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Tauberian Remainder Theorems for the Weighted Mean Method of Summability

Sefa Anıl Sezer[a,b] **and İbrahim Çanak**[a]

[a] *Department of Mathematics, Ege University*
35100 İzmir, Turkey
[b] *Department of Mathematics, Istanbul Medeniyet University*
34720 İstanbul, Turkey
E-mail(*corresp.*): sefaanilsezer@gmail.com
E-mail: sefaanil.sezer@medeniyet.edu.tr
E-mail: ibrahimcanak@yahoo.com
E-mail: ibrahim.canak@ege.edu.tr

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Abstract. Using the weighted general control modulo, we prove several Tauberian remainder theorems for the weighted mean method of summability. Our results generalize the results proved by Meronen and Tammeraid [Math. Model. Anal. 18 (1) 2013, 97–102].

Keywords: weighted general control modulo, Tauberian remainder theorem, weighted mean method, λ-bounded sequence.

AMS Subject Classification: 40C05; 40E05; 40G05.

1 Introduction

Let $x = \{\xi_n\}$ be a sequence of real numbers and any term with a nonnegative index be zero. Assume that $p = \{p_n\}$ is a sequence of nonnegative numbers with $p_0 > 0$ such that

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as } n \to \infty.$$

The n^{th} weighted mean of the sequence $x = \{\xi_n\}$ is defined by

$$\sigma_{n,p}^{(1)}(x) := \frac{1}{P_n} \sum_{k=0}^{n} p_k \xi_k.$$

A sequence $x = \{\xi_n\}$ is said to be summable by the weighted mean method determined by the sequence p, in short; $(\overline{N}, p)$ summable to a finite number $\sigma(x)$

if

$$\lim_{n\to\infty} \sigma^{(1)}_{n,p}(x) = \sigma(x).$$

If $p_n = 1$ for all nonnegative n, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

The weighted Kronecker identity [1] is given by

$$\xi_n - \sigma^{(1)}_{n,p}(x) = V^{(0)}_{n,p}(\Delta x), \tag{1.1}$$

where

$$V^{(0)}_{n,p}(\Delta x) := \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} \Delta \xi_k.$$

The weighted classical control modulo of $\{\xi_n\}$ is given by

$$\omega^{(0)}_{n,p}(x) = \frac{P_{n-1}}{p_n} \Delta \xi_n$$

and the weighted general control modulo [1] of integer order $m \geq 1$ of $\{\xi_n\}$ is defined by

$$\omega^{(m)}_{n,p}(x) = \omega^{(m-1)}_{n,p}(x) - \sigma^{(1)}_{n,p}\big(\omega^{m-1}(x)\big).$$

For each integer $m \geq 0$, we define $\sigma^{(m)}_{n,p}(x)$ by

$$\sigma^{(m)}_{n,p}(x) = \begin{cases} \dfrac{1}{P_n} \displaystyle\sum_{k=0}^{n} p_k \sigma^{(m-1)}_{k,p}(x), & m \geq 1, \\ \xi_n, & m = 0. \end{cases}$$

In the recent years different classes of sequences associated with multiplier sequences have been introduced and their different algebraic and topological properties have been investigated by researchers with specific objectives. The studies on sequence spaces associated with multiplier sequences was started by Goes and Goes [3], followed by Tripathy and Mahanta [14], Tripathy and Hazarika [13], Tripathy and Chandra [12] and others.

A sequence $x = \{\xi_n\}$ is called λ-bounded with the rapidty $\lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ if

$$\lambda_n(\xi_n - \xi) = O(1)$$

with $\lim \xi_n = \xi$. We denote the set of all λ-bounded sequences by m^λ.

A sequence $x = \{\xi_n\}$ is called λ-bounded by the weighted mean method of summability $(\overline{N}, p)$ if $(\overline{N}, p)x$ is λ-bounded. This is equivalent to saying that

$$\lambda_n\big(\sigma^{(1)}_{n,p}(x) - \sigma(x)\big) = O(1)$$

with

$$\sigma^{(1)}_{n,p}(x) := \frac{1}{P_n} \sum_{k=0}^{n} p_k \xi_k \quad \text{and} \quad \lim_{n\to\infty} \sigma^{(1)}_{n,p}(x) = \sigma(x).$$

In short, we write $x \in \big((\overline{N}, p), m^\lambda\big)$.

A number of authors including Kangro [4], Tammeraid [11] and Šeletski and Tali [10] proved Tauberian remainder theorems for some summability methods using summability with given rapidity. Meronen and Tammaraid [5] obtained a Tauberian condition under which λ-boundedness of $\{\xi_n\}$ transformed by the generalized Euler–Knopp method implies λ-boundedness of $\{\xi_n\}$. In [6], they obtained sufficient conditions for the generalized Nörlund method and Tauberian conditions to deduce $x \in m_X^\lambda$ from $Nx \in m_X^\lambda$. Meronen and Tammaraid [7] also proved gap Tauberian theorems for the generalized linear methods. Moreover, they proved some results on λ-summable series by the Cesàro method of order one and by the weighted mean method $(\overline{N}, p)$ in [8]. Recently, Meronen and Tammeraid [9] have proved Tauberian remainder theorems for Cesàro summability method using the concept of the general control modulo of integer order $m \geq 1$, defined by Dik [2]. In this paper we obtain some Tauberian conditions to deduce $x \in m^\lambda$ from $(\overline{N}, p)x \in m^\lambda$. Our results generalize the results proved by Meronen and Tammeraid [9].

