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Global Stability of a Three-Species Food-Chain Model with Diffusion and Nonlocal Delays*

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Abstract. In this paper, a three species reaction-diffusion food-chain system with nonlocal delays is investigated. Sufficient conditions are derived for the global stability of a positive steady state and boundary steady states of the system by using the energy function method. Numerical simulations are carried out to illustrate the theoretical results.

Keywords: food-chain model, reaction-diffusion, nonlocal delay, global stability.

AMS Subject Classification: 35B35; 92D25.

1 Introduction

The classical Lotka–Volterra type systems are very important in the models of multi-species populations interactions. Recently, three-species food chain models have been studied by many authors (see, for example, [3, 8, 9, 11, 15, 16]). In [8], Lin studied the following three-species food-chain system with time delays

$$
\begin{aligned}
&\frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_{11}u_1 - a_{12}u_2(x, t-\tau_2)),\\
&\frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + u_2(-r_2 + a_{21}u_1(x, t-\tau_1) - a_{22}u_2 - a_{23}u_3(x, t-\tau_3)),\\
&\frac{\partial u_3}{\partial t} = d_3 \frac{\partial^2 u_3}{\partial x^2} + u_3(-r_3 + a_{32}u_2(x, t-\tau_2) - a_{33}u_3),\\
&\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \frac{\partial u_3}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0, +\infty),\\
&u_i(x,\theta) = \phi_i(x,\theta) \geq 0 \quad (i = 1,2,3), \quad (x,\theta) \in \Omega \times [-\tau_i, 0],
\end{aligned} \tag{1.1}
$$

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where $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ represent the densities of prey, predator and top predator populations at location x and time t, respectively. In [8], the author considered the asymptotic behavior of solution of system (1.1), by using the method of upper and lower solutions.

We note that the time delays of system (1.1) are space-independent. However, in many realistic ecological models, any delays should be spatially inhomogeneous, that is, the delays affect both the temporal and spatial variables. This is due to the fact that any given individual may not necessarily have been at the same spatial location at previous times. Such delays are called *spatio-temporal delays* or *nonlocal delays*. The effect of nonlocal delays on the dynamics of ecological models has been taken into account in several papers (see, for example, [1, 2, 4, 6, 5, 7, 10, 13, 12, 14, 17, 18, 19]). In [6], Gourley and So introduced a nonlocal delay with the form

$$\int_{-\infty}^{t}\int_{0}^{\pi} G(x,y,t-s)k(t-s)u(y,s)\,dy\,ds,$$

where $G(x,y,t)=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}e^{-dn^2t}\cos nx\cos ny$ is solution of

$$\frac{\partial G}{\partial t}=d\frac{\partial^2 G}{\partial x^2}$$

subject to $G(x,y,0)=\delta(x-y)$, the function $k(t)$ is called the delay kernel and satisfies $k(t)\ge 0$ for all $t\ge 0$ together with the normalization condition $\int_0^{+\infty}k(t)\,dt=1$.

Motivated by the work of Lin [8] and Gourley and So [6], in this paper, we study the following reaction-diffusion food-chain model with nonlocal delays

$$\begin{aligned}
\frac{\partial u_1}{\partial t}&=d_1\frac{\partial^2 u_1}{\partial x^2}+u_1(r_1-a_{11}u_1-a_{12}u_2),\\
\frac{\partial u_2}{\partial t}&=d_2\frac{\partial^2 u_2}{\partial x^2}+u_2\big(-r_2+a_{21}\int_{-\infty}^{t}\int_{0}^{\pi}G_1(x,y,t-s)k_1(t-s)u_1(y,s)\,dy\,ds\\
&\quad-a_{22}u_2-a_{23}u_3\big),\\
\frac{\partial u_3}{\partial t}&=d_3\frac{\partial^2 u_3}{\partial x^2}+u_3\big(-r_3+a_{32}\int_{-\infty}^{t}\int_{0}^{\pi}G_2(x,y,t-s)k_2(t-s)u_2(y,s)\,dy\,ds\\
&\quad-a_{33}u_3\big)
\end{aligned}\tag{1.2}$$

for $t>0, x\in(0,\pi)$, with homogeneous Neumann boundary conditions

$$\frac{\partial u_1}{\partial n}=\frac{\partial u_2}{\partial n}=\frac{\partial u_3}{\partial n}=0,\quad x=0,\pi \tag{1.3}$$

and initial conditions

$$u_i(x,\theta)=\phi_i(x,\theta)\ge 0\quad(i=1,2,3),\quad (x,\theta)\in[0,\pi]\times(-\infty,0], \tag{1.4}$$

where

$$G_i(x,y,t)=\frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}e^{-d_in^2t}\cos nx\cos ny,\quad i=1,2,$$

G_i is the weight function describing the distribution at past times of the individual of the species u_i who is at position x and time t. We recall that in system (1.2), $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ represent the densities of prey, predator and top predator populations at location x and time t, respectively. All the parameters in system (1.2) are positive constants.

This paper is organized as follows. In Section 2, we introduce some notations and several lemmas which will be essential to our proofs. In Section 3, we discuss the global stability of steady states for system (1.2). Finally, some numerical simulations are given to illustrate the main theoretical results.

2 Preliminaries

In this section, we introduce some notations and several results which will be useful in next section.

Let $R = (-\infty, +\infty)$, $\Omega = (0, \pi)$. For $1 \le p \le \infty$, let $L^p(\Omega)$ denote the Banach space of measurable functions u on Ω satisfying

$$\|u\|_p = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} < \infty & \text{if } 1 \le p < \infty, \\ ess \sup\limits_{x \in \Omega} |u(x)| < \infty & \text{if } p = \infty. \end{cases} \tag{2.1}$$

In particular, if $p = 2$, $L^2(\Omega)$ becomes a Hilbert space with usual inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2^2 = \langle \cdot, \cdot \rangle$. Let $\| \cdot \|_T$ denote a norm in $L^2((0,T); L^2(\Omega, R))$, i.e.,

$$\| \cdot \|_T = \left(\int_0^T \|u(s)\|_2^2 \, ds \right)^{\frac{1}{2}}. \tag{2.2}$$

Let $(u_1(t,x), u_2(t,x), u_3(t,x))$ be a solution of system (1.2). Then by a comparison theorem we have the following result.

