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Modified SSOR-Like Method for Augmented Systems*

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Abstract. For solving the augmented system, Golub, Wu and Yuan and Zheng, Wang and Wu have presented the SOR-like methods and SSOR-like methods, respectively. In this paper, the SSOR-like method with two real parameters ω and α is established for solving the augmented system, which is the extension of the SSOR iteration method, and the new method is called the modified SSOR-like method (MSSOR-like method). The convergence of the MSSOR-like method is studied, and the function equation relating the parameters and eigenvalues of the iteration matrix of this method is obtained. Numerical experiments show that the MSSOR-like method with proper preconditioning matrix and parameters is better than the SOR-like method and the SSOR-like method.

Keywords: SOR-like method, SSOR-like method, augmented system, saddle point problem, MSSOR-like method.

AMS Subject Classification: 65F10.

1 Introduction

In this article, the iteration method for the large sparse two-by-two block linear system

$$\begin{pmatrix} A & B \\ B^{\mathrm{T}} & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix} \tag{1.1}$$

is considered, where the matrix $A \in \mathbb{R}^{m\times m}$ is symmetric and positive definite (SPD), $B \in \mathbb{R}^{m\times n}$ is of full column rank, e.g., if we let $m \geqslant n$, then it has $\mathrm{rank}(B) = n$, B^{T} is the transposed matrix of B, and vectors x, $b \in \mathbb{R}^m$, y, $q \in \mathbb{R}^n$ with x, y unknown and b, q known. Under these assumptions, the

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system (1.1) has a unique solution. It appears in many different applications of scientific computing, such as the finite element approximation and mixed finite element methods to solve the Navier-Stokes equation, the constrained and generalized least squares problems, constrained optimization, and fluid dynamics, etc. [2, 3, 4, 11, 14].

For the sake of simplicity, we rewrite the system (1.1) as

$$\begin{pmatrix} A & B \\ -B^{\mathrm{T}} & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix}. \tag{1.2}$$

Since the coefficient matrix of the system (1.2) is large and sparse, iteration methods are often used to solve (1.2) as a result of the storage requirements and the preservations of sparsity. It is well known that the successive over relaxation (SOR) method and the symmetric successive over relaxation (SSOR) method [13] cannot be applied directly to the system (1.2) because of the singularity of the block diagonal of the coefficient matrix, even though these methods are popular in engineering applications as simple iteration methods. Recently, several proposals have been developed in [5, 6, 7, 8, 9, 10] for generalizing the SOR method to solve the above system, in which the most practical and important scheme is the SOR-like method presented by Golub et al. [5]. The SOR-like method is more closely related to the normal SOR splitting

$$\begin{pmatrix} A & B \\ -B^{\mathrm{T}} & O \end{pmatrix} \equiv D_1 - L_1 - U_1, \tag{1.3}$$

where

$$D_1 = \begin{pmatrix} A & O \\ O & Q \end{pmatrix}, \quad L_1 = \begin{pmatrix} O & O \\ B^{\mathrm{T}} & O \end{pmatrix}, \quad U_1 = \begin{pmatrix} O & -B \\ O & Q \end{pmatrix},$$

and the matrix $Q \in \mathbb{R}^{n\times n}$ is symmetric and nonsingular. Then the SOR-like procedure is

$$(D_1 - \omega L_1) \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = [(1-\omega)D_1 + \omega U_1] \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \omega \begin{pmatrix} b \\ -q \end{pmatrix}.$$

Thus, the SOR-like iteration takes the following form in [5]:

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(b - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \omega Q^{-1}(B^{\mathrm{T}}x^{(k+1)} - q). \end{cases} \tag{1.4}$$

About the optimum parameters for the SOR-like method, we could see an excellent article [6], where Li and co-authors have given an explicit expression for the optimum parameters in each case. Besides, they have considered the Chebyshev acceleration of the SOR-like method by the proper choices of the auxiliary or preconditioning matrix Q in [8].