2 Tauberian Remainder Theorems For $(\overline{N}, p)$

Using the classical control modulo, we prove the following Tauberian theorem for $(\overline{N}, p)$ summability method.

Theorem 1. *Let* $x \in ((\overline{N}, p), m^\lambda)$. *If*

$$\lambda_n V_{n,p}^{(0)}(\Delta x) = O(1), \tag{2.1}$$

then $x \in m^\lambda$.

Proof. Assume that $x \in ((\overline{N}, p), m^\lambda)$. By the weighted Kronecker identity

$$\xi_n - \sigma_{n,p}^{(1)}(x) = V_{n,p}^{(0)}(\Delta x),$$

we have

$$\lambda_n\big(\xi_n - \sigma(x)\big) = \lambda_n\big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big) + \lambda_n V_{n,p}^{(0)}(\Delta x).$$

Taking (2.1) into account, we obtain $x \in m^\lambda$. □

Lemma 1. *The following assertion is valid:*

$$\omega_{n,p}^{(1)}(x) = \omega_{n,p}^{(0)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x). \tag{2.2}$$

Proof. By the weighted general control modulo of integer order $m = 1$ and (1.1), we have

$$\begin{aligned} \omega_{n,p}^{(1)}(x) &= \omega_{n,p}^{(0)}(x) - \sigma_{n,p}^{(1)}\big(\omega^{(0)}(x)\big) \\ &= \omega_{n,p}^{(0)}(x) - \frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{P_{k-1}}{p_k}\Delta\xi_k \\ &= \omega_{n,p}^{(0)}(x) - V_{n,p}^{(0)}(\Delta x) = \omega_{n,p}^{(0)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x). \end{aligned}$$

□

Theorem 2. *Let $x \in ((\overline{N},p),m^{\lambda})$. If the conditions*

$$\lambda_n \omega_{n,p}^{(0)}(x) = O(1), \quad \lambda_n \omega_{n,p}^{(1)}(x) = O(1) \tag{2.3}$$

are satisfied, then $x \in m^{\lambda}$.

Proof. Assume that $x \in ((\overline{N},p),m^{\lambda})$. By Lemma 1, we have

$$\xi_n - \sigma(x) = \omega_{n,p}^{(0)}(x) - \omega_{n,p}^{(1)}(x) + \big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big). \tag{2.4}$$

It follows from (2.4) that

$$\lambda_n \big(\xi_n - \sigma(x)\big) = \lambda_n \omega_{n,p}^{(0)}(x) - \lambda_n \omega_{n,p}^{(1)}(x) + \lambda_n \big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big).$$

Taking (2.3) into account, we obtain $x \in m^{\lambda}$. □

The next statement gives an identity for the weighted general control modulo of integer order $m = 2$ in terms of the classical control modulo and the n^{th} weighted means of x.

Lemma 2. *The following assertion is valid:*

$$\omega_{n,p}^{(2)}(x) = \omega_{n,p}^{(0)}(x) - 2\xi_n + 3\sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x). \tag{2.5}$$

Proof. By the weighted general control modulo of integer order $m = 2$ and (2.2), we have

$$\begin{aligned}
\omega_{n,p}^{(2)}(x) &= \omega_{n,p}^{(1)}(x) - \sigma_{n,p}^{(1)}\big(\omega^{(1)}(x)\big) \\
&= \omega_{n,p}^{(0)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x) - \frac{1}{P_n}\sum_{k=0}^{n} p_k \big(\omega_{k,p}^{(0)}(x) - \xi_k + \sigma_{k,p}^{(1)}(x)\big) \\
&= \omega_{n,p}^{(0)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x) - \frac{1}{P_n}\sum_{k=0}^{n} p_k \omega_{k,p}^{(0)}(x) + \frac{1}{P_n}\sum_{k=0}^{n} p_k \xi_k \\
&\quad - \frac{1}{P_n}\sum_{k=0}^{n} p_k \sigma_{k,p}^{(1)}(x) \\
&= \omega_{n,p}^{(0)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x) - V_{n,p}^{(0)}(\Delta x) + \sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x) \\
&= \omega_{n,p}^{(0)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x) + \sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x) \\
&= \omega_{n,p}^{(0)}(x) - 2\xi_n + 3\sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x).
\end{aligned}$$

□

Theorem 3. *Let $x \in ((\overline{N},p),m^{\lambda})$. If the conditions* (2.3),

$$\lambda_n \omega_{n,p}^{(2)}(x) = O(1), \tag{2.6}$$

$$\lambda_n \big(\sigma_{n,p}^{(2)}(x) - \sigma(x)\big) = O(1) \tag{2.7}$$

are satisfied, then $x \in m^{\lambda}$.

