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Asymptotical Analysis of Some Coupled Nonlinear Wave Equations

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Abstract. We consider coupled nonlinear equations modelling a family of travelling wave solutions. The goal of our work is to show that the method of internal averaging along characteristics can be used for wide classes of coupled non-linear wave equations such as Korteweg-de Vries, Klein – Gordon, Hirota – Satsuma, etc. The asymptotical analysis reduces a system of coupled non-linear equations to a system of integro – differential averaged equations. The averaged system with the periodical initial conditions disintegrates into independent equations in non-resonance case. These equations describe simple weakly non-linear travelling waves in the non-resonance case. In the resonance case the integro – differential averaged systems describe interaction of waves and give a good asymptotical approximation for exact solutions.

Keywords: Non-linear waves, resonances, averaging, asymptotical integration.

AMS Subject Classification: 35L60; 34E13.

1 Introduction

It is well-known that (see e. g., [5, 27]) Burgers, Korteweg – de Vries, Klein – Gordon and other nonlinear evolution and wave equations are obtained from mathematical models of real physical phenomena in gas and fluid dynamics, acoustics, nonlinear optics, plasma physics, etc. For example, the method of construction of asymptotic expansions has been presented in [25, 26] (see also [6]). The basic idea of this method is to reduce a class of nonlinear partial differential equations to independent nonlinear equations such as Burgers and Korteweg – de Vries. Applications of this method in the hydrodynamics and the plasma physics were discussed in [25, 26]. A rigorous mathematical analysis of the deriving the nonlinear equations from systems of partial differential equations has been presented in [11]. Since 80's the coupled systems of nonlinear evolution and wave equations are considered as important mathematical models. They are used to describe various physical phenomena. For example, in [8] it is shown how such coupled Korteweg – de Vries type system as Ito, Kaup – Boussinesq, Broer – Kaup system, Hirota – Satsuma system, Nutku – Oguz and others can be derived from the models describing flows in geodesic. In [19], the system of two coupled nonlinear Klein – Gordon equations describes the dynamics of a twisted elastic rod.

There are various aspects of investigation of the nonlinear coupled systems. In this paper we consider coupled nonlinear equations, which can be transformed into the following form:

$$u'_{it} + \lambda_i u'_{ix} = \varepsilon f_i \left(u_1, \ldots, u_n, \ldots, u'_{jx}, \ldots, u''_{kxx}, \ldots \right), \tag{1.1}$$

where ε is a small parameter. On the one hand, we can give an objective context to the small parameter ε, for example, such as Mach, Reynolds, Rossby and other known in wave theory numbers (for more reasonings see [16]) and on the other hand the ε can be an abstract mathematical parameter (for example, a measure of weakness of dispersion and nonlinearity for equations in [3, 4]).

Let us first consider the Korteweg–de Vries equation

$$u_t - u u_x + u_{xxx} = 0. \tag{1.2}$$

We use the transformation:

$$u(t,x) = u_0 + \tilde{\varepsilon} u_1 \left(\overline{t}, \overline{x}; \tilde{\varepsilon} \right), \ \overline{t} = \tilde{\varepsilon}^{\alpha} t, \overline{x} = \tilde{\varepsilon}^{\alpha} x \tag{1.3}$$

and obtain the equation

$$\tilde{\varepsilon}^{\alpha+1} u'_{1\overline{t}} - \tilde{\varepsilon}^{\alpha+1} u_0 u'_{1\overline{x}} + \tilde{\varepsilon}^{1+3\alpha} u'''_{1\overline{x}\,\overline{x}\,\overline{x}} = 0. \tag{1.4}$$

Therefore with $\lambda = -u_0$, $\alpha = \frac{1}{2}$, $f = -u'_{1\overline{x}\,\overline{x}\,\overline{x}}$ and $\varepsilon = \tilde{\varepsilon}^2$ we have equation given in form (1.1).

The other class of problems, which can be transformed to (1.1) form, is analyzed in Section 3. Let us say that we have equation

$$u_{tt} - u_{xx} = \varepsilon f \left(u_t, u_x \right). \tag{1.5}$$

Let take $u_t = r_1, u_x = r_2$, then

$$\begin{cases} r_{2t} - r_{1x} = 0, \\ r_{1t} - r_{2x} = \varepsilon f \left(r_1, r_2 \right). \end{cases} \tag{1.6}$$

Equation (1.6) can be rewritten in the form (1.1).

