frac{\delta_3}{4}(M_0 + M_1)^2(1 + \delta_4) + \frac{1}{2\delta_3}(1 + \delta_5)\Big]\|\nabla\theta_1^n\|_0^2 \\ &\qquad + C\|\phi_1^n\|_0^2 + C\|\nabla\theta_1^{n-1}\|_0^2. \end{aligned} \tag{3.19}$$

Here δ_i $(i = 1, 2, \ldots, 5)$ are chosen appropriately such that the coefficient of $\|\nabla\theta_1^n\|_0^2$ on the right side of (3.17) is less than ν. Combining (3.17)–(3.19), using (3.6), adding all equations for each n with $1 \le n \le N$ and from Lemma 5, we have

$$\|\theta_1^n\|_0 \le CH^2\Big(\|u^0\|_2 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t\Big) + C\tau^2\Big(\int_0^{t_n} \|u_{ttt}\|_0^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{3.20}$$

Consequently, using (3.6), (3.20), and the triangle inequality, we complete the proof of (3.4).

Next, we can give the error estimates of Step 2.

Theorem 4. *Let $(p_h, u_h) \in V_h \times W_h$ be the solution of Eqs. (3.2), for the $P_0^2 - P_1$ finite element pair, there exists a positive constant C such that*

$$\begin{aligned}
&\left\|u^n - u_h^n\right\|_1 + \left\|p^n - p_h^n\right\|_0 \\
&\quad \le CH^2\biggl(\left\|u^0\right\|_2 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t\biggr) + C\tau^2\biggl(\int_0^{t_n} \|u_{ttt}\|_0^2 \,\mathrm{d}t\biggr)^{\frac{1}{2}} \\
&\qquad + Ch\biggl(\left\|u^0\right\|_2 + \left\|p^0\right\|_1 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t + \int_0^{t_n} \|p_t\|_1 \,\mathrm{d}t\biggr).
\end{aligned} \tag{3.21}$$

Proof. Let

$$\begin{aligned}
u^n - u_h^n &= u^n - \Lambda u^n + \Lambda u^n - u_h^n = \phi_2^n + \theta_2^n = \varepsilon_2^n, \\
p^n - p_h^n &= p^n - \Pi p^n + \Pi p^n - p_h^n = \rho_2^n + \xi_2^n = \eta_2^n.
\end{aligned}$$

From (2.6) and (2.8), we have

$$\left\|\phi_2^n\right\|_1 = \left\|u^n - \Lambda u^n\right\|_1 \le Ch\biggl(\left\|u^0\right\|_2 + \int_0^{t_n} \|u_t\|_2 \,\mathrm{d}t\biggr), \tag{3.22}$$

$$\left\|\rho_2^n\right\|_0 = \left\|p^n - \Pi p^n\right\|_0 \le Ch\biggl(\left\|p^0\right\|_1 + \int_0^{t_n} \|p_t\|_1 \,\mathrm{d}t\biggr).$$

Using (2.1) and (3.2), for any $q \in V_h$ and $v \in W_h$, we obtain the error equations as follows:

$$\begin{aligned}
&\left(\frac{\varepsilon_2^n - \varepsilon_2^{n-1}}{\tau}, v\right) - \nu\left(\frac{\eta_2^n + \eta_2^{n-1}}{2}, \nabla v\right) - (\varphi_H^n, v) - \bigl(\varphi_H^{n-1}, v\bigr) \\
&\quad = \left(\frac{u^n - u^{n-1}}{\tau} - u_t^{n-\frac{1}{2}}, v\right) - \left(\frac{u_t^n + u_t^{n-1}}{2} - u_t^{n-\frac{1}{2}}, v\right), \\
&\left(\frac{\eta_2^n + \eta_2^{n-1}}{2}, q\right) + \left(q, \nabla\frac{\varepsilon_2^n + \varepsilon_2^{n-1}}{2}\right) = 0.
\end{aligned}$$

From (2.4) and (2.9), we get

$$\begin{aligned}
&\left(\frac{\theta_2^n - \theta_2^{n-1}}{\tau}, v\right) - \nu\left(\frac{\xi_2^n + \xi_2^{n-1}}{2}, \nabla v\right) - (\varphi_H^n, v) - \bigl(\varphi_H^{n-1}, v\bigr) \\
&\quad = \left(\frac{u^n - u^{n-1}}{\tau} - u_t^{n-\frac{1}{2}}, v\right) - \left(\frac{u_t^n + u_t^{n-1}}{2} - u_t^{n-\frac{1}{2}}, v\right),
\end{aligned} \tag{3.23}$$

$$\left(\frac{\xi_2^n + \xi_2^{n-1}}{2}, q\right) + \left(q, \nabla\frac{\theta_2^n + \theta_2^{n-1}}{2}\right) = 0. \tag{3.24}$$