Proof. Assume that $x \in ((\overline{N}, p), m^{\lambda})$. By Lemma 2, we have

$$2\big(\xi_n - \sigma(x)\big) = \omega_{n,p}^{(0)}(x) - \omega_{n,p}^{(2)}(x) + 3\big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big) - \big(\sigma_{n,p}^{(2)}(x) - \sigma(x)\big).$$

Then it follows that

$$2\lambda_n\big(\xi_n - \sigma(x)\big) = \lambda_n\big(\omega_{n,p}^{(0)}(x) - \omega_{n,p}^{(2)}(x) + 3\big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big) - \big(\sigma_{n,p}^{(2)}(x) - \sigma(x)\big)\big).$$

Taking (2.3) and (2.6) and (2.7) into account, we obtain $x \in m^{\lambda}$. □

Lemma 3. *The following assertion is valid:*

$$\omega_{n,p}^{(3)}(x) = \omega_{n,p}^{(0)}(x) - 3\xi_n + 6\sigma_{n,p}^{(1)}(x) - 4\sigma_{n,p}^{(2)}(x) + \sigma_{n,p}^{(3)}(x). \tag{2.8}$$

Proof. By the weighted general control modulo of integer order $m = 3$ and (2.5), we have

$$\begin{aligned}
\omega_{n,p}^{(3)}(x) &= \omega_{n,p}^{(2)}(x) - \sigma_{n,p}^{(1)}\big(\omega^{(2)}(x)\big) = \omega_{n,p}^{(0)}(x) - 2\xi_n + 3\sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x) \\
&\quad - \frac{1}{P_n}\sum_{k=0}^{n} p_k\big(\omega_{k,p}^{(0)}(x) - 2\xi_k + 3\sigma_{k,p}^{(1)}(x) - \sigma_{k,p}^{(2)}(x)\big) \\
&= \omega_{n,p}^{(0)}(x) - 2\xi_n + 3\sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x) - \frac{1}{P_n}\sum_{k=0}^{n} p_k\omega_{k,p}^{(0)}(x) + \frac{2}{P_n}\sum_{k=0}^{n} p_k\xi_k \\
&\quad - \frac{3}{P_n}\sum_{k=0}^{n} p_k\sigma_{k,p}^{(1)}(x) + \frac{1}{P_n}\sum_{k=0}^{n} p_k\sigma_{k,p}^{(2)}(x) \\
&= \omega_{n,p}^{(0)}(x) - 2\xi_n + 3\sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x) - V_{n,p}^{(0)}(\Delta x) + 2\sigma_{n,p}^{(1)}(x) \\
&\quad - 3\sigma_{n,p}^{(2)}(x) + \sigma_{n,p}^{(3)}(x) \\
&= \omega_{n,p}^{(0)}(x) - 2\xi_n + 3\sigma_{n,p}^{(1)}(x) - \sigma_{n,p}^{(2)}(x) - \xi_n + \sigma_{n,p}^{(1)}(x) + 2\sigma_{n,p}^{(1)}(x) \\
&\quad - 3\sigma_{n,p}^{(2)}(x) + \sigma_{n,p}^{(3)}(x) \\
&= \omega_{n,p}^{(0)}(x) - 3\xi_n + 6\sigma_{n,p}^{(1)}(x) - 4\sigma_{n,p}^{(2)}(x) + \sigma_{n,p}^{(3)}(x).
\end{aligned}$$

□

Theorem 4. *Let* $x \in ((\overline{N}, p), m^{\lambda})$*. If the conditions* (2.3), (2.7),

$$\lambda_n\omega_{n,p}^{(3)}(x) = O(1), \quad \lambda_n\big(\sigma_{n,p}^{(3)}(x) - \sigma(x)\big) = O(1) \tag{2.9}$$

are satisfied, then $x \in m^{\lambda}$.

Proof. Assume that $x \in ((\overline{N}, p), m^{\lambda})$. By Lemma 3, we have

$$\begin{aligned}
3\big(\xi_n - \sigma(x)\big) &= \omega_{n,p}^{(0)}(x) - \omega_{n,p}^{(3)}(x) + 6\big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big) - 4\big(\sigma_{n,p}^{(2)}(x) - \sigma(x)\big) \\
&\quad + \big(\sigma_{n,p}^{(3)}(x) - \sigma(x)\big).
\end{aligned}$$

Then it follows that

$$\begin{aligned}
3\lambda_n\big(\xi_n - \sigma(x)\big) &= \lambda_n\omega_{n,p}^{(0)}(x) - \lambda_n\omega_{n,p}^{(3)}(x) + 6\lambda_n\big(\sigma_{n,p}^{(1)}(x) - \sigma(x)\big) \\
&\quad - 4\lambda_n\big(\sigma_{n,p}^{(2)}(x) - \sigma(x)\big) + \lambda_n\big(\sigma_{n,p}^{(3)}(x) - \sigma(x)\big).
\end{aligned}$$

Taking (2.3), (2.7) and (2.9) into account, we obtain $x \in m^{\lambda}$. □

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