Moreover, several proposals have also been developed in [1, 12, 15, 16] for generalizing the SSOR method to solve the system (1.2), in which the SSOR-like method for saddle point problems [15] is the most primacy. From [16], the SSOR-like method is found to have the same splitting as (1.3). Then the SSOR-like procedure is

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = H_\omega \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \omega(2-\omega)(D_1-\omega U_1)^{-1}D_1(D_1-\omega L_1)^{-1}\begin{pmatrix} b \\ -q \end{pmatrix},$$

where

$$H_\omega = (D_1-\omega U_1)^{-1}[(1-\omega)D_1+\omega L_1](D_1-\omega L_1)^{-1}[(1-\omega)D_1+\omega U_1].$$

Thus, the SSOR-like iteration takes the following form in [16]:

$$\begin{cases} y^{(k+1)} = y^{(k)} + \omega(2-\omega)Q^{-1}[B^{\mathrm{T}}x^{(k)} - \frac{\omega B^{\mathrm{T}}A^{-1}By^{(k)}}{1-\omega} + \frac{\omega B^{\mathrm{T}}A^{-1}b}{1-\omega} - \frac{q}{1-\omega}], \\ x^{(k+1)} = (1-\omega)^2x^{(k)} - \omega A^{-1}B[(1-\omega)y^{(k)} + y^{(k+1)}] + \omega(2-\omega)A^{-1}b. \end{cases} \tag{1.5}$$

The main aim of this paper is to present a new SSOR-like method for the augmented linear system, which has two real parameters ω and α, and is called the modified SSOR-like (MSSOR-like) method. Here we will discuss its convergence and establish the relations between the parameters and the eigenvalues of the iteration matrix of this method. Numerical results will show that the MSSOR-like method for solving the augmented linear system is more efficient than the SOR-like method and the SSOR-like method. For the special case when $\alpha = 0$ and $\beta = 1$, our method will be identical with the SSOR-like method in [16].

The outline of this paper is given as follows. In Section 2, we establish the modified SSOR-like method for solving the augmented system (1.2). In Section 3, we give out some basic functional equations and lemmas. Then the convergence analysis of the MSSOR-like method is discussed in Section 4, and numerical experiments, with proper choices of the parameters ω and α, are presented in Section 5. Finally, conclusions are made for this paper.

2 Modified SSOR-Like Method

For the coefficient matrix of the augmented system (1.2), we consider the following splitting:

$$\begin{pmatrix} A & B \\ -B^{\mathrm{T}} & O \end{pmatrix} \equiv D - L - U, \tag{2.1}$$

where

$$D = \begin{pmatrix} A & O \\ O & Q \end{pmatrix}, \quad L = \begin{pmatrix} O & O \\ B^{\mathrm{T}} & \alpha Q \end{pmatrix}, \quad U = \begin{pmatrix} O & -B \\ O & \beta Q \end{pmatrix},$$

and the matrix $Q \in \mathbb{R}^{n\times n}$ is symmetric and nonsingular, $\alpha + \beta = 1$.

From $\alpha + \beta = 1$, we have $\beta = 1 - \alpha$. Note that

$$D - \omega L = \begin{pmatrix} A & O \\ -\omega B^{\mathrm{T}} & (1-\alpha\omega)Q \end{pmatrix},$$

$$D - \omega U = \begin{pmatrix} A & \omega B \\ O & (1-\beta\omega)Q \end{pmatrix} = \begin{pmatrix} A & \omega B \\ O & (1-\omega+\alpha\omega)Q \end{pmatrix}.$$

Since the matrix A is SPD and Q is nonsingular, we obtain that

$$\det(D-\omega L) = (1-\alpha\omega)^n \det(A)\det(Q) \neq 0,$$
$$\det(D-\omega U) = (1-\omega+\alpha\omega)^n \det(A)\det(Q) \neq 0$$

if and only if $1-\alpha\omega \neq 0$ and $1-\omega+\alpha\omega \neq 0$, i.e., $(1-\alpha\omega)(1-\omega+\alpha\omega) \neq 0$.

Let $z^{(k)} = \binom{x^{(k)}}{y^{(k)}}$ be the kth approximate solution of the system (1.2), then by the use of the SOR method with the splitting (2.1), it has

$$z^{(k+1/2)} = L_\omega z^{(k)} + \omega(D-\omega L)^{-1}c, \tag{2.2}$$

where

$$L_\omega = (D-\omega L)^{-1}[(1-\omega)D+\omega U] = \begin{pmatrix} (1-\omega)I_m & -\omega A^{-1}B \\ \frac{\omega(1-\omega)Q^{-1}B^{\mathrm{T}}}{1-\alpha\omega} & I_n - \frac{\omega^2 Q^{-1}B^{\mathrm{T}}A^{-1}B}{1-\alpha\omega} \end{pmatrix}$$

and $c = \begin{pmatrix} b \\ -q \end{pmatrix}$. On the other hand, with the backward generalized SOR method, we can compute $z^{(k+1)}$ from $z^{(k+1/2)}$ as

$$z^{(k+1)} = U_\omega z^{(k+1/2)} + \omega(D-\omega U)^{-1}c, \tag{2.3}$$

where

$$U_\omega = (D-\omega U)^{-1}[(1-\omega)D+\omega L] = \begin{pmatrix} (1-\omega)I_m - \frac{\omega^2 A^{-1}BQ^{-1}B^{\mathrm{T}}}{1-\omega+\alpha\omega} & -\omega A^{-1}B \\ \frac{\omega Q^{-1}B^{\mathrm{T}}}{1-\omega+\alpha\omega} & I_n \end{pmatrix}.$$