We consider

$$\left(\frac{\xi_2^n - \xi_2^{n-1}}{\tau}, q\right) + \left(q, \nabla\frac{\theta_2^n - \theta_2^{n-1}}{\tau}\right) = 0, \quad \forall q \in V_h,\ n = 1, 2, \ldots, N \tag{3.25}$$

instead of (3.24). From (3.24) and taking $q = \nabla\frac{\theta_2^n-\theta_2^{n-1}}{2}$, applying the Cauchy–Schwartz and Young inequality, we obtain

$$\begin{aligned}\left\|\nabla\theta_2^n\right\|_0^2 - \left\|\nabla\theta_2^{n-1}\right\|_0^2 &\le \left\|\xi_2^n+\xi_2^{n-1}\right\|_0\left\|\nabla\theta_2^n+\nabla\theta_2^{n-1}\right\|_0\\ &\le \frac{\|\xi_2^n+\xi_2^{n-1}\|_0^2}{2\delta}+\frac{\delta\|\nabla\theta_2^n+\nabla\theta_2^{n-1}\|_0^2}{2}.\end{aligned}$$

Choosing $\delta \ge 0$ such that $1-\delta>0$ and due to Poincaré inequality, we have

$$\left\|\theta_2^n\right\|_1^2 \le C\big(\left\|\theta_2^{n-1}\right\|_1^2+\left\|\xi_2^n\right\|_0^2\big).$$

Considering $u_h^0 = \Lambda u^0$ and adding all equations for each n with $1\le n\le N$ and from Lemma 5, we get

$$\left\|\theta_2^n\right\|_1 \le C_1\left\|\xi_2^n\right\|_0. \tag{3.26}$$

From the sum of (3.23) with $v=\frac{\theta_2^n-\theta_2^{n-1}}{\tau}$ and (3.25) with $q=\frac{\xi_2^n+\xi_2^{n-1}}{2}$, applying the Cauchy–Schwartz and Young inequality and Lemma 6, we obtain

$$\begin{aligned}\nu\big(\left\|\xi_2^n\right\|_0^2-\left\|\xi_2^{n-1}\right\|_0^2\big) &\le \big([u^{n-1},u^{n-1}]p^{n-1}-[u_H^{n-1},u_H^{n-1}]p_H^{n-1},\theta_2^n-\theta_2^{n-1}\big)\\ &+\big(\big[u^n,u^n\big]p^n-\big[u_H^n,u_H^n\big]p_H^n,\theta_2^n-\theta_2^{n-1}\big)+C\tau^4\int_{t_{n-1}}^{t_n}\|u_{ttt}\|_0^2\,\mathrm{d}t. \end{aligned}\tag{3.27}$$

From Theorem 1 and Theorem 2, using (3.26) and Green's formula, applying the Cauchy–Schwartz and Young inequality again, we obtain

$$\begin{aligned}&\big([u^{n-1},u^{n-1}]p^{n-1}-[u_H^{n-1},u_H^{n-1}]p_H^{n-1},\theta_2^n-\theta_2^{n-1}\big)\\ &=\big([u_H^{n-1},u_H^{n-1}]\nabla u_H^{n-1},\theta_2^n-\theta_2^{n-1}\big)-\big([u^{n-1},u^{n-1}]\nabla u^{n-1},\theta_2^n-\theta_2^{n-1}\big)\\ &=\big([(u^{n-1})^2-(u_H^{n-1})^2,(u^{n-1})^2-(u_H^{n-1})^2],\nabla(\theta_2^n-\theta_2^{n-1})\big)\\ &\le C\left\|u^{n-1}-u_H^{n-1}\right\|_0^2+\delta_1C_1\left\|\xi_2^n\right\|_0^2+C\left\|\xi_2^{n-1}\right\|_0^2. \end{aligned}\tag{3.28}$$