Lemma 1. $(u_1(t,x), u_2(t,x), u_3(t,x))$ *exists globally and satisfies*

$$0 \le u_j(x,t) \le \max\left\{ M_j, \sup_{\theta \le 0} \|\phi_j(\theta, \cdot)\|_{C(\bar{\Omega}; R)} \right\}, \quad j = 1, 2, 3, \tag{2.3}$$

where $M_1 = r_1/a_{11}$, $M_2 = (a_{21}M_1 - r_2)/a_{22}$, $M_3 = (a_{32}M_2 - r_3)/a_{33}$. *Also, by the strong maximum principle, if* $\phi_i(x,0) \ne 0$ $(i = 1,2,3)$, *we have* $u_i(x,t) > 0$ $(i = 1,2,3)$, *for all* $x \in \bar{\Omega}$ *and* $t > 0$.

Moreover, we recall (Lemma 2.1 in [6]).

Lemma 2. *Let* $K(x,y,t) = G(x,y,t)k(t)$, *for the term* $\int_{-\infty}^t \int_\Omega K(x,y,t - s)u(y,s) \, dy \, ds$, *we have the following inequality*

$$\left\| \int_{-\infty}^t \int_\Omega K(x,y,t-s)u(y,s) \, dy \, ds \right\|_2 \le \int_{-\infty}^t k(t-s) \|u(s)\|_2 \, ds \tag{2.4}$$

for any function $u(x,t)$ *such that* $\frac{\partial u}{\partial n} = 0$ *on* $\partial\Omega$.

3 Global Stability

In this section, we study the global stability of a positive steady state and boundary steady states of system (1.2) by using the energy function method.

It is easy to show that system (1.2) has two steady states $E_0(0,0,0)$ and $E_1(r_1/a_{11},0,0)$. Moreover, if the condition (H1) $a_{21}r_1 > a_{11}r_2$ holds, then system (1.2) has another semi-trivial steady state $E_2(\tilde{u}_1,\tilde{u}_2,0)$, where

$$\tilde{u}_1 = \frac{a_{22}r_1 + a_{12}r_2}{a_{11}a_{22} + a_{12}a_{21}}, \qquad \tilde{u}_2 = \frac{a_{21}r_1 - a_{11}r_2}{a_{11}a_{22} + a_{12}a_{21}}.$$

Further, if the condition (H2) $a_{21}a_{32}r_1 - a_{11}a_{32}r_2 - a_{11}a_{22}r_3 - a_{12}a_{21}r_3 > 0$ holds, then system (1.2) has a positive steady state $E^*(u_1^*,u_2^*,u_3^*)$, where

$$\begin{aligned} u_1^* &= \frac{a_{22}a_{33}r_1 + a_{23}a_{32}r_1 + a_{12}a_{33}r_2 - a_{12}a_{23}r_3}{a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}}, \\ u_2^* &= \frac{a_{21}a_{33}r_1 - a_{11}a_{33}r_2 + a_{11}a_{23}r_3}{a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}}, \\ u_3^* &= \frac{a_{21}a_{32}r_1 - a_{11}a_{32}r_2 - a_{11}a_{22}r_3 - a_{12}a_{21}r_3}{a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}}. \end{aligned} \tag{3.1}$$

We are now in a position to state and prove our main result on the global stability of the positive steady state of system (1.2).

Theorem 1. *Let $(u_1(x,t),u_2(x,t),u_3(x,t))$ be a solution of system* (1.2) *with boundary conditions* (1.3) *and initial conditions* (1.4). *If* (H2) *holds and $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} > 0$, then*

$$\lim_{t\to\infty} \big(u_1(x,t),u_2(x,t),u_3(x,t)\big) = (u_1^*,u_2^*,u_3^*) \quad \textit{uniformly for } x \in [0,\pi].$$

Proof. Define

$$\begin{aligned} E(u_i) &= \int_\Omega \Big(u_i - u_i^* - u_i^* \ln\frac{u_i}{u_i^*}\Big)\,dx \quad (i=1,2,3), \\ K_i(x,y,t-s) &= G_i(x,y,t-s)k_i(t-s) \quad (i=1,2). \end{aligned} \tag{3.2}$$

For some constants $\alpha > 0, \beta > 0$ to be determined later, we have

$$\begin{aligned} &\frac{d}{dt}[E(u_1) + \alpha E(u_2) + \beta E(u_3)] \\ &= \int_\Omega \frac{\partial u_1}{\partial t}\Big(1 - \frac{u_1^*}{u_1}\Big)\,dx + \alpha\int_\Omega \frac{\partial u_2}{\partial t}\Big(1 - \frac{u_2^*}{u_2}\Big)\,dx + \beta\int_\Omega \frac{\partial u_3}{\partial t}\Big(1 - \frac{u_3^*}{u_3}\Big)\,dx \\ &= -d_1u_1^*\int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx - \alpha d_2u_2^*\int_\Omega \frac{|\nabla u_2|^2}{u_2^2}\,dx - \beta d_3u_3^*\int_\Omega \frac{|\nabla u_3|^2}{u_3^2}\,dx \\ &\quad - a_{11}\int_\Omega (u_1-u_1^*)^2\,dx - \alpha a_{22}\int_\Omega (u_2-u_2^*)^2\,dx - \beta a_{33}\int_\Omega (u_3-u_3^*)^2\,dx \\ &\quad - a_{12}\int_\Omega (u_1-u_1^*)(u_2-u_2^*)\,dx - \alpha a_{23}\int_\Omega (u_2-u_2^*)(u_3-u_3^*)\,dx \end{aligned}$$

$$+ \alpha a_{21} \int_{\Omega} \int_{-\infty}^{t} \int_{0}^{\pi} K_1(x,y,t-s)(u_1(y,s) - u_1^*)\,dy\,ds(u_2 - u_2^*)\,dx$$
$$+ \beta a_{32} \int_{\Omega} \int_{-\infty}^{t} \int_{0}^{\pi} K_2(x,y,t-s)(u_2(y,s) - u_2^*)\,dy\,ds(u_3 - u_3^*)\,dx. \quad (3.3)$$