Deleting $z^{(k+1/2)}$ from (2.2) and (2.3), we get the MSSOR-like method of the form $z^{(k+1)} = H_{\omega,\alpha}z^{(k)} + C_{\omega,\alpha}$, where

$$H_{\omega,\alpha} = U_\omega L_\omega = \begin{pmatrix} (1-\omega)^2 I_m - A_{11} & -\omega(2-\omega)A^{-1}B + A_{12} \\ \frac{\omega(1-\omega)(2-\omega)Q^{-1}B^{\mathrm{T}}}{(1-\alpha\omega)(1-\omega+\alpha\omega)} & I_n - \frac{\omega^2(2-\omega)Q^{-1}B^{\mathrm{T}}A^{-1}B}{(1-\alpha\omega)(1-\omega+\alpha\omega)} \end{pmatrix}$$

$$A_{11} = \frac{\omega^2(1-\omega)(2-\omega)A^{-1}BQ^{-1}B^{\mathrm{T}}}{(1-\alpha\omega)(1-\omega+\alpha\omega)}, \quad A_{12} = \frac{\omega^3(2-\omega)A^{-1}BQ^{-1}B^{\mathrm{T}}A^{-1}B}{(1-\alpha\omega)(1-\omega+\alpha\omega)}$$

and

$$\begin{aligned} C_{\omega,\alpha} &= \omega(2-\omega)(D-\omega U)^{-1}D(D-\omega L)^{-1}\begin{pmatrix} b \\ -q \end{pmatrix} \\ &= \omega(2-\omega)\begin{pmatrix} A^{-1}b - \frac{\omega^2 A^{-1}BQ^{-1}B^{\mathrm{T}}A^{-1}b}{(1-\alpha\omega)(1-\omega+\alpha\omega)} + \frac{\omega A^{-1}BQ^{-1}q}{(1-\alpha\omega)(1-\omega+\alpha\omega)} \\ \frac{\omega Q^{-1}B^{\mathrm{T}}A^{-1}b}{(1-\alpha\omega)(1-\omega+\alpha\omega)} - \frac{Q^{-1}q}{(1-\alpha\omega)(1-\omega+\alpha\omega)} \end{pmatrix}. \end{aligned}$$

Hence, the MSSOR-like method can be written in the following form ($\omega \neq 1$):

$$\begin{cases} y^{(k+1)} = y^{(k)} + \frac{\omega(2-\omega)(1-\omega)Q^{-1}}{(1-\alpha\omega)(1-\omega+\alpha\omega)}[B^{\mathrm{T}}x^{(k)} - \frac{\omega B^{\mathrm{T}}A^{-1}By^{(k)}}{1-\omega} + \frac{\omega B^{\mathrm{T}}A^{-1}b}{1-\omega} - \frac{q}{1-\omega}], \\ x^{(k+1)} = (1-\omega)^2 x^{(k)} - \omega A^{-1}B[(1-\omega)y^{(k)} + y^{(k+1)}] + \omega(2-\omega)A^{-1}b. \end{cases} \tag{2.4}$$

Evidently, if let $\alpha = 0$ and $\beta = 1$, then the MSSOR-like method becomes the SSOR-like method [16], and the equations (2.4) become the equations (1.5). Therefore, we can say that our method contains the SSOR-like method, in other words, the SSOR-like method is just a special case of our method. For analyzing the convergence of the MSSOR-like method, we first give several basic functional equations and lemmas in the next section.

3 Basic Functional Equations and Lemmas

In this part, our goal is to establish several basic functional equations and lemmas for the latter use. Now, we can prove the following results.

