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Higher-Order Families of Multiple Root Finding Methods Suitable for Non-Convergent Cases and their Dynamics

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Abstract. In this paper, we present many new one-parameter families of classical Rall's method (modified Newton's method), Schröder's method, Halley's method and super-Halley method for the first time which will converge even though the guess is far away from the desired root or the derivative is small in the vicinity of the root and have the same error equations as those of their original methods respectively, for multiple roots. Further, we also propose an optimal family of iterative methods of fourth-order convergence and converging to a required root in a stable manner without divergence, oscillation or jumping problems. All the methods considered here are found to be more effective than the similar robust methods available in the literature. In their dynamical study, it has been observed that the proposed methods have equal or better stability and robustness as compared to the other methods.

Keywords: multiple roots, Rall's method, Schröder's method, super-Halley's method, basins of attraction.

AMS Subject Classification: 41A25; 65H05.

1 Introduction

One topic which has always been of paramount importance in computational mathematics is that of approximating efficiently multiple roots of nonlinear equations of the form

$$f(x) = 0, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear sufficiently differentiable function in an interval I.

To solve nonlinear equation (1.1), one can use classical iterative methods such as Rall's method (modified Newton's method) [12], [11], Schröder's method [13], Halley's and super-Halley method [5]. Perhaps, the most celebrated of all such iterative methods is the classical Rall's method (also known as modified Newton's method) for finding multiple roots of nonlinear equation (1.1), given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n \geq 0.$$

It converges quadratically and requires the prior knowledge of multiplicity m.

However, as it is well-known that the prominent one-point modified Newton's method has some drawbacks. This method may be sensitive to the quality of the initial guess. Moreover, the iteration can be aborted due to overflow or leads to divergence, if the derivative of a function at an iterative point is singular or almost singular ($f'(x_n) = 0$), which restricts their practical applications. Therefore, more effective globally convergent algorithms are still needed.

In order to overcome these problems, Kanwar et al. [4] proposed the following modification over Rall's method as

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n) - \gamma f(x_n)}, \ \gamma \in \mathbb{R}. \tag{1.2}$$

It satisfies the following error equation

$$e_{n+1} = \frac{(c_1 - \gamma)}{m} e_n^2 + O(e_n^3),$$

$c_k = \frac{m!}{(m+k)!}\frac{f^{(m+k)}(r_m)}{f^{(m)}(r_m)}$, $k = 1, 2, 3, \ldots$ and r_m is a multiple root which is different from Rall's method. This family converges quadratically under the condition that $f'(x_n) - \gamma f(x_n) \neq 0$, while $f'(x_n) = 0$ is permitted at some points. This technique provides an alternative to the failure situation of existing classical Rall's method. Unfortunately, this technique does not have the same error equation as that of existing classical Rall's method.

Therefore, we intend to develop a scheme that will converge to the required root even though the guess is far away from the required root or derivative is very small in the vicinity of the required root and also has the same error equation as that of original Rall's method. Further, we present many new highly efficient families of Schröder's method, super-Halley method and Halley's method respectively which will not only converge to the required root but also have the same error equations as those of existing classical methods.

Furthermore, a new fourth-order optimal family of methods has been developed by discretization of the second-order derivative involved in the family of super-Halley method.

2 Development of iterative schemes

Let us consider a curve in the following form

$$y = a_1(x - x_n) + a_2 e^{\gamma(x - x_n)}, \tag{2.1}$$

where $\gamma \in \mathbb{R}$, a_1 and a_2 are arbitrary constants to be determined. To be osculating, we require

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n),$$

which lead to

$$a_1 = f'(x_n) - \gamma f(x_n) \text{ and } a_2 = f(x_n).$$

Suppose the curve (2.1) cuts the x-axis at x_{n+1}, then $y(x_{n+1}) = 0$ and it follows from (2.1) that

$$a_1(x_{n+1} - x_n) + a_2 e^{\gamma(x_{n+1} - x_n)} = 0. \tag{2.2}$$

From (2.2), we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \gamma f(x_n)} e^{\gamma(x_{n+1} - x_n)}. \tag{2.3}$$