Using Young's inequality, we have that

$$\begin{aligned}
&\frac{d}{dt}[E(u_1) + \alpha E(u_2) + \beta E(u_3)] \\
&\quad \le -d_1 u_1^* \int_{\Omega} \frac{|\nabla u_1|^2}{u_1^2}\,dx - \alpha d_2 u_2^* \int_{\Omega} \frac{|\nabla u_2|^2}{u_2^2}\,dx - \beta d_3 u_3^* \int_{\Omega} \frac{|\nabla u_3|^2}{u_3^2}\,dx \\
&\qquad - \left(a_{11} - \frac{a_{12}\eta_1}{2}\right) \int_{\Omega} (u_1 - u_1^*)^2\,dx - \left(\alpha a_{22} - \frac{a_{12}}{2\eta_1} - \frac{\alpha a_{23}\eta_2}{2}\right) \\
&\qquad \times \int_{\Omega} (u_2 - u_2^*)^2\,dx - \left(\beta a_{33} - \frac{\alpha a_{23}}{2\eta_2}\right) \int_{\Omega} (u_3 - u_3^*)^2\,dx \\
&\qquad + \alpha a_{21} \int_{\Omega} \int_{-\infty}^{t} \int_{0}^{\pi} K_1(x,y,t-s)(u_1(y,s) - u_1^*)\,dy\,ds(u_2 - u_2^*)\,dx \\
&\qquad + \beta a_{32} \int_{\Omega} \int_{-\infty}^{t} \int_{0}^{\pi} K_2(x,y,t-s)(u_2(y,s) - u_2^*)\,dy\,ds(u_3 - u_3^*)\,dx, \quad (3.4)
\end{aligned}$$

for any $\eta_1, \eta_2 > 0$. It follows from (3.4) that

$$\begin{aligned}
&\frac{d}{dt}[E(u_1) + \alpha E(u_2) + \beta E(u_3)] + d_1 u_1^* \int_{\Omega} \frac{|\nabla u_1|^2}{u_1^2}\,dx + \alpha d_2 u_2^* \int_{\Omega} \frac{|\nabla u_2|^2}{u_2^2}\,dx \\
&\qquad + \beta d_3 u_3^* \int_{\Omega} \frac{|\nabla u_3|^2}{u_3^2}\,dx + \left(a_{11} - \frac{a_{12}\eta_1}{2}\right) \|u_1 - u_1^*\|_2^2 \\
&\qquad + \left(\alpha a_{22} - \frac{a_{12}}{2\eta_1} - \frac{\alpha a_{23}\eta_2}{2}\right) \|u_2 - u_2^*\|_2^2 + \left(\beta a_{33} - \frac{\alpha a_{23}}{2\eta_2}\right) \|u_3 - u_3^*\|_2^2 \\
&\quad \le \alpha a_{21} \left\langle \int_{-\infty}^{t} \int_{0}^{\pi} K_1(x,y,t-s)(u_1(y,s) - u_1^*)\,dy\,ds, u_2 - u_2^* \right\rangle \\
&\qquad + \beta a_{32} \left\langle \int_{-\infty}^{t} \int_{0}^{\pi} K_2(x,y,t-s)(u_2(y,s) - u_2^*)\,dy\,ds, u_3 - u_3^* \right\rangle. \quad (3.5)
\end{aligned}$$

By Lemma 2, for $i = 1, 2$, we have that

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} \int_{0}^{\pi} K_i(x,y,t-s)(u_i(y,s) - u_i^*)\,dy\,ds \right\|_2 \\
&\quad \le \int_{-\infty}^{t} k_i(t-s) \|u_i(s) - u_i^*\|_2\,ds \\
&\quad \le \sup_{s \le 0} \|u_i(s) - u_i^*\|_2 \int_{t}^{\infty} k_i(s)\,ds + \int_{0}^{t} k_i(t-s) \|u_i(s) - u_i^*\|_2\,ds. \quad (3.6)
\end{aligned}$$

For any $T > 0$, we have

$$\begin{aligned}
&\Big| \int_0^T \Big\langle \int_{-\infty}^t \int_0^\pi K_1(x,y,t-s)(u_1(y,s)-u_1^*)\,dy\,ds, u_2(t)-u_2^* \Big\rangle\,dt \Big| \\
&\quad \le \int_0^T \|u_2(t)-u_2^*\|_2 \Big\| \int_{-\infty}^t \int_0^\pi K_1(x,y,t-s)(u_1(y,s)-u_1^*)\,dy\,ds \Big\|_2\,dt \\
&\quad \le \sup_{s\le 0} \|u_1(s)-u_1^*\|_2 \sup_{0\le t\le T} \|u_2(s)-u_2^*\|_2 \int_0^\infty sk_1(s)\,ds \\
&\qquad + \int_0^T \|u_2(t)-u_2^*\|_2 \int_0^t k_1(t-s)\|u_1(s)-u_1^*\|_2\,ds\,dt. \qquad (3.7)
\end{aligned}$$

We estimate the second term in (3.7) that

$$\begin{aligned}
&\int_0^T \|u_2(t)-u_2^*\|_2 \int_0^t k_1(t-s)\|u_1(s)-u_1^*\|_2\,ds\,dt \\
&\quad \le \|u_2-u_2^*\|_T \Big(\int_0^T \Big(\int_0^t k_1(t-s)\|u_1(s)-u_1^*\|_2\,ds \Big)^2 dt \Big)^{1/2} \\
&\quad \le \|u_2-u_2^*\|_T \Big(\int_0^T \Big(\int_0^t k_1(t-s)\,ds \Big) \int_0^t k_1(t-s)\|u_1(s)-u_1^*\|_2^2\,ds\,dt \Big)^{1/2} \\
&\quad \le \|u_2-u_2^*\|_T \Big(\int_0^T \int_0^t k_1(t-s)\|u_1(s)-u_1^*\|_2^2\,ds\,dt \Big)^{1/2} \\
&\quad = \|u_2-u_2^*\|_T \Big(\int_0^T \|u_1(s)-u_1^*\|_2^2 \int_s^T k_1(t-s)\,dt\,ds \Big)^{1/2} \\
&\quad \le \|u_2-u_2^*\|_T \|u_1-u_1^*\|_T, \qquad (3.8)
\end{aligned}$$

where $\|\cdot\|_T$ denotes the norm defined as in (2.2). Therefore, for any $T > 0$, we have

$$\begin{aligned}
&\Big| \int_0^T \Big\langle \int_{-\infty}^t \int_0^\pi K_1(x,y,t-s)(u_1(y,s)-u_1^*)\,dy\,ds, u_2(t)-u_2^* \Big\rangle\,dt \Big| \\
&\quad \le \sup_{s\le 0} \|u_1(s)-u_1^*\|_2 \sup_{0\le t\le T} \|u_2(s)-u_2^*\|_2 \int_0^\infty sk_1(s)\,ds \\
&\qquad + \|u_2-u_2^*\|_T \|u_1-u_1^*\|_T. \qquad (3.9)
\end{aligned}$$