Lemma 1. *Let $H_{\omega,\alpha}$ be the iteration matrix of the MSSOR-like method, $\omega \neq 0$, 1, 2 and $m \geqslant n$. Then*

(i) $\lambda = (1-\omega)^2$ *is an eigenvalue of $H_{\omega,\alpha}$ if $m > n$.*

(ii) $\lambda = (1-\omega)^2$ *is not an eigenvalue $H_{\omega,\alpha}$ if $m = n$.*

(iii) *For any eigenvalue $\lambda \neq (1-\omega)^2$ of $H_{\omega,\alpha}$, there exists an eigenvalue μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ such that λ, μ, and ω satisfy the function equation*

$$(\lambda-1)(1-\alpha\omega)(1-\omega+\alpha\omega)[(1-\omega)^2-\lambda] = \lambda\omega^2(2-\omega)^2\mu. \tag{3.1}$$

(iv) *For any eigenvalue μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$, if λ is different from $(1-\omega)^2$ and λ, μ, and ω satisfy the above function equation, then λ is an eigenvalue of $H_{\omega,\alpha}$.*

Proof. Suppose λ is an eigenvalue of $H_{\omega,\alpha}$, $\lambda \neq 0$ and the corresponding eigenvector is $(x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$. Then, we have

$$H_{\omega,\alpha}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

it follows that

$$\begin{aligned} &[(1-\omega)^2-\lambda]x_1 - \omega(2-\omega)A^{-1}Bx_2 \\ &\quad = \frac{\omega^2(2-\omega)A^{-1}BQ^{-1}B^{\mathrm{T}}}{(1-\alpha\omega)(1-\omega+\alpha\omega)}[(1-\omega)x_1 - \omega A^{-1}Bx_2] \end{aligned}$$

and

$$\frac{\omega(2-\omega)(1-\omega)Q^{-1}B^{\mathrm{T}}}{(1-\alpha\omega)(1-\omega+\alpha\omega)}x_1 + x_2 - \frac{\omega^2(2-\omega)Q^{-1}B^{\mathrm{T}}A^{-1}B}{(1-\alpha\omega)(1-\omega+\alpha\omega)}x_2 = \lambda x_2.$$

By calculation, we obtain

$$(1-\omega)^2x_1 - \lambda x_1 = \omega(1+\lambda-\omega)A^{-1}Bx_2, \tag{3.2}$$

$$\begin{aligned} &\omega(2-\omega)(1-\omega)Q^{-1}B^{\mathrm{T}}x_1 \\ &\quad = \omega^2(2-\omega)Q^{-1}B^{\mathrm{T}}A^{-1}Bx_2 + (1-\alpha\omega)(1-\omega+\alpha\omega)(\lambda-1)x_2. \end{aligned} \tag{3.3}$$

First, we assume $\lambda = (1-\omega)^2$, then from (3.2) and previous assumptions, it has $A^{-1}Bx_2 = 0$ and

$$(1-\omega)Q^{-1}B^{\mathrm{T}}x_1 = \omega Q^{-1}B^{\mathrm{T}}A^{-1}Bx_2 - (1-\alpha\omega)(1-\omega+\alpha\omega)x_2.$$

Since the matrix B is of full column rank, and A is SPD, the above systems are equivalent to

$$x_2 = 0 \quad \text{and} \quad Q^{-1}B^{\mathrm{T}}x_1 = 0.$$

Note that B^{T} is a $n \times m$ matrix with $\mathrm{rank}(B^{\mathrm{T}}) = n$. Hence, if $m > n$, $Q^{-1}B^{\mathrm{T}}x_1 = 0$ has $m-n$ independent non-zero solutions, that is to say, the corresponding eigenvector of $\lambda = (1-\omega)^2$ is non-zero vector, which has proved our conclusion (i).

On the other hand, if $m = n$, $Q^{-1}B^{\mathrm{T}}x_1 = 0$ has no non-zero solutions. Thus, $\lambda = (1-\omega)^2$ is not an eigenvalue of $H_{\omega,\alpha}$, and conclusion (ii) is proved.

Second, we assume $\lambda \neq (1-\omega)^2$, then from (3.2), we have

$$x_1 = \frac{\omega(1+\lambda-\omega)}{(1-\omega)^2-\lambda}A^{-1}Bx_2.$$

Substituting this equation to the above equations, we get

$$\lambda\omega^2(2-\omega)^2Q^{-1}B^{\mathrm{T}}A^{-1}Bx_2 = (\lambda-1)(1-\alpha\omega)(1-\omega+\alpha\omega)\big[(1-\omega)^2-\lambda\big]x_2.$$