Let us notice that function f in equation (1.5) can depend not only on u_t, u_x, but also on function u:

$$u_{tt} - u_{xx} = \varepsilon f \left(u, u_t, u_x \right). \tag{1.7}$$

We denote $u_t + u_x = \tilde{\varepsilon} U$, where $\tilde{\varepsilon} = \sqrt{\varepsilon}$, then equation (1.7) has the following form:

$$\begin{cases} u_t + u_x = \tilde{\varepsilon} U, \\ U_t - U_x = \tilde{\varepsilon} f(u). \end{cases} \tag{1.8}$$

It is easy to see that nonperturbed (with $\varepsilon = 0$) system (1.1) describes independent travelling waves $u_i = \varphi_i(t - \lambda_i t)$. Perturbed system (1.1) usually has differentiable solution $u_i(t, x; \varepsilon) \in C^p(\Omega_\varepsilon)$, where Ω_ε is a large domain as $\varepsilon \to 0$: $\Omega_\varepsilon = \{(t, x) : \ t + |x| = O(\varepsilon^{-1})\}$. The construction of uniformly valid asymptotic solutions of system (1.1) in the domain Ω_ε is a nontrivial problem of asymptotic integration. It is particularly complicated in the periodical case [17].

The periodical problems with quadratic non-linearity are reduced to analogical averaged integro – differential systems [2, 20, 21, 24]. A general form of non-linearity requires special analysis. In this case the relation of dispersion and additional requirements for solutions should be studied (for example, in [19] a coupled Klein – Gordon system is investigated, in [12], non-linear waves in typical mechanical systems are analyzed; in [22], an analysis of the four-wave resonant interactions in shallow water is presented).

Our method doesn't require special limitations for non-linearity type and allows to construct the averaged systems using general averaging scheme. In general case, the analysis of asymptotic methods is complicated. Usually the theorems of existence and uniqueness can not be proved. Therefore the construction of asymptotic expansions without secular terms are the main result for many problems. It is important to note that the obtained averaged systems do not have problems of asymptotic integration for a long time interval. The theorems of existence and uniqueness of exact and asymptotic solution and their accuracy estimates in a long time interval are proved [11, 15].

In this paper some quite non-trivial nonlinear problems are analyzed, therefore a full asymptotic proof is not done there. However, the constructed asymptotic expansions do not have secular terms and they are uniformly valid in a long time interval in both, resonance and non-resonance cases. Also our method allows to construct higher order expansions.

2 Multicomponent Korteweg – de Vries equation

We consider weakly nonlinear coupled Korteweg – de Vries equation with dispersion [28], which was introduced in [18]:

$$\begin{cases} \dfrac{\partial u}{\partial t} - 6u_0 \dfrac{\partial u}{\partial x} - 2 \sum\limits_{j=1}^{n} v_{0j} \dfrac{\partial v_j}{\partial x} = \varepsilon f_u[u, v], \ 0 < \varepsilon \ll 1, \\ \dfrac{\partial v_j}{\partial t} - 2u_0 \dfrac{\partial v_j}{\partial x} - 2v_{0j} \dfrac{\partial u}{\partial x} = \varepsilon f_j[u, v], \ j = 1, 2, \ldots, n, \end{cases} \tag{2.1}$$

where the right–hand side of (2.1) is given by

$$f_u\left[u,v\right]=a_u u\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^3}+b_u\sum_{j=1}^{n}v_j\frac{\partial v_j}{\partial x}+\sum_{j=1}^{n}c_j\frac{\partial^2 v_j}{\partial x^2},$$
$$f_j\left[u,v\right]=a_j u\frac{\partial v_j}{\partial x}+b_j v_j\frac{\partial u}{\partial x}+d_j\frac{\partial^2 u}{\partial x^2}.$$

System (2.1) is hyperbolic and can be rewritten in Riemann invariants

$$\frac{\partial r_j}{\partial t}-\lambda_j\frac{\partial r_j}{\partial x}=\varepsilon F_j\left[r,r_x,r_{xx},r_{xxx}\right],\ j=1,2,\ldots,n,n+1, \tag{2.2}$$

where

$$u(t,x;\varepsilon)=u_0+\varepsilon u_1(t,x;\varepsilon),\ v_j(t,x;\varepsilon)=v_{0j}+\varepsilon v_{1j}(t,x;\varepsilon),\ j=1,2,\ldots,n.$$