In a similar way, we have that

$$\begin{aligned}
&\Big| \int_0^T \Big\langle \int_{-\infty}^t \int_0^\pi K_2(x,y,t-s)(u_2(y,s)-u_2^*)\,dy\,ds, u_3(t)-u_3^* \Big\rangle\,dt \Big| \\
&\quad \le \sup_{s\le 0} \|u_2(s)-u_2^*\|_2 \sup_{0\le t\le T} \|u_3(s)-u_3^*\|_2 \int_0^\infty sk_2(s)\,ds \\
&\qquad + \|u_2-u_2^*\|_T \|u_3-u_3^*\|_T. \qquad (3.10)
\end{aligned}$$

Integrating (3.5) over $[0,T]$, and noting that $\sup_{0\le t\le T}\|u_1(s)-u_1^*\|_2$, $\sup_{0\le t\le T}\|u_2(s)-u_2^*\|_2$ and $\sup_{0\le t\le T}\|u_3(s)-u_3^*\|_2$ can be bounded independently of T (by Lemma 1), we obtain that there exists a positive constant C independent of T such that

$$\begin{aligned} d_1u_1^*\Big\|\frac{\nabla u_1}{u_1}\Big\|_T^2 + \alpha d_2u_2^*\Big\|\frac{\nabla u_2}{u_2}\Big\|_T^2 + \beta d_3u_3^*\Big\|\frac{\nabla u_3}{u_3}\Big\|_T^2 + \Big(a_{11}-\frac{a_{12}\eta_1}{2}\Big)\|u_1-u_1^*\|_T^2 \\ + \Big(\alpha a_{22}-\frac{a_{12}}{2\eta_1}-\frac{\alpha a_{23}\eta_2}{2}\Big)\|u_2-u_2^*\|_T^2 + \Big(\beta a_{33}-\frac{\alpha a_{23}}{2\eta_2}\Big)\|u_3-u_3^*\|_T^2 \\ \le C+\alpha a_{21}\|u_2-u_2^*\|_T\|u_1-u_1^*\|_T+\beta a_{32}\|u_2-u_2^*\|_T\|u_3-u_3^*\|_T. \end{aligned} \tag{3.11}$$

By using the Young's inequality, we have

$$\begin{aligned} d_1u_1^*\Big\|\frac{\nabla u_1}{u_1}\Big\|_T^2 + \alpha d_2u_2^*\Big\|\frac{\nabla u_2}{u_2}\Big\|_T^2 + \beta d_3u_3^*\Big\|\frac{\nabla u_3}{u_3}\Big\|_T^2 + \Big(a_{11}-\frac{a_{12}\eta_1}{2}\Big)\|u_1-u_1^*\|_T^2 \\ + \Big(\alpha a_{22}-\frac{a_{12}}{2\eta_1}-\frac{\alpha a_{23}\eta_2}{2}\Big)\|u_2-u_2^*\|_T^2 + \Big(\beta a_{33}-\frac{\alpha a_{23}}{2\eta_2}\Big)\|u_3-u_3^*\|_T^2 \\ \le C+\alpha a_{21}\Big(\frac{\varepsilon_1}{2}\|u_1-u_1^*\|_T^2+\frac{1}{2\varepsilon_1}\|u_2-u_2^*\|_T^2\Big) \\ + \beta a_{32}\Big(\frac{\varepsilon_2}{2}\|u_2-u_2^*\|_T^2+\frac{1}{2\varepsilon_2}\|u_3-u_3^*\|_T^2\Big), \end{aligned} \tag{3.12}$$

for any $\varepsilon_1,\varepsilon_2>0$. We choose $\varepsilon_1=\eta_1=2a_{11}/(a_{12}+\alpha a_{21})$, $\varepsilon_2=\eta_2=(\alpha a_{23}+\beta a_{32})/2\beta a_{33}$. Then (3.12) becomes

$$\begin{aligned} d_1u_1^*\Big\|\frac{\nabla u_1}{u_1}\Big\|_T^2 + \alpha d_2u_2^*\Big\|\frac{\nabla u_2}{u_2}\Big\|_T^2 + \beta d_3u_3^*\Big\|\frac{\nabla u_3}{u_3}\Big\|_T^2 + \Big(\alpha a_{22}-\frac{a_{12}}{2\eta_1}-\frac{\alpha a_{23}\eta_2}{2}\Big) \\ \times\|u_2-u_2^*\|_T^2 \le C+\Big(\alpha a_{21}\frac{1}{2\varepsilon_1}+\beta a_{32}\frac{\varepsilon_2}{2}\Big)\|u_2-u_2^*\|_T^2. \end{aligned} \tag{3.13}$$

If $\alpha a_{22}-\frac{a_{12}}{2\eta_1}-\frac{\alpha a_{23}\eta_2}{2}>\alpha a_{21}\frac{1}{2\varepsilon_1}+\beta a_{32}\frac{\varepsilon_2}{2}$, from (3.13) we can conclude that

$$\Big\|\frac{\nabla u_1}{u_1}\Big\|_T\le C_1,\quad \Big\|\frac{\nabla u_2}{u_2}\Big\|_T\le C_2,\quad \Big\|\frac{\nabla u_3}{u_3}\Big\|_T\le C_3,\quad \|u_2-u_2^*\|_T\le C_4, \tag{3.14}$$

for some constants C_i $(i=1,2,3,4)$ independent of T.