As a result of $\omega \neq 0,1,2$, and $\lambda \neq 0,1,(1-\omega)^2$. Thus we can assume μ is an eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ such that we have

$$(\lambda-1)(1-\alpha\omega)(1-\omega+\alpha\omega)\big[(1-\omega)^2-\lambda\big] = \lambda\omega^2(2-\omega)^2\mu.$$

Conversely, for any eigenvalue μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$, if $\lambda \neq (1-\omega)^2$ and λ, μ, ω satisfy the equation (3.1), and $\omega \neq 0,1,2$, which implies $\lambda \neq 0$, 1, then we can prove the conclusion (iv) by reversing the above processes. Hence, Lemma 1 is proved. □

Corollary 1. Let $\rho(H_{\omega,\alpha})$ be the spectral radius of the MSSOR-like iteration matrix $H_{\omega,\alpha}$, $m > n$, then we have

$$\rho(H_{\omega,\alpha}) \geqslant |(1-\omega)^2|.$$

Proof. Suppose λ is the eigenvalue of the iteration matrix $H_{\omega,\alpha}$, since $\lambda = (1-\omega)^2$ is an eigenvalue of $H_{\omega,\alpha}$ as indicated in the first conclusion of Lemma 1, then we have $\rho(H_{\omega,\alpha}) \geqslant |\lambda|$. Thus, this corollary holds. □

Furthermore, we obtain the necessary condition for the convergence of the MSSOR-like method when $m > n$ is $0 < \omega < 2$. It is easy to obtain the following corollary if $\alpha = 0$, $\beta = 1$.

Corollary 2 [see [16]]. Let λ be an eigenvalue of H_ω, if $\lambda \neq 0,1,(1-\omega)^2$ and $\omega \neq 0,1,2$, then there exists an eigenvalue μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ such that

$$(\lambda-1)(1-\omega)\big[(1-\omega)^2-\lambda\big] = \lambda\omega^2(2-\omega)^2\mu. \tag{3.4}$$

Conversely, for any eigenvalue μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ and $\omega \neq 1$, if λ satisfies the equation (3.4), then λ is an eigenvalue of H_ω.

The following result is quoted for the use of the next section.

Lemma 2 [see [13]]. *Both roots of the real quadratic equation* $\lambda^2 - b\lambda + c = 0$ *are less than unity in modulus if and only if* $|c| < 1$ *and* $|b| < 1 + c$.

4 Convergence Analysis

In [16], Zheng et al. have illustrated that the SSOR-like method is suitable to the case when all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are negative, and the case when all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are positive. Hence, the convergence expressions for parameters ω and α will be similarly discussed for these two cases. The main result about the convergence of the MSSOR-like method is given as follows.

Theorem 1. *Assume that the parameters* ω *and* α *satisfy* $(1-\alpha\omega)(1-\omega+\alpha\omega) \neq 0$, *if we choose a nonsingular matrix* Q *such that all eigenvalues* μ *of* $Q^{-1}B^{\mathrm{T}}A^{-1}B$ *are real, and let* $\mu_{\min} = \min\mu$, $\mu_{\max} = \max\mu$. *Then*

(i) *If* $\mu_{\min} > 0$, *the MSSOR-like method is convergent if and only if*

$$\begin{aligned} &0 < \omega < 2, (1-\alpha\omega)(1-\omega+\alpha\omega) > 0, \\ &\frac{\omega^2(2-\omega)^2\mu_{\max}}{(1-\alpha\omega)(1-\omega+\alpha\omega)} < 2 + 2(1-\omega)^2; \end{aligned} \tag{4.1}$$

(ii) *If* $\mu_{\max} < 0$, *the MSSOR-like method is convergent if and only if*

$$\begin{aligned} &0 < \omega < 2, (1-\alpha\omega)(1-\omega+\alpha\omega) < 0, \\ &\frac{\omega^2(2-\omega)^2\mu_{\min}}{(1-\alpha\omega)(1-\omega+\alpha\omega)} < 2 + 2(1-\omega)^2. \end{aligned} \tag{4.2}$$

Proof. It follows from Lemma 1, the equation (3.1) can be rewritten as

$$\begin{aligned} &\lambda^2 - \Big[1 + (1-\omega)^2 - \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega+\alpha\omega)}\Big]\lambda + (1-\omega)^2 = 0, \\ &(1-\alpha\omega)(1-\omega+\alpha\omega) \neq 0. \end{aligned}$$

According to Lemma 2, $|\lambda| < 1$ if and only if

$$|(1-\omega)^2| < 1, \quad \left|1 + (1-\omega)^2 - \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega+\alpha\omega)}\right| < 1 + (1-\omega)^2. \tag{4.3}$$