When n=2, then we have

$$u_1=r_3-\frac{v_{02}}{v_{01}}r_2,\ \ v_{11}=\left(\frac{u_0}{v_{02}}+\frac{q}{2v_{02}}\right)r_1+\frac{v_{01}}{2v_{02}}r_2+\frac{1}{2}r_3, \tag{2.3}$$
$$v_{12}=\left(\frac{u_0}{v_{02}}-\frac{q}{2v_{02}}\right)r_1+\frac{v_{01}}{2v_{02}}r_2+\frac{1}{2}r_3,$$
$$\lambda_{1,2}=4u_0\pm q,\ \ q=2\Big(u_0^2+\sum_{j=1}^{n}v_{0j}^2\Big)^{1/2},\quad \lambda_{3,4,\ldots,n,n+1}=2u_0,$$
$$F_j\left[r,r_x,r_{xx},r_{xxx}\right]=\sum_{i=1}^{n+1}\sum_{k=1}^{n+1}a_{jik}r_i\frac{\partial r_k}{\partial x}+\sum_{k=1}^{n+1}b_{jk}\frac{\partial^2 r_k}{\partial x^2}+\sum_{k=1}^{n+1}c_{jk}\frac{\partial^3 r_k}{\partial x^3}.$$

Coefficients a_{jik}, b_{jk}, c_{jk} can be written by using the coefficients of system (2.1). When $n = 2$, then we get:

$$a_{111} = \frac{b_u}{v_{02}^2}\left(2u_0^2 + \frac{q^2}{2}\right), \quad a_{121} = \frac{2b_u u_0 v_{01}}{v_{02}^2}, \quad a_{131} = \frac{2b_u u_0 v_{02}}{v_{02}^2},$$
$$a_{112} = \frac{2b_u u_0 v_{01}}{v_{02}^2}, \quad a_{122} = \frac{2b_u v_{01}^4 + a_u v_{02}^4}{v_{02}^2 v_{01}^2}, \quad a_{132} = \frac{2b_u v_{01}^2 - a_u v_{02}^2}{v_{01} v_{02}},$$
$$a_{123} = \frac{2b_u v_{01}^2 - a_u v_{02}^2}{v_{02} v_{02}}, \quad a_{212} = -\frac{b_1 (2u_0 + q)}{2v_{01}}, \quad a_{213} = \frac{b_1 (2u_0 + q)}{2v_{02}},$$
$$a_{222} = \frac{a_1 v_{02}^2 - v_{01}^2 b_1}{v_{01}^2}, \quad a_{133} = 2b_u + a_u, \quad a_{223} = \frac{b_1 v_{01}^2 - a_1 v_{02}^2}{v_{01} v_{02}},$$
$$a_{233} = a_1 + b_1, \quad a_{312} = -\frac{b_2 (2u_0 + q)}{2v_{01}}, \quad a_{322} = \frac{a_2 v_{02}^2 - v_{01}^2 b_2}{v_{01}^2},$$
$$a_{332} = -\frac{v_{02} (a_2 + b_2)}{v_{01}}, \quad a_{313} = \frac{b_2 (2u_0 - q)}{2v_{02}}, \quad a_{323} = \frac{b_2 v_{01}^2 - a_2 v_{02}^2}{v_{01} v_{02}},$$
$$a_{333} = a_2 + b_2, \quad a_{232} = -\frac{v_{02} (a_1 + b_1)}{v_{01}}, \quad a_{113} = \frac{2b_u u_0}{v_{02}},$$
$$b_{11} = \frac{u_0^2 (c_1 + c_2)}{v_{02}^2}, \quad b_{12} = \frac{v_{01}^2 (c_1 + c_2)}{v_{02}^2}, \quad b_{13} = c_1 + c_2, \quad b_{22} = \frac{d_1 v_{02}^2}{v_{01}^2},$$
$$b_{23} = d_1, \quad b_{32} = \frac{d_2 v_{02}^2}{v_{01}^2}, \quad b_{33} = d_2, c_{12} = -\frac{v_{02}^3}{v_{01}^3}, \quad c_{13} = -1.$$