We can choose $\alpha,\beta>0$ satisfying $\alpha a_{22}-\frac{a_{12}}{2\eta_1}-\frac{\alpha a_{23}\eta_2}{2}>\alpha a_{21}\frac{1}{2\varepsilon_1}+\beta a_{32}\frac{\varepsilon_2}{2}$. Noting that $\varepsilon_1=\eta_1=\frac{2a_{11}}{a_{12}+\alpha a_{21}}$, $\varepsilon_2=\eta_2=\frac{\alpha a_{23}+\beta a_{32}}{2\beta a_{33}}$, we obtain that

$$\beta^2a_{32}^2+2\beta\Big(\frac{a_{33}(a_{12}+\alpha a_{21})^2}{2a_{11}}-2\alpha a_{22}a_{33}+\alpha a_{32}a_{23}\Big)+\alpha^2a_{23}^2<0. \tag{3.15}$$

Denote

$$\begin{aligned} \varDelta_1&=4\Big(\frac{a_{33}(a_{12}+\alpha a_{21})^2}{2a_{11}}-2\alpha a_{22}a_{33}+\alpha a_{32}a_{23}\Big)^2-4a_{32}^2\alpha^2a_{23}^2,\\ B_1&=\frac{a_{33}(a_{12}+\alpha a_{21})^2}{2a_{11}}-2\alpha a_{22}a_{33}+\alpha a_{32}a_{23}. \end{aligned}$$

If $\Delta_1 > 0$, $B_1 < 0$, then there is a $\beta > 0$ such that the inequality (3.15) holds. For $\Delta_1 > 0$, $B_1 < 0$, we get that

$$\begin{aligned} &a_{33}(a_{12} + \alpha a_{21})^2/2a_{11} - 2\alpha a_{22}a_{33} + 2\alpha a_{32}a_{23} < 0 \\ &a_{33}(a_{12} + \alpha a_{21})^2 - 4\alpha a_{11}a_{22}a_{33} + 4\alpha a_{11}a_{32}a_{23} < 0 \\ &\alpha^2 a_{21}^2 a_{33} + 2\alpha(2a_{11}a_{23}a_{32} - 2a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33}) + a_{12}^2 a_{33} < 0. \end{aligned} \tag{3.16}$$

Denote

$$\begin{aligned} \Delta_2 &= 4(2a_{11}a_{23}a_{32} - 2a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33})^2 - 4a_{21}^2 a_{33}^2 a_{12}^2 \\ &= 16(a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33} + a_{21}a_{12}a_{33})(a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33}), \\ B_2 &= 2a_{11}a_{23}a_{32} - 2a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33}. \end{aligned} \tag{3.17}$$

Noting that $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} > 0$, we obtain that $\Delta_2 > 0$, $B_2 < 0$. Then there exists $\alpha > 0$ such that (3.16) holds. Therefore, the inequality (3.15) holds. Because of (2.3) we may deduce from (3.14) that, for some constant C_5 independent of T,

$$\|\nabla u_2\|_T \le C_5. \tag{3.18}$$

Inequalities (3.14) and (3.18) imply that $u_2 - u_2^* \in L^2((0,\infty); W^{1,2}(\Omega; R))$ and thus

$$\lim_{t\to\infty} \|u_2 - u_2^*\|_{W^{1,2}} = 0. \tag{3.19}$$

Therefore,

$$\lim_{t\to\infty} \|u_2 - u_2^*\|_{C(\bar{\Omega};R)} = 0. \tag{3.20}$$

In a similar way, we derive that

$$\lim_{t\to\infty} \|u_1 - u_1^*\|_{C(\bar{\Omega};R)} = 0, \qquad \lim_{t\to\infty} \|u_3 - u_3^*\|_{C(\bar{\Omega};R)} = 0.$$

The proof is complete. □

Next, we consider the global stability of the boundary steady state of system (1.2).

Theorem 2. *Let* $(u_1(x,t), u_2(x,t), u_3(x,t))$ *be a solution of system* (1.2) *with boundary conditions* (1.3) *and initial conditions* (1.4). *If* $a_{21}r_1 \le a_{11}r_2$ *and* $a_{11}a_{22} > a_{12}a_{21}$, *then*

$$\lim_{t\to\infty} (u_1(x,t), u_2(x,t), u_3(x,t)) = \big(r_1/a_{11}, 0, 0\big) \quad \textit{uniformly for } x \in [0,\pi].$$

Proof. Define

$$\begin{aligned} &E(u_1) = \int_\Omega \Big(u_1 - \frac{r_1}{a_{11}} - \frac{r_1}{a_{11}} \ln \frac{a_{11}u_1}{r_1}\Big)\,dx, \quad F(u_i) = \int_\Omega u_i\,dx, \quad i = 2,3, \\ &K_i(x,y,t-s) = G_i(x,y,t-s)k_i(t-s) \quad (i = 1,2). \end{aligned} \tag{3.21}$$

For some constants $\alpha, \beta > 0$ to be determined later, we have

$$\begin{aligned}
&\frac{d}{dt}[E(u_1) + \alpha F(u_2) + \beta F(u_3)] \\
&\quad= \int_\Omega \frac{\partial u_1}{\partial t}\Big(1 - \frac{r_1}{a_{11}u_1}\Big)\,dx + \alpha \int_\Omega \frac{\partial u_2}{\partial t}\,dx + \beta \int_\Omega \frac{\partial u_3}{\partial t}\,dx \\
&\quad= -\frac{d_1 r_1}{a_{11}} \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx - a_{11}\int_\Omega \Big(u_1 - \frac{r_1}{a_{11}}\Big)^2 dx - \alpha a_{22}\int_\Omega u_2^2\,dx \\
&- \beta a_{33}\int_\Omega u_3^2\,dx - a_{12}\int_\Omega \Big(u_1 - \frac{r_1}{a_{11}}\Big)u_2\,dx - \alpha \int_\Omega r_2 u_2\,dx - \alpha a_{23}\int_\Omega u_2 u_3\,dx \\
&\quad- \beta \int_\Omega r_3 u_3\,dx + \alpha a_{21}\int_\Omega \Big(\int_{-\infty}^t \int_0^\pi K_1(x,y,t-s)u_1(y,s)\,dy\,ds\Big)u_2\,dx \\
&\qquad+ \beta a_{32}\int_\Omega \Big(\int_{-\infty}^t \int_0^\pi K_2(x,y,t-s)u_2(y,s)\,dy\,ds\Big)u_3\,dx. \qquad (3.22)
\end{aligned}$$