From the first inequality of (4.3), we get $0 < \omega < 2$. In addition, it is clear that the second inequality in (4.3) is equivalent to

$$-1 - (1-\omega)^2 < 1 + (1-\omega)^2 - \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega+\alpha\omega)} < 1 + (1-\omega)^2,$$

that is,

$$0 < \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega+\alpha\omega)} < 2 + 2(1-\omega)^2. \tag{4.4}$$

Hence, from (4.4), if $\mu_{\min} > 0$, then we have the following inequalities

$$(1-\alpha\omega)(1-\omega+\alpha\omega) > 0,$$
$$0 < \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega+\alpha\omega)} \leqslant \frac{\omega^2(2-\omega)^2\mu_{\max}}{(1-\alpha\omega)(1-\omega+\alpha\omega)} < 2+2(1-\omega)^2.$$

On the other hand, if $\mu_{\max} < 0$, then from (4.4) we obtain

$$(1-\alpha\omega)(1-\omega+\alpha\omega) < 0,$$
$$0 < \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega+\alpha\omega)} \leqslant \frac{\omega^2(2-\omega)^2\mu_{\min}}{(1-\alpha\omega)(1-\omega+\alpha\omega)} < 2+2(1-\omega)^2.$$

Therefore, the conclusions (i) and (ii) in Theorem 1 have been proved. □

Remark 1. From the equations (4.1) and (4.2) in Theorem 1, we see that it is quite difficult to obtain the explicit expressions about the convergence domain for the parameters ω and α for the MSSOR-like method. However, we can obtain the following corollary when $\alpha = 0$ and $\beta = 1$, which has the same results as in [16].

Corollary 3 [see [16]]. Suppose that all eigenvalues μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are real, let $\mu_{\min} = \min\mu$, $\mu_{\max} = \max\mu$. Then

(i) If $\mu_{\min} > 0$, the SSOR-like method is convergent if and only if

$$0 < \omega < 1 - \frac{2\mu_{\max}}{\sqrt{1+4\mu_{\max}^2} + 1 + \sqrt{2+2\sqrt{1+4\mu_{\max}^2}}};$$

(ii) If $\mu_{\max} < 0$, the SSOR-like method is convergent if and only if

$$1 + \frac{-2\mu_{\min}}{\sqrt{1+4\mu_{\min}^2} + 1 + \sqrt{2+2\sqrt{1+4\mu_{\min}^2}}} < \omega < 2.$$

Proof. We can see [16] for details. □

Remark 2. For the equation (4.1) in Theorem 1, if we let $\alpha > 0$, then according to the conditions $(1-\alpha\omega)(1-\omega+\alpha\omega) > 0$ and $0 < \omega < 2$, it is easy to get that

$$(1-\alpha\omega)(1-\omega+\alpha\omega) \leqslant \big(1-\frac{\omega}{2}\big)\big(1-\frac{\omega}{2}\big) = \big(1-\frac{\omega}{2}\big)^2,$$
$$\omega^2(2-\omega)^2\mu_{\max} < \big(1-\frac{\omega}{2}\big)^2\big[2+2(1-\omega)^2\big].$$

It follows that $0 < \omega < \frac{1}{1-\alpha} < 2$, if $0 < \alpha < \frac{1}{2}$, and $0 < \omega < \frac{1}{\alpha}$, if $\alpha > \frac{1}{2}$. Thus, in this case, we have the following corollary.

Corollary 4 [see [12]]. Assume that all eigenvalues μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are real and positive, let $\mu_{\max} = \max\mu$. Then

(i) If $0 < \mu \leqslant \frac{1}{4}$, the MSSOR-like method converges for all ω such that $0 < \omega < 2$.

(ii) If $\mu = \frac{1}{2}$, the MSSOR-like method converges for all ω such that $0 < \omega < 1$.

(iii) If $\mu > \frac{1}{4}$ and $\mu \neq \frac{1}{2}$, the MSSOR-like method converges for all ω such that

$$0 < \omega < \frac{1 - \sqrt{4\mu_{\max} - 1}}{1 - 2\mu_{\max}} = \frac{2}{1 + \sqrt{4\mu_{\max} - 1}} < 2.$$

Proof. The proof can be found in [12], where Wu et al. have discussed the case when $\alpha = \beta = \frac{1}{2}$. □

5 Numerical Example

In this section, we will apply the MSSOR-like method to solve one augmented system, and we will illustrate by the numerical results that our method is quiet effective and converges faster than the SOR-like method [5] and the SSOR-like method [16].