Let $\tau = \varepsilon t$, $y = x + \lambda_1 t$, $z = x + \lambda_2 t$, $w = x + 2u_0 t$. We construct asymptotic solution of system (2.2) as the following expansions

$$r_1(t, x; \varepsilon) = h_{01}(\tau, y) + \sum_{k=1}^{m} \varepsilon^k \left(h_{k1}(\tau, y) + s_{k1}(\tau, y, z, w)\right) + O\left(\varepsilon^{m+1}\right), \quad (2.4)$$

$$r_2(t, x; \varepsilon) = h_{02}(\tau, z) + \sum_{k=1}^{m} \varepsilon^k \left(h_{k2}(\tau, z) + s_{k2}(\tau, y, z, w)\right) + O\left(\varepsilon^{m+1}\right),$$
$$r_{3,4,\dots,n+1}(t, x; \varepsilon) = h_{0;3,4,\dots,n+1}(\tau, w) + \sum_{k=1}^{m} \varepsilon^k \Big(h_{k;3,4,\dots,n+1}(\tau, w) + s_{k;3,4,\dots,n+1}(\tau, y, z, w)\Big) + O\left(\varepsilon^{m+1}\right).$$

For finding functions h_{ij} in (2.4) we solve the averaged systems

$$\frac{\partial h_{ij}}{\partial \tau} = M_j\left[F_{ij}\left[h_{i1}, \dots, h_{i,n+1}, \tau, y, z, w\right]\right], \quad (2.5)$$

where M_j are the following operators of averaging along characteristics:

$$M_1\left[g(\tau,y,z,w)\right]=\lim_{T\to+\infty}\frac{1}{T}\int_0^T g(\tau,y,y-\alpha t,y+\beta t)\,dt,$$
$$M_2\left[g(\tau,y,z,w)\right]=\lim_{T\to+\infty}\frac{1}{T}\int_0^T g(\tau,z,z+\alpha t,z+\gamma t)\,dt,$$
$$M_{3,4,\dots,n+1}\left[g(\tau,y,z,w)\right]=\lim_{T\to+\infty}\frac{1}{T}\int_0^T g(\tau,w+\delta t,w+\kappa t,w)\,dt,$$

and

$$\alpha=\lambda_1+\lambda_2=8u_0,\ \ \beta=\lambda_1-2u_0=2u_0+q,\ \ \gamma=\lambda_2-2u_0=2u_0-q,$$
$$\delta=\lambda_1+2u_0=6u_0+q,\quad \kappa=\lambda_2+2u_0=6u_0-q.$$

2.1 Periodical waves

Let $h_{0j}\left(\tau,x+2\pi\right)\equiv h_{0j}\left(\tau,x+2\pi\right)$ and $\int_0^{2\pi}h_{0j}\left(0,x+2\pi\right)\,dx=0$. Then [15]

$$(\forall i\neq j)\ M_j\left[D^k h_{0i}\right]\equiv 0,\ \ D\equiv\frac{\partial^k}{\partial x^k},\ \ k=0,1,2,3.$$

Note also, that $M_j\left[D^k h_{0j}\right]\equiv D^k h_{0j}$. Now we can write an averaged system for functions h_{01}, h_{02},…, $h_{0,n+1}$:

$$\frac{\partial h_{j0}}{\partial\tau}-a_{jjj}h_{j0}\frac{\partial h_{j0}}{\partial x_j}-b_{jj}\frac{\partial^2 h_{j0}}{\partial x_j^2}-c_{jj}\frac{\partial^3 h_{j0}}{\partial x_j^3}=\sum_{i\neq j}\sum_{k\neq j}a_{jik}M_j\left[h_{0i}\frac{\partial h_{0k}}{\partial x}\right],$$
$$j=1,2,\dots,n+1,\quad x_1=y,\ x_2=z,\ x_j=w,\ j>2. \tag{2.6}$$

For finding functions h_{k1}, h_{k2},…, $h_{k,n+1}$ for $k>0$ we construct analogous averaged systems. Functions s_{ij} in (2.4) can be express directly as Fourier series:

$$s_{ij}\left(\tau,y,z,w\right)=\sum_{\vec{l}=(l_y,l_z,l_w)\in\mathbf{Z}^3}s_{ij\vec{l}}\left(\tau\right)e^{\mathbf{i}(l_y y+l_z z+l_w w)}.$$

2.1.1 Case $n=1$

In this case in (2.6) $i=j$ or $k=j$ and the right hand side of (2.6) is equal to zero. Thus we have two independent Burgers – Korteweg-de Vries equations.