Noting that $a_{11}r_2 \geq a_{21}r_1$, by using the Young's inequality we have

$$\begin{aligned}
&\frac{d}{dt}[E(u_1) + \alpha F(u_2) + \beta F(u_3)] \\
&\quad\leq -\frac{r_1 d_1}{a_{11}} \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx - \Big(a_{11} - \frac{a_{12}\eta_1}{2}\Big)\int_\Omega \Big(u_1 - \frac{r_1}{a_{11}}\Big)^2 dx \\
&\qquad- \Big(\alpha a_{22} - \frac{a_{12}}{2\eta_1}\Big)\int_\Omega u_2^2\,dx - \beta a_{33}\int_\Omega u_3^2\,dx \\
&\qquad+ \alpha a_{21}\int_\Omega \Big(\int_{-\infty}^t \int_0^\pi K_1(x,y,t-s)\Big(u_1(y,s) - \frac{r_1}{a_{11}}\Big)\,dy\,ds\Big)u_2\,dx \\
&\qquad+ \beta a_{32}\int_\Omega \Big(\int_{-\infty}^t \int_0^\pi K_2(x,y,t-s)u_2(y,s)\,dy\,ds\Big)u_3\,dx. \qquad (3.23)
\end{aligned}$$

It follows from (3.23) that

$$\begin{aligned}
&\frac{d}{dt}[E(u_1) + \alpha F(u_2) + \beta F(u_3)] + \frac{d_1 r_1}{a_{11}}\int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx \\
&\qquad+ \Big(a_{11} - \frac{a_{12}\eta_1}{2}\Big)\Big\|u_1 - \frac{r_1}{a_{11}}\Big\|_2^2 + \Big(\alpha a_{22} - \frac{a_{12}}{2\eta_1}\Big)\|u_2\|_2^2 + \beta a_{33}\|u_3\|_2^2 \\
&\quad\leq \alpha a_{21}\Big\langle \int_{-\infty}^t \int_0^\pi K_1(x,y,t-s)\Big(u_1(y,s) - \frac{r_1}{a_{11}}\Big)\,dy\,ds, u_2\Big\rangle \\
&\qquad+ \beta a_{32}\Big\langle \int_{-\infty}^t \int_0^\pi K_2(x,y,t-s)u_2(y,s)\,dy\,ds, u_3\Big\rangle. \qquad (3.24)
\end{aligned}$$

By Lemma 2, we have that

$$\Big\| \int_{-\infty}^{t} \int_0^{\pi} K_1(x,y,t-s)\Big(u_1(y,s) - \frac{r_1}{a_{11}}\Big)\,dy\,ds\Big\|_2 \leq \int_{-\infty}^{t} k_1(t-s)\Big\|u_1(s) - \frac{r_1}{a_{11}}\Big\|_2\,ds$$

$$\le \sup_{s\le 0}\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\int_t^\infty k_1(s)\,ds+\int_0^t k_1(t-s)\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\,ds. \quad (3.25)$$

For any $T>0$, we have

$$\begin{aligned}
&\Big|\int_0^T\Big\langle\int_{-\infty}^t\int_0^\pi K_1(x,y,t-s)\Big(u_1(y,s)-\frac{r_1}{a_{11}}\Big)\,dy\,ds,u_2(t)\Big\rangle\,dt\Big|\\
&\quad\le\int_0^T\|u_2(t)\|_2\Big\|\int_{-\infty}^t\int_0^\pi K_1(x,y,t-s)\Big(u_1(y,s)-\frac{r_1}{a_{11}}\Big)\,dy\,ds\Big\|_2\,dt\\
&\quad\le\sup_{s\le 0}\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\sup_{0\le t\le T}\|u_2(s)\|_2\int_0^\infty sk_1(s)\,ds\\
&\qquad+\int_0^T\|u_2(t)\|_2\int_0^t k_1(t-s)\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\,ds\,dt. \qquad (3.26)
\end{aligned}$$

We estimate the second term in (3.26) that

$$\begin{aligned}
&\int_0^T\|u_2(t)\|_2\int_0^t k_1(t-s)\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\,ds\,dt \qquad (3.27)\\
&\le\|u_2\|_T\Big(\int_0^T\Big(\int_0^t k_1(t-s)\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\,ds\Big)^2\,dt\Big)^{1/2}\\
&\le\|u_2\|_T\Big(\int_0^T\Big(\int_0^t k_1(t-s)\,ds\Big)\int_0^t k_1(t-s)\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2^2\,ds\,dt\Big)^{1/2}\\
&\le\|u_2\|_T\Big(\int_0^T\int_0^t k_1(t-s)\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2^2\,ds\,dt\Big)^{1/2}\\
&=\|u_2\|_T\Big(\int_0^T\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2^2\int_s^T k_1(t-s)\,dt\,ds\Big)^{1/2}\le\|u_2\|_T\Big\|u_1-\frac{r_1}{a_{11}}\Big\|_T,
\end{aligned}$$

where $\|\cdot\|_T$ denotes the norm defined as in (2.2). Therefore, for any $T>0$, we have

$$\begin{aligned}
&\Big|\int_0^T\Big\langle\int_{-\infty}^t\int_0^\pi K_1(x,y,t-s)\Big(u_1(y,s)-\frac{r_1}{a_{11}}\Big)\,dy\,ds,u_2(t)\Big\rangle\,dt\Big| \qquad (3.28)\\
&\quad\le\sup_{s\le 0}\Big\|u_1(s)-\frac{r_1}{a_{11}}\Big\|_2\sup_{0\le t\le T}\|u_2(s)\|_2\int_0^\infty sk_1(s)\,ds+\|u_2\|_T\Big\|u_1-\frac{r_1}{a_{11}}\Big\|_T.
\end{aligned}$$