Example 1 [See Darvishi and Hessari [1]]. Consider the augmented system (1.1), where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & O \\ O & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2}, B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2},$$

and

$$T = \frac{1}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p \times p}, \quad F = \frac{1}{h}\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{p \times p},$$

with $\otimes$ being the Kronecker product symbol and $h = \frac{1}{1+p}$ the discretization mesh size and $S = $tridiag$(a, b, c)$ is a tridiagonal matrix with $S_{ii} = b$, $S_{i-1,i} = a$ and $S_{i,i+1} = c$ for the appropriate i.

For this example, we set $m = 2p^2$ and $n = p^2$. Hence, the total number of variables is $m + n = 3p^2$. Here we consider the following three cases when $p = 8$, $p = 16$, and $p = 24$. All the computations are performed on a PC with a 1.86GHz 64-bit processor and 2GB memory.

In our experiments, all runs with respect to the SOR-like method, the SSOR-like method and the proposed MSSOR-like method are started from an initial vector $({x^{(0)}}^{\mathrm{T}}, {y^{(0)}}^{\mathrm{T}})^{\mathrm{T}} = 0$, and terminated when the current iteration satisfies RES $< 10^{-9}$, where RES $= \text{norm}({x^{(k)}}^{\mathrm{T}} - {x^{(0)}}^{\mathrm{T}}, {y^{(k)}}^{\mathrm{T}} - {y^{(0)}}^{\mathrm{T}})^{\mathrm{T}}$ with $({x^{(k)}}^{\mathrm{T}}, {y^{(k)}}^{\mathrm{T}})^{\mathrm{T}}$ the final approximate solution. Additionaly, we choose the right hand-side vector $(b^{\mathrm{T}}, q^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m+n}$ such that the exact solution of the augmented system (1.1) is $((x^*)^{\mathrm{T}}, (y^*)^{\mathrm{T}})^{\mathrm{T}} = (1, 1, \cdots, 1)^{\mathrm{T}} \in \mathbb{R}^{m+n}$. The number of iterations (denoted by IT) and the RES defined as above are reported in the following tables in order to show the efficiency of the MSSOR-like method.

Table 1. The minimum $\mu_{\min}$ and the maximum $\mu_{\max}$ eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$.

m		128	512	1152
n		64	256	576
$m+n$		192	768	1728
$Q = B^{\mathrm{T}}B(\tau = 1)$	$\mu_{\min}$	0.0016	4.3633e−4	2.0008e−4
	$\mu_{\max}$	0.0425	0.0402	0.0394
$Q = B^{\mathrm{T}}B(\tau = -1)$	$\mu_{\min}$	−0.0425	−0.0402	−0.0394
	$\mu_{\max}$	−0.0016	−4.3633e−4	−2.0008e−4
$Q = 10I(\gamma = 10)$	$\mu_{\min}$	0.0153	0.0091	0.0065
	$\mu_{\max}$	0.1000	0.1000	0.1000
$Q = -I(\gamma = -1)$	$\mu_{\min}$	−1.0000	−1.0000	−1.0000
	$\mu_{\max}$	−0.1525	−0.0907	−0.0651

Table 2. The optimum parameters, IT and RES for this example when $Q = B^{\mathrm{T}}B$ $(\tau = 1)$.

m		128	512	1152
n		64	256	576
$m+n$		192	768	1728
SOR	$\omega = \omega_{opt}$	1.9188	1.9248	1.9266
	IT	7674	29099	64190
	RES	9.9934e−10	9.9941e−10	9.9982e−10
SSOR	ω	0.9775	0.9791	0.9800
	IT	186	566	1114
	RES	9.4418e−10	9.9602e−10	9.9016e−10
MSSOR	ω	1.5000	1.8000	1.8000
	α	0.6500	0.45	0.5510
	IT	133	146	287
	RES	9.3139e−10	9.0696e−10	9.7157e−10

For the preconditioning matrix Q, we mainly provide two possible choices of Q to carry out the MSSOR method and the SOR-like method and the SSOR-like method. The choices of Q are similar to those presented by Golub et al. in [5]. One is $Q = \tau B^{\mathrm{T}}B$ $(\tau = -1, 1)$, and the other is $Q = \gamma I$ $(\gamma = -1, 10)$, where I is a $n \times n$ identity matrix. Since the matrix A is SPD, and B is of full column rank, it follows that $B^{\mathrm{T}}A^{-1}B$ is a positive definite matrix. Thus for any eigenvalues μ of $Q^{-1}B^{\mathrm{T}}A^{-1}B$, if $\tau > 0$, $\gamma > 0$, then we get $\mu > 0$, conversely, if $\tau < 0$, $\gamma < 0$, then we have $\mu < 0$.