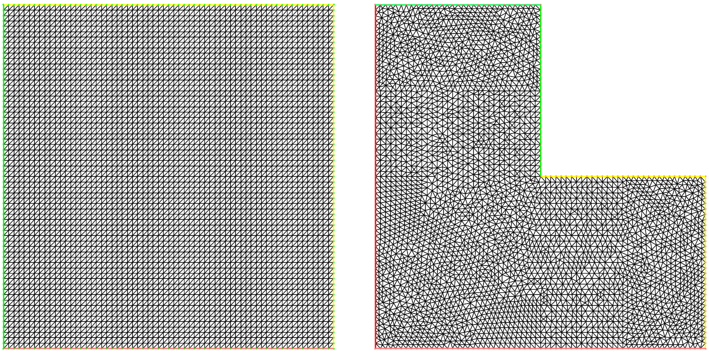
Similarly,

$$\begin{aligned}&\big([u^n,u^n]p^n-[u_H^n,u_H^n]p_H^n,\theta_2^n-\theta_2^{n-1}\big)\\ &=\big([u_H^n,u_H^n]\nabla u_H^n,\theta_2^n-\theta_2^{n-1}\big)-\big([u^n,u^n]\nabla u^n,\theta_2^n-\theta_2^{n-1}\big)\\ &\le C\left\|u^n-u_H^n\right\|_0^2+\delta_2C_1\left\|\xi_2^n\right\|_0^2+C\left\|\xi_2^{n-1}\right\|_0^2. \end{aligned}\tag{3.29}$$

Here δ_i $(i=1,2)$ are chosen appropriately such that the coefficient of $\left\|\xi_2^n\right\|^2$ in the right side of (3.27) is less than ν. Combining (3.27)–(3.29), using (3.4), adding all equations for each n with $1\le n\le N$ and from Lemma 5, we have

$$\left\|\xi_2^n\right\|_0 \le CH^2\bigg(\left\|u^0\right\|_2+\int_0^{t_n}\|u_t\|_2\,\mathrm{d}t\bigg)+C\tau^2\bigg(\int_0^{t_n}\|u_{ttt}\|_0^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}. \tag{3.30}$$

Consequently, using (3.22), (3.26), (3.30) and the triangle inequality, we complete the proof of (3.21). $\square$



(a) (b)

Figure 1. Domain. (a) the base triangles of Example 1; (b) the base triangles of Example 2.

Table 1. Relative error and convergence rate of the two-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 1$.

$1/H$	$1/h$	$\frac{\\|u-u_h\\|_1}{\\|u\\|_1}$	u_{H^1}-rate	$\frac{\\|p-p_h\\|_0}{\\|p\\|_0}$	p_{L^2}-rate	CPU time
4	16	0.1037240	—	0.0966421	—	0.344s
6	36	0.0463592	0.9931	0.0430196	0.9981	2.64s
8	64	0.0261144	0.9975	0.0242029	0.9997	10.578s
10	100	0.0169334	0.9707	0.0156916	0.9710	32.515s
12	144	0.0116199	1.0327	0.0107611	1.0344	83.531s

4 Numerical Experiments

In this section, we report two test problems for 2D Burgers' equation using a new mixed finite element method based on the Crank–Nicolson scheme with $H^2 = h$ in unite-square domain and L-shape domain respectively. The accuracy and the numerical stability of our method are checked, then we compare the results obtained by our method with those obtained by one-grid method. Our algorithms are implemented using the public domain finite element software [12].

Example 1. The exact solution u is given as follows:

$$u = \cos(t)x(x-1)y(y-1).$$

The initial condition in (1.2) is set according to the exact solution and the right-hand side $f(x, y, t)$ determined by (1.1). Here, the final time $T = 1$. In this experiment, $\Omega = [0, 1] \times [0, 1]$ in R^2. The mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square. Fig. 1 (a) gives the unite-square domain $[0, 1] \times [0, 1]$ in R^2.

In Tables 1–6, we show relative errors and the convergence of two-grid method and one-grid method when we take $\tau^2 = h$, $\nu = 1, 0.1, 0.01$ for $P_0^2 - P_1$

Table 2. Relative error and convergence rate of the one-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 1$.

$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
16	0.1018810	—	0.0942280	—	0.375s
36	0.0453472	0.9982	0.0418430	1.0011	2.953s
64	0.0255155	0.9995	0.0235345	1.0002	11.812s
100	0.0164971	0.9772	0.0152148	0.9774	36.875s
144	0.0113430	1.0273	0.0104611	1.0273	93.515s

Table 3. Relative error and convergence rate of the two-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 0.1$.