2.1.2 Case $n>1$

In this case the right hand side of (2.6) can be equal to zero in non-resonance case. In resonance case, the averaging operators are described by the following integrals

$$M_j\left[h_{0i}\frac{\partial h_{0k}}{\partial x}\right]=\frac{1}{\Lambda}\int_0^{\Lambda}h_{0i}\left(\tau,x+\mu s\right)\frac{\partial h_{0k}\left(\tau,x+\nu s\right)}{\partial x}\,ds,$$

where Λ, μ, ν depend on α, β, γ, δ, κ and j. Such systems can be solved numerically (see, [17]).

Let $u_0 \neq 0$. Then system (2.6) is non-resonance (its right hand side is equal to zero) if coefficients α, β, γ δ and κ satisfy restrictions

$$\frac{\alpha}{\beta} \notin \mathbf{Q}, \quad \frac{\alpha}{\gamma} \notin \mathbf{Q}, \quad \frac{\delta}{\kappa} \notin \mathbf{Q}, \tag{2.7}$$

where $\mathbf{Q}$ is a set of rational numbers.

3 System of n weakly nonlinear wave equations

We consider the following system of weakly nonlinear wave equations

$$u_{jtt} - a_j^2 u_{jxx} = \varepsilon f_j \left(u_{1t}, u_{1x}, \dots, u_{nt}, u_{nx}\right), \quad j = 1, 2, \dots, n. \tag{3.1}$$

System (3.1) can be rewritten as

$$r_{jt}^{\pm} \mp a_j r_{jx}^{\pm} = \varepsilon \overline{f}_j \left(r^+, r^-\right), \tag{3.2}$$

where $r_j^{\pm} = u_{jt} \pm u_{jx}$, $\overline{f}_j\left(r^+, r^-\right) = f_j\big(\dots, \frac{1}{2}\left(r_i^+ + r_i^-\right), \frac{1}{2a_i}\left(r_i^+ - r_i^-\right), \dots\big)$.

There are various aspects of asymptotic analysis for system (3.1) (see, for example, [1, 9]). In order to construct asymptotic solution of system (3.2) we use the following ansatz

$$r_j^{\pm}(t, x; \varepsilon) = r_{0j}^{\pm}\left(\tau, y_j^{\pm}\right) + \sum_{k=1}^{m} \varepsilon^k \Big(r_{kj}^{\pm}\left(\tau, y_j^{\pm}\right) + s_{kj}^{\pm}\left(\tau, y_1^+, y_1^-, \dots, y_n^+, y_n^-\right)\Big) + O\left(\varepsilon^{m+1}\right), \tag{3.3}$$

where $\tau = \varepsilon t$, $y^{\pm} = x \pm a_j t$. Let all functions in (3.3) be 2π-periodical and a_j are integer numbers. Then the averaged system for functions $r_{kj}^{\pm}$ is given by

$$\frac{\partial r_{kj}^{\pm}}{\partial \tau} = \frac{1}{2\pi} \int_0^{2\pi} f_{kj}\left(r_{k1}^+\left(\tau, y_1^+\right), r_{k1}^-\left(\tau, y_1^-\right), \dots, r_{kn}^+\left(\tau, y_n^+\right), r_{kn}^-\left(\tau, y_n^-\right), \tau, y_1^+, y_1^-, \dots, y_n^+, y_n^-\right)\Big|_{\substack{y_i^+ = y_j^{\pm} + (a_i \mp a_j)\, t \\ y_i^- = y_j^{\pm} - (a_i \pm a_j)\, t}} dt. \tag{3.4}$$

Functions s_{ij} can be computed directly by using Fourier series.