Note that the SOR-like method requires all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ be positive, while the SSOR-like method and our MSSOR-like method are not only suitable to the case when all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are positive but also suitable to the case when all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are negative. Specifically, the minimum $\mu_{\min}$ and the maximum $\mu_{\max}$ eigenvalue of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are listed for different values of m and n in Table 1 such that the optimum parameters ω_{opt} of the SOR-like method and the SSOR-like method can be computed according to the results of Li et al. [6] and Zheng et al. [16]. However, since the explicit expressions of parameters ω and α cannot be obtained for the MSSOR-like method, we only choose them by trial and error.

Table 3. The optimum parameters, IT and RES for this example when $Q = -B^{\mathrm{T}}B$ ($\tau = -1$).

m		128	512	1152
n		64	256	576
$m+n$		192	768	1728
SSOR	ω	1.0227	1.0205	1.0199
	IT	183	560	1107
	RES	9.7234e−10	9.7391e−10	9.9566e−10
MSSOR	ω	1.4998	1.7998	1.7993
	α	0.6798	0.4400	0.5600
	IT	115	124	288
	RES	9.4829e−10	8.2673e−10	9.9632e−10

Table 4. The optimum parameters, IT and RES for this example when $Q = 10I$ ($\gamma = 10$).

m		128	512	1152
n		64	256	576
$m+n$		192	768	1728
SOR	$\omega = \omega_{opt}$	1.8110	1.8195	1.8230
	IT	808	1419	$*$
	RES	9.7665e−10	9.8692e−10	$*$
SSOR	ω	0.9400	0.9455	0.9465
	IT	76	123	172
	RES	8.1954e−10	8.7625e−10	8.9776e−10
MSSOR	ω	1.6139	1.7010	1.7023
	α	0.4983	0.5030	0.5600
	IT	52	75	78
	RES	5.3953e−10	8.5559e−10	8.5468e−10

In Tables 2, 3, 4 and 5 we have presented the numerical results for the SOR-like, SSOR-like and MSSOR-like methods with different values of m and n and preconditioning matrices Q. In particular, Tables 2 and 4 have presented the comparisons of the IT and RES for the SOR-like, SSOR-like and MSSOR-like methods when all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are positive. Tables 3 and 5 have supplied the numerical performance of the SSOR-like and MSSOR-like methods when all eigenvalues of $Q^{-1}B^{\mathrm{T}}A^{-1}B$ are negative. Note that the symbol "$*$" in Table 4 denotes that the SOR-like method with the optimum parameter ω_{opt} cannot arrive the prescribed accuracy.

Clearly, from Tables 2, 3, 4 and 5, it is not difficult to find that the number of iterations of the MSSOR-like method is much less than those of the SOR-like and SSOR-like methods. Moreover, we see that the RES of our MSSOR-like method is also less than those of the SOR-like and SSOR-like methods, except for the cases when the problem size of this test problem is 1728 in Table 3 and 512 in Table 5. Hence, we can say that our MSSOR-like method, with proper choices of parameters ω and α, is superior to the SOR-like method with its optimal parameter ω_{opt} and the SSOR-like method in reducing the iteration counts and RES.

Table 5. The optimum parameters, IT and RES for this example when $Q = -I$ ($\gamma = -1$).

m		128	512	1152
n		64	256	576
$m+n$		192	768	1728
SSOR	ω	1.3800	1.3650	1.3605
	IT	50	92	131
	RES	7.8701e−10	9.0287e−10	9.0907e−10
MSSOR	ω	1.5240	1.5876	1.5998
	α	0.8523	0.7985	0.7865
	IT	41	52	63
	RES	5.1928e−10	9.3926e−10	7.8155e−10

6 Conclusions

In this paper, we have established the modified SSOR-like method by introducing two real parameters ω and α. The main result about the convergence of the MSSOR-like method has been given in this context. Numerical results in Tables 2, 3, 4 and 5 have illustrated that the MSSOR-like method, with proper choices of parameters ω and α, is effective and superior to the SOR-like and SSOR-like methods in the sense of the IT and RES. However, the theoretical determination of the optimum parameters for the MSSOR-like is underway, which needs further in-depth study.

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