$1/H$	$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
4	16	0.1048050	—	0.0975728	—	0.344s
6	36	0.0471750	0.9843	0.0438312	0.9868	2.672s
8	64	0.0268127	0.9820	0.0249302	0.9807	10.672s
10	100	0.0174972	0.9564	0.0162864	0.9540	32.672s
12	144	0.0121805	0.9933	0.0113550	0.9891	84.078s

Table 4. Relative error and convergence rate of the one-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 0.1$.

$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
16	0.1021570	—	0.0945552	—	0.375s
36	0.0456450	0.9934	0.0421535	0.9962	2.906s
64	0.0258208	0.9902	0.0238557	0.9895	11.687s
100	0.0167999	0.9631	0.0155349	0.9611	36.078s
144	0.0116479	1.0044	0.0107838	1.0011	92.485s

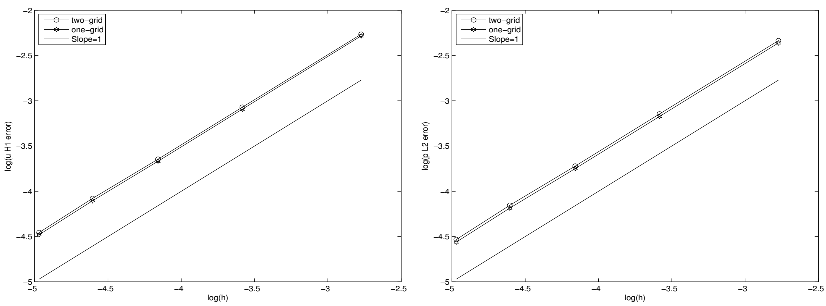
Table 5. Relative error and convergence rate of the two-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 0.01$.

$1/H$	$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
4	16	0.2284800	—	0.2246870	—	0.344s
6	36	0.1177040	0.8179	0.1162330	0.8128	2.625s
8	64	0.0700781	0.9013	0.0692935	0.8990	10.578s
10	100	0.0464995	0.9191	0.0459989	0.9181	32.547s
12	144	0.0330057	0.9400	0.0326662	0.9387	82.547s

finite element pair based on the Crank–Nicolson scheme in time, respectively. We obtain the optimal error estimates in Theorem 4. And the two methods keep the same convergence rates. We also give the CPU time of two methods in Tables 1–6. From these tables, we know that computing the Burgers' equation by using two-grid method is less than by using one-grid method in CPU time. Obviously, the computed time of our method is not much less than the one-grid method.

Table 6. Relative error and convergence rate of the one-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 0.01$.

$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
16	0.1047980	—	0.0972054	—	0.359s
36	0.0483947	0.9528	0.0450431	0.9485	2.922s
64	0.0285286	0.9185	0.0267061	0.9085	11.687s
100	0.0194087	0.8631	0.0182741	0.8501	36.109s
144	0.0141890	0.8591	0.0134418	0.8422	92.562s



(a) (b)

Figure 2. Convergence analysis using one- and two-grid methods for $\nu = 1$. (a) H^1 error for the velocity; (b) L^2 error for the flux.

According to the numerical results in Tables 1–2, the velocity of the H^1 norm error convergence order and flux of the L^2 norm error convergence order are shown in Fig. 2 by using the different methods. From Fig. 2 we can see that the convergence orders of the two methods are substantially coincident, and this shows that the results are reasonable. Moreover, we take $1/h = 100$ and $1/H = 10$.

In Fig. 3, we give plots of the numerical solutions of the velocity and pressure which are obtained by using one-grid method and two-grid method when we take $x = 0.25$ in square domain. Seen from this figure, there are not any negative effect for the Burgers' equation by two methods in the range of allowable error if compared with the exact solution. In brief, our method can get the same convergence rate of the one-grid method and with less time.

Example 2. We take an example of the Burgers' equation (1.1)–(1.3) the right-hand side $f(x, y, t)$ of which is determined by the exact solution u of trigonometric function:

$$u = \left(t^2 + 1\right) \sin(2\pi x) \sin(2\pi y).$$

In this experiment, Ω is the L-shape domain $[0, 1] \times [0, 1]$ in R^2, are shown in Fig. 1 (b). The final time $T = 1$. In Tables 7–8, relative errors and the convergence of two-grid method and one-grid method for $P_0^2 - P_1$ finite element pair based on the Crank–Nicolson scheme in time, respectively. And the two methods keep the same convergence rates. We also give the CPU time of two