3.1 Example

Let be $n = 2$ and $f_j = \alpha_j u_{1x} u_{2x}$ in (3.1). Then we get in (3.2):

$$\overline{f}_j = \frac{\alpha_j}{4a_1 a_2}\left(r_1^+ r_2^+ - \left(r_1^+ r_2^- + r_2^+ r_1^-\right) + r_1^- r_2^-\right).$$

The averaging system is defined by

$$\begin{cases} \dfrac{\partial r_{01}^{+}}{\partial \tau} = \dfrac{\alpha_1}{4a_1a_2}\left(r_{01}^{+}\left[r_{02}^{+}\right]_1^{+} - r_{01}^{+}\left[r_{02}^{-}\right]_1^{+} - \left[r_{01}^{-}r_{02}^{+}\right]_1^{+} + \left[r_{01}^{-}r_{02}^{-}\right]_1^{+}\right), \\ \dfrac{\partial r_{01}^{-}}{\partial \tau} = \dfrac{\alpha_1}{4a_1a_2}\left(\left[r_{01}^{+}r_{02}^{+}\right]_1^{-} - \left[r_{01}^{+}r_{02}^{-}\right]_1^{-} - r_{01}^{-}\left[r_{02}^{+}\right]_1^{-} + r_{01}^{-}\left[r_{02}^{-}\right]_1^{-}\right), \\ \dfrac{\partial r_{02}^{+}}{\partial \tau} = \dfrac{\alpha_2}{4a_1a_2}\left(r_{02}^{+}\left[r_{01}^{+}\right]_2^{+} - \left[r_{01}^{+}r_{02}^{-}\right]_2^{+} - r_{02}^{+}\left[r_{01}^{-}\right]_2^{+} + \left[r_{01}^{-}r_{02}^{-}\right]_2^{+}\right), \\ \dfrac{\partial r_{02}^{-}}{\partial \tau} = \dfrac{\alpha_2}{4a_1a_2}\left(\left[r_{01}^{+}r_{02}^{+}\right]_2^{-} - r_{02}^{-}\left[r_{01}^{+}\right]_2^{-} - \left[r_{01}^{-}r_{02}^{+}\right]_2^{-} + r_{02}^{-}\left[r_{01}^{-}\right]_2^{-}\right), \end{cases}$$

where $[\,]_{1,2}^{\pm}$ are the following averaging operators:

$$\left[f\left(\tau, y_1^{+}, y_1^{-}, y_2^{+}, y_2^{-}\right)\right]_1^{+} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f\big(\tau, y_1^{+}, y_1^{+} - 2a_1t, y_1^{+} + (a_2 - a_1)t, y_1^{+} - (a_2 + a_1)t\big)\,dt,$$

$$\left[f\left(\tau, y_1^{+}, y_1^{-}, y_2^{+}, y_2^{-}\right)\right]_1^{-} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f\big(\tau, y_1^{-} + 2a_1t, y_1^{-}, y_1^{-} + (a_2 - a_1)t, y_1^{-} - (a_2 + a_1)t\big)\,dt,$$

$$\left[f\left(\tau, y_1^{+}, y_1^{-}, y_2^{+}, y_2^{-}\right)\right]_2^{+} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f\big(\tau, y_2^{+} - (a_2 - a_1)t, y_2^{+} - (a_2 + a_1)t, y_2^{+}, y_2^{+} - 2a_2t\big)\,dt,$$

$$\left[f\left(\tau, y_1^{+}, y_1^{-}, y_2^{+}, y_2^{-}\right)\right]_2^{-} = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f\big(\tau, y_2^{-} + (a_2 + a_1)t, y_2^{-} + (a_2 - a_1)t, y_2^{-} + 2a_2t, y_2^{-}\big)\,dt.$$

The condition of the resonance in this case is $a_1/a_2 \in \mathbf{Q}$.

4 Asymptotical analysis of Hirota – Satsuma type system

We consider Hirota – Satsuma type system, which was introduced in [10] (see also [7, 23]):

$$\begin{cases} u_t + uu_x = \delta\,(vw)_x + au_{xxx}, \ \ \delta \neq 0, \\ v_t - uv_x = bv_{xxx}, \\ w_t - uw_x = cw_{xxx}. \end{cases} \tag{4.1}$$

We are interested in finding a small-amplitude wave solution of (4.1)

$$\begin{cases} u\,(t, x; \varepsilon) = u_0 + \varepsilon u_1\left(\sqrt{\varepsilon}t, \sqrt{\varepsilon}x; \varepsilon\right), \\ v\,(t, x; \varepsilon) = v_0 + \varepsilon v_1\left(\sqrt{\varepsilon}t, \sqrt{\varepsilon}x; \varepsilon\right), \\ w\,(t, x; \varepsilon) = w_0 + \varepsilon w_1\left(\sqrt{\varepsilon}t, \sqrt{\varepsilon}x; \varepsilon\right). \end{cases} \tag{4.2}$$