In a similar way, we have that

$$\begin{aligned}
&\Big|\int_0^T\Big\langle\int_{-\infty}^t\int_0^\pi K_2(x,y,t-s)u_2(y,s)\,dy\,ds,u_3(t)\Big\rangle\,dt\Big|\\
&\qquad\le\sup_{s\le 0}\|u_2(s)\|_2\sup_{0\le t\le T}\|u_3(s)\|_2\int_0^\infty sk_2(s)\,ds+\|u_2\|_T\|u_3\|_T. \qquad (3.29)
\end{aligned}$$

Integrating (3.24) over $[0,T]$, and noting that $\sup_{0\le t\le T}\|u_1-\frac{r_1}{a_{11}}\|_2$, $\sup_{0\le t\le T}\|u_2(s)\|_2$, $\sup_{0\le t\le T}\|u_3(s)\|_2$ can be bounded independently of T (by Lemma 1), we obtain that there exists a positive constant C independent of T such that

$$\frac{r_1 d_1}{a_{11}}\Big\|\frac{\nabla u_1}{u_1}\Big\|_T^2+\Big(a_{11}-\frac{a_{12}\eta_1}{2}\Big)\Big\|u_1-\frac{r_1}{a_{11}}\Big\|_T^2+\Big(\alpha a_{22}-\frac{a_{12}}{2\eta_1}\Big)\|u_2\|_T^2 \\ +\beta a_{33}\|u_3\|_T^2\le C+\alpha a_{21}\|u_2\|_T\Big\|u_1-\frac{r_1}{a_{11}}\Big\|_T+\beta a_{32}\|u_2\|_T\|u_3\|_T. \quad (3.30)$$

By using the Young's inequality, we have

$$\frac{r_1 d_1}{a_{11}}\Big\|\frac{\nabla u_1}{u_1}\Big\|_T^2+\Big(a_{11}-\frac{a_{12}\eta_1}{2}\Big)\Big\|u_1-\frac{r_1}{a_{11}}\Big\|_T^2+\Big(\alpha a_{22}-\frac{a_{12}}{2\eta_1}\Big)\|u_2\|_T^2+\beta a_{33}\|u_3\|_T^2 \\ \le C+\alpha a_{21}\Big(\frac{\varepsilon_1}{2}\Big\|u_1-\frac{r_1}{a_{11}}\Big\|_T^2+\frac{1}{2\varepsilon_1}\|u_2\|_T^2\Big)+\beta a_{32}\Big(\frac{\varepsilon_2}{2}\|u_2\|_T^2+\frac{1}{2\varepsilon_2}\|u_3\|_T^2\Big), \quad (3.31)$$

for any $\varepsilon_1,\varepsilon_2>0$. We choose $\varepsilon_1=\eta_1=2a_{11}/a_{12}+\alpha a_{21}$, $\varepsilon_2=a_{32}/2a_{33}$. Then (3.31) becomes

$$\frac{r_1 d_1}{a_{11}}\Big\|\frac{\nabla u_1}{u_1}\Big\|_T^2+\Big(\alpha a_{22}-\frac{a_{12}}{2\eta_1}\Big)\|u_2\|_T^2\le C+\Big(\alpha a_{21}\frac{1}{2\varepsilon_1}+\beta a_{32}\frac{\varepsilon_2}{2}\Big)\|u_2\|_T^2. \quad (3.32)$$

If $\alpha a_{22}-a_{12}/2\eta_1>\alpha a_{21}/2\varepsilon_1+\beta a_{32}\varepsilon_2/2$, from (3.32) we can conclude that

$$\Big\|\frac{\nabla u_1}{u_1}\Big\|_T\le C_1,\qquad \|u_2\|_T\le C_4, \quad (3.33)$$

for some constants C_i $(i=1,2,3,4)$ independent of T. We can choose $\alpha,\beta>0$ satisfying $\alpha a_{22}-a_{12}/2\eta_1>\alpha a_{21}/2\varepsilon_1+\beta a_{32}\varepsilon_2/2$. Noting that $\varepsilon_1=\eta_1=2a_{11}/a_{12}+\alpha a_{21}$, $\varepsilon_2=a_{32}/2a_{33}$, we obtain that

$$\alpha^2 a_{21}^2 a_{33}+2a_{33}\alpha(a_{12}a_{21}-2a_{11}a_{22})+\beta a_{11}a_{32}^2+a_{33}a_{12}^2<0. \quad (3.34)$$

Denote $\Delta_1=4a_{33}^2(a_{12}a_{21}-2a_{11}a_{22})^2-4a_{21}^2a_{33}(\beta a_{11}a_{32}^2+a_{33}a_{12}^2)$. Noting that $a_{11}a_{22}>a_{12}a_{21}$, we know that if $\Delta_1>0$, there exists an $\alpha>0$ such that inequality (3.34) holds. For $\Delta_1>0$, we get that

$$4a_{11}a_{22}a_{33}^2(a_{11}a_{22}-a_{12}a_{21})>\beta a_{11}a_{21}^2a_{32}^2a_{33}. \quad (3.35)$$

So we can choose $0<\beta<4a_{22}a_{33}(a_{11}a_{22}-a_{12}a_{21})/a_{21}^2a_{32}^2$. Therefore, inequality (3.34) holds. Hence, $\lim_{t\to\infty}\|u_2\|_{C(\bar\Omega;R)}=0$.

In a similar way, we derive that

$$\lim_{t\to\infty}\big\|u_1-r_1/a_{11}\big\|_{C(\bar\Omega;R)}=0,\quad \lim_{t\to\infty}\|u_3\|_{C(\bar\Omega;R)}=0.$$

The proof is complete. □

Using a similar argument, we can also prove the following results.