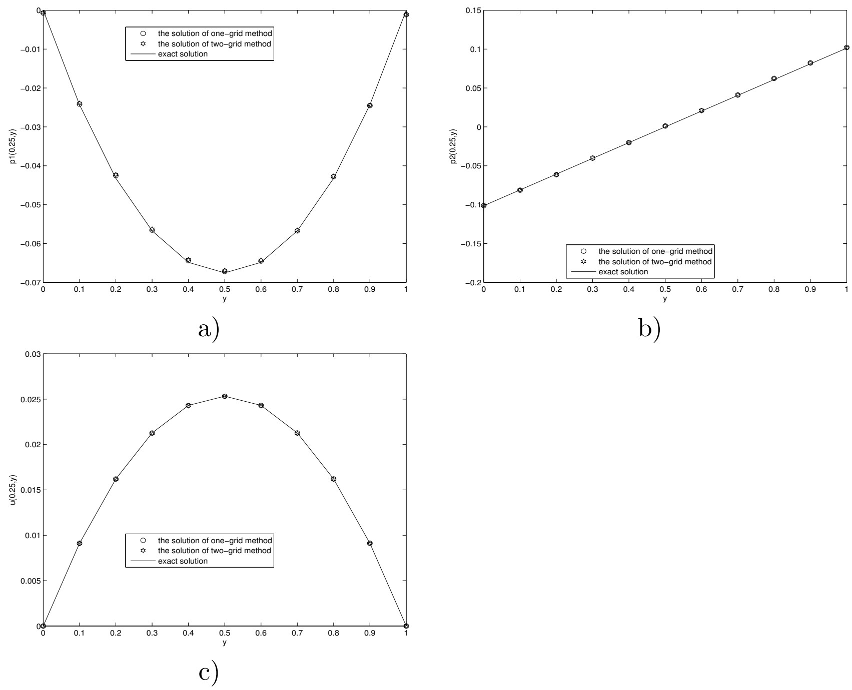


Figure 3. Numerical solutions and exact solutions for $\nu = 1$. (a) the solutions of p_1, (b) the solutions of p_2, (c) the solutions of u.

Table 7. Relative error and convergence rate of the two-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 1$.

$1/H$	$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
4	16	0.2177770	—	0.1953060	—	0.344s
6	36	0.0927220	1.0529	0.0823992	1.0642	2.406s
8	64	0.0539628	0.9408	0.0484577	0.9227	10.015s
10	100	0.0349448	0.9736	0.0315188	0.9637	30.672s
12	144	0.0245384	0.9695	0.0222284	0.9577	77.078s

methods in Tables 7–8, and are shown in Fig. 4. From the plot, we can see that the CPU time of two-grid method for solving the Burgers' equation is much shorter than the CPU time for the one-grid method.

5 Conclusions

In this work, we have extended and studied the two-grid method for 2D Burgers' equation discretized by a new mixed finite element method based on the Crank–Nicolson scheme. A priori error estimate has been derived and numerical results agreeing with the estimates have been presented. Obviously, this method can be expanded to the case of three dimensions. And further developments can

Table 8. Relative error and convergence rate of the one-grid method for the velocity and flux with $\tau^2 = h$ and $\nu = 1$.

$1/h$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	u_{H^1}-rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	p_{L^2}-rate	CPU time
16	0.1633090	—	0.1340180	—	0.375s
36	0.0730782	0.9916	0.0599128	0.9928	5.203s
64	0.0411831	0.9968	0.0338208	0.9938	21.922s
100	0.0262965	1.0052	0.0216241	1.0022	67.234s
144	0.0182218	1.0060	0.0150274	0.9981	168.562s

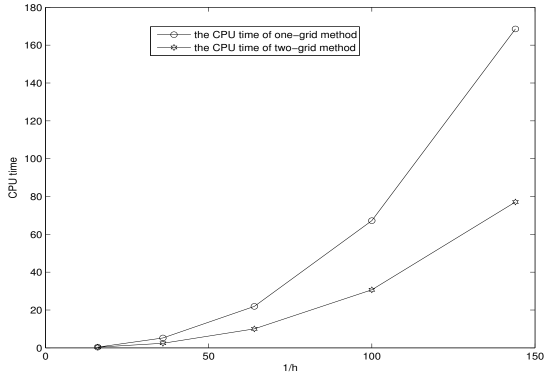


Figure 4. CPU time analysis using one- and two-grid methods.

extend these techniques and ideas to the other nonlinear problems, for example, Cahn–Hilliard equation, MBE models, etc. see [16, 25, 28] and the references therein. Furthermore, the $P_0^2 - P_1$ pair combined with the LDG method can be expanded to solve a shock problem.

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