Let us denote $\sqrt{\varepsilon}t = \overline{t}$, $\sqrt{\varepsilon}x = \overline{x}$ and insert (4.2) in (4.1), then we get a system with a small positive parameter ε:

$$\begin{cases} u_{1\overline{t}}+u_0u_{1\overline{x}}-\delta w_0 v_{1\overline{x}}-\delta v_0 w_{1\overline{x}} = \varepsilon\left(-u_1u_{1\overline{x}}+\delta\left(v_1w_1\right)_{\overline{x}}+au_{1\overline{x}\overline{x}\overline{x}}\right), \\ v_{1\overline{t}}-u_0v_{1\overline{x}} = \varepsilon\left(u_1v_{1\overline{x}}+bv_{1\overline{x}\overline{x}\overline{x}}\right), \\ w_{1\overline{t}}-u_0w_{1\overline{x}} = \varepsilon\left(u_1w_{1\overline{x}}+cw_{1\overline{x}\overline{x}\overline{x}}\right). \end{cases} \tag{4.3}$$

We define several new functions $(\delta \neq 0,\ v_0 \neq 0,\ w_0 \neq 0)$

$$u_1 = r_1+r_2+r_3, \quad v_1 = \frac{2u_0}{\delta w_0}r_2, \quad w_1 = \frac{2u_0}{\delta v_0}r_3.$$

Then system (4.3) can be rewritten in Riemann invariants r_1,r_2,r_3 (the line above variables t and x will be not written):

$$\begin{cases} r_{1t}+r_{2t}+r_{3t}+u_0\left(r_{1x}+r_{2x}+r_{3x}\right)-2u_0r_{2x}-2u_0r_{3x} = \varepsilon F_u, \\ r_{2t}-u_0r_{2x} = \varepsilon F_v, \\ r_{3t}-u_0r_{3x} = \varepsilon F_w, \end{cases} \tag{4.4}$$

where

$$F_u = -\left(r_1+r_2+r_3\right)\left(r_{1x}+r_{2x}+r_{3x}\right)+\frac{4u_0^2}{\delta w_0 v_0}\left(r_2r_3\right)_x + a\left(r_{1xxx}+r_{2xxx}+r_{3xxx}\right),$$

$$F_v = \left(r_1+r_2+r_3\right)r_{2x}+br_{2xxx}, \quad F_w = \left(r_1+r_2+r_3\right)r_{3x}+cr_{3xxx}.$$

So we can simplify the first equation of system (4.4)

$$\begin{cases} r_{1t}+u_0r_{1x} = \varepsilon F_1, \\ r_{2t}-u_0r_{2x} = \varepsilon F_2, \\ r_{3t}-u_0r_{3x} = \varepsilon F_3, \end{cases} \tag{4.5}$$

where $F_1 = F_u - F_v - F_w$, $F_2 = F_v$, $F_3 = F_w$. We find the asymptotic solution in a long time interval $t \in \left[0, O\left(\varepsilon^{-1}\right)\right]$

$$r_1\left(t,x;\varepsilon\right) = \overline{r}_1\left(\tau,y\right)+O\left(\varepsilon\right),$$
$$r_{2,3}\left(t,x;\varepsilon\right) = \overline{r}_{2,3}\left(\tau,z\right)+O\left(\varepsilon\right),$$

where $\tau = \varepsilon t$, $y = x-u_0t$, $z = x+u_0t$. We construct the averaged system:

$$\frac{\partial \overline{r}_j}{\partial \tau} = \left\langle F_j\right\rangle_j, \quad j = 1,2,3. \tag{4.6}$$

It can be written in the form (the line above variables r_j will be not written)

$$r_{1\tau}+r_1r_{1y}-ar_{1yyy} = -\left\langle\left(r_2+r_3\right)\left(r_{2z}+r_{3z}\right)\right\rangle_1 - \left\langle r_2+r_3\right\rangle_1 r_{1y} + \left(\frac{4u_0^2}{\delta w_0 v_0}-1\right)\left(\left\langle r_3r_{2z}\right\rangle_1+\left\langle r_2r_{3z}\right\rangle_1\right), \tag{4.7}$$

$$r_{2\tau}-r_2r_{2z}-3\left\langle r_1+r_3\right\rangle_2 r_{2z}-br_{2zzz} = 0, \tag{4.8}$$

$$r_{3\tau}-r_3r_{3z}-3\left\langle r_1+r_2\right\rangle_3 r_{3z}-cr_{3zzz} = 0. \tag{4.9}$$