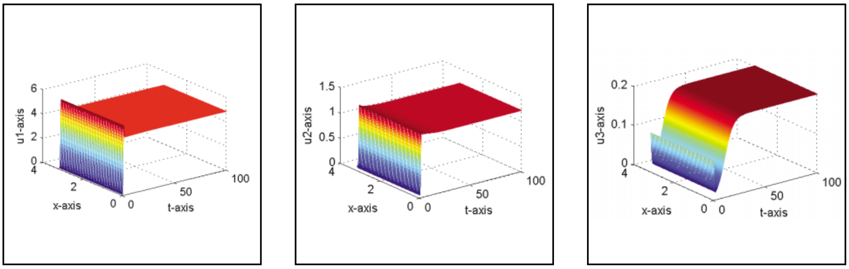
Theorem 3. *Let* $(u_1(x,t),u_2(x,t),u_3(x,t))$ *be a solution of system* (1.2) *with boundary conditions* (1.3) *and initial conditions* (1.4). *If*

$$a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}>0, \\ a_{21}a_{32}r_1-a_{11}a_{32}r_2-a_{11}a_{22}r_3-a_{12}a_{21}r_3\le 0$$

hold and (H1) *is valid, then* $\lim_{t\to\infty}(u_1(x,t),u_2(x,t),u_3(x,t))=(\tilde u_1,\tilde u_2,0)$ *uniformly for* $x\in[0,\pi]$.

4 Numerical Simulations

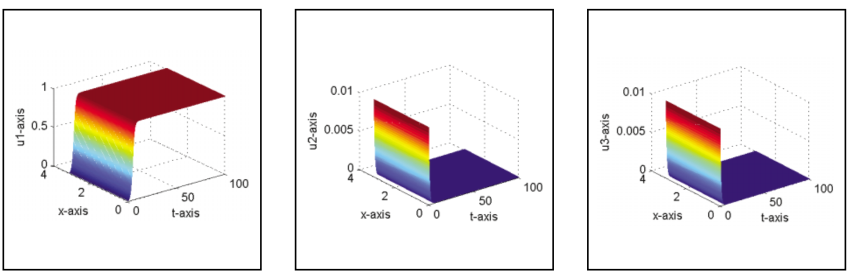
In this section, we present some numerical simulations to illustrate the results in Section 3. In system (1.2), we let $f_i(t) = \frac{1}{\tau_i} \exp(-t/\tau_i)$, $i = 1, 2$.



(a) (b) (c)

Figure 1. The time dependent solution found by numerical integration with $r_1 = 6$, $r_2 = r_3 = 1$, $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = 3$, $a_{23} = a_{32} = a_{33} = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\tau_1 = \tau_2 = 0.1$; $\phi_i(x, 0) = 0.1$ $(i = 1, 2, 3)$; $\partial u_i/\partial x = 0$ $(i = 1, 2, 3)$, $t \geq 0$, $x = 0, \pi$.

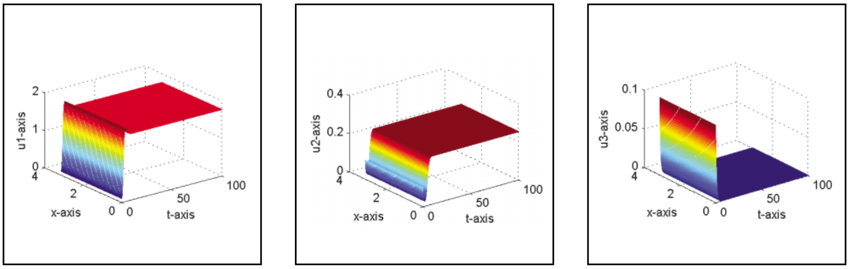
Example 1. In system (1.2), we let $r_1 = 6$, $r_2 = r_3 = 1$, $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = 3$, $a_{23} = a_{32} = a_{33} = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\tau_1 = \tau_2 = 0.1$. Then system (1.2) has four steady states $E_0(0, 0, 0)$, $E_1(6, 0, 0)$, $E_2(4.75, 1.25, 0)$ and $E^*(4.8, 1.2, 0.2)$. It is easy to see that $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} = 1 > 0$ and (H2) holds. By Theorem 1, we know that the positive equilibrium E^* of system (1.2) is globally stable (see Fig. 1).



(a) (b) (c)

Figure 2. The time dependent solution found by numerical integration with $r_1 = r_2 = r_3 = 1$, $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = 3$, $a_{23} = a_{32} = a_{33} = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\tau_1 = \tau_2 = 0.1$; $\phi_i(x, 0) = 0.01$ $(i = 1, 2, 3)$; $\partial u_i/\partial x = 0$ $(i = 1, 2, 3)$, $t \geq 0$, $x = 0, \pi$.

Example 2. In system (1.2), we let $r_1 = r_2 = r_3 = 1$, $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = 3$, $a_{23} = a_{32} = a_{33} = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\tau_1 = \tau_2 = 0.1$. Then system (1.2) has two steady states $E_0(0, 0, 0)$, $E_1(1, 0, 0)$. It is easy to see that $a_{11}a_{22} - a_{12}a_{21} = 2 > 0$, $a_{21}r_1 = a_{11}r_2$. By Theorem 2, we know that the equilibrium E_1 of system (1.2) is globally stable (see Fig. 2).



(a) (b) (c)

Figure 3. The time dependent solution found by numerical integration with $r_1 = 2$, $r_2 = r_3 = 1$, $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = 3$, $a_{23} = a_{32} = a_{33} = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\tau_1 = \tau_2 = 0.1$; $\phi_i(x, 0) = 0.1$ $(i = 1, 2, 3)$; $\partial u_i/\partial x = 0$ $(i = 1, 2, 3)$, $t \geq 0$, $x = 0, \pi$.

Example 3. In system (1.2), we let $r_1 = 2$, $r_2 = r_3 = 1$, $a_{11} = a_{12} = a_{21} = 1$, $a_{22} = 3$, $a_{23} = a_{32} = a_{33} = 1$, $d_1 = 1$, $d_2 = 2$, $d_3 = 1$, $\tau_1 = \tau_2 = 0.1$. Then system (1.2) has three steady states $E_0(0, 0, 0)$, $E_1(2, 0, 0)$, $E_2(1.75, 0.25, 0)$. It is easy to see that $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} = 1 > 0$, $a_{21}r_1 - a_{11}r_2 = 1 > 0$, $a_{21}a_{32}r_1 - a_{11}a_{32}r_2 - a_{11}a_{22}r_3 - a_{12}a_{21}r_3 = -3 < 0$. By Theorem 3, we know that the equilibrium E_2 of system (1.2) is globally stable (see Fig. 3).

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