We solve Cauchy problem, when $r_1(\tau, y)$ and $r_{2,3}(\tau, z)$ are 2π-periodic functions with respect to variables y and z. Then we get that

$$r_1(\tau, y) = r_{10}(\tau) + \sum_{l \neq 0} r_{1l}(\tau) e^{ily},$$
$$r_j(\tau, z) = r_{j0}(\tau) + \sum_{m \neq 0} r_{jm}(\tau) e^{imz}, \quad j = 2, 3.$$

Averaging operators are the following:

$$\langle r_1 \rangle_z = \lim_{T \to \infty} \frac{1}{T} \int_0^T r_1(\tau, z - 2u_0 s)\, ds,$$
$$\langle r_{2,3} \rangle_y = \lim_{T \to \infty} \frac{1}{T} \int_0^T r_{2,3}(\tau, y + 2u_0 s)\, ds.$$

Let's assume, that λ is an integer number. Then

$$\langle r_1 \rangle_y = \frac{1}{2\pi} \int_0^{2\pi} r_1(\tau, x - u_0 s)\, ds = \frac{1}{2\pi} \int_0^{2\pi} r_1(\tau, y)\, dy = r_{10}(\tau),$$
$$\langle r_j \rangle_z = \frac{1}{2\pi} \int_0^{2\pi} r_j(\tau, x + u_0 s)\, ds = \frac{1}{2\pi} \int_0^{2\pi} r_j(\tau, z)\, dz = r_{j0}(\tau), \quad j = 2, 3.$$

It is assumed that the initial conditions $[r_j(0)] \equiv 0$ are valid, i. e.

$$r_{10}(0) = r_{20}(0) = r_{30}(0) = 0. \tag{4.10}$$

We get

$$\left[\langle r_k r_{mz} \rangle_y\right] \equiv \frac{1}{2\pi} \int_0^{2\pi} \langle r_k r_{mz} \rangle (\tau, y)\, dy \equiv 0, \quad k, m = 2, 3.$$

When r_j are periodic functions, then we have:

$$\int_0^{2\pi} \frac{\partial^r r_1(\tau, y)}{\partial y^r} dy \equiv 0, \quad \int_0^{2\pi} \frac{\partial^r r_{2,3}(\tau, z)}{\partial z^r} dz \equiv 0, \quad r = 1, 2, \ldots \tag{4.11}$$

Integrating (4.8) and (4.9) from 0 to 2π along z we get that $[r_2] = [r_3] = const$. The averaged system reduces to three independent Korteweg – de Vries equations:

$$\begin{aligned} r_{1\tau} + r_1 r_{1y} - a r_{1yyy} &= 0, \\ r_{2\tau} - r_2 r_{2z} - b r_{2zzz} &= 0, \\ r_{3\tau} - r_3 r_{3z} - c r_{3zzz} &= 0. \end{aligned} \tag{4.12}$$

This case is non-resonance and the solution is expressed as a sum of three simple waves

$$u(t, x; \varepsilon) = r_1(\varepsilon t, x - u_0 t) + r_2(\varepsilon t, x + u_0 t) + r_3(\varepsilon t, x + u_0 t) + O(\varepsilon),$$

where waves r_j satisfy the independent Korteweg – de Vries equations (4.12).

5 Conclusion

In this paper it is presented that the method of internal averaging along characteristics [13, 14] can be used for wide classes of coupled non-linear wave equations. Also it is shown how to construct the asymptotic expansions which are uniformly valid in the region $t \sim O\left(\varepsilon^{-1}\right)$. The averaged system disintegrates into independent wave equations such as Burger's and Korteweg – de Vries in the non-resonance case. In the resonance case the averaged systems describe interaction of waves. Moreover the averaged systems does not have problems of asymptotic integration and can be solved numerically, similar to [17]. In the literature these systems usually are not solved numerically and they are treated only as a particular theoretical result of asymptotical analysis [2, 11, 20, 21, 24, 26].

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