Approximating $(x_{n+1} - x_n)$ on the right-hand side of equation (2.3) by the correction factor: $-\frac{f(x_n)}{f'(x_n) - \gamma f(x_n)}$ (given in formula (1.2) for $m = 1$), one gets

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \gamma f(x_n)} e^{\frac{-\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}}. \tag{2.4}$$

The error equation of scheme (2.4) is given by

$$e_{n+1} = c_1 e_n^2 + O(e_n^3).$$

This is a new one-parameter family of Newton's method. The beauty of this family is that it has the same error equation as Newton's method. In addition, scheme (2.4) does not fail even the guess is far away from the required root or become very small in the vicinity of the required root. Now, we want to extend this idea for multiple roots. Consider the following modification to the above mentioned one-point iterative family (2.4)

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n) - \gamma f(x_n)} e^{\frac{-\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}}. \tag{2.5}$$

Further, we want to simplify the body structure of the above family and also introduce some higher-order variants of this family (2.5). Therefore, we consider $\left|\frac{\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}\right| << 1$ and using Taylor series expansion for $e^{\frac{-\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}}$, we obtain

$$e^{\frac{-\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}} \approx 1 - \frac{\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}.$$

Using this approximate value of $\mu_n = e^{\frac{-\gamma f(x_n)}{f'(x_n)-\gamma f(x_n)}}$ in formula (2.5), one gets

$$x_{n+1} = x_n - m\frac{f(x_n)\Big(f'(x_n) - 2\gamma f(x_n)\Big)}{\Big(f'(x_n) - \gamma f(x_n)\Big)^2}. \tag{2.6}$$

This is another new one-parameter family of Newton's method. Let us assume that the corrector factor as $g(x) = \frac{mf(x_n)}{f'(x_n)-\gamma f(x_n)}\mu_n$, then we have

$$g'(x) = m\left(\frac{f'(x_n) - 2\gamma f(x_n)}{f'(x_n) - \gamma f(x_n)}\right)\left(\frac{f'(x_n)^2 - f(x_n)f''(x_n)}{\big(f'(x_n) - \gamma f(x_n)\big)^2}\right)e^{\frac{-\gamma f(x_n)}{f'(x_n)-\gamma f(x_n)}}.$$

Now, by applying Newton's method to the correction factor $\frac{mf(x_n)}{f'(x_n)-\gamma f(x_n)}\mu_n$ of formula (2.5), one gets

$$x_{n+1} = x_n - \frac{f(x_n)\big(f'(x_n) - \gamma f(x_n)\big)^2}{\big(f'(x_n)^2 - f(x_n)f''(x_n)\big)\big(f'(x_n) - 2\gamma f(x_n)\big)}. \tag{2.7}$$

This is a one-parameter modified family of Schröder's method [13] for an equation having multiple roots of multiplicity $m \geq 1$ unknown. It is easy to verify that this method has quadratic convergence. The order of convergence of family (2.6) is analyzed in Theorem 1.

Further, from formula (2.6) and (2.7), one gets

$$\begin{aligned} x_{n+1} =& x_n - \frac{1}{2}\left[\frac{mf(x_n)\big(f'(x_n) - 2\gamma f(x_n)\big)}{\big(f'(x_n) - \gamma f(x_n)\big)^2}\right. \\ &\left.+\frac{f(x_n)\big(f'(x_n) - \gamma f(x_n)\big)^2}{\big(f'(x_n)^2 - f(x_n)f''(x_n)\big)\big(f'(x_n) - 2\gamma f(x_n)\big)}\right]. \end{aligned} \tag{2.8}$$

Note that this is a new family of famous cubically convergent super-Halley method and can be viewed as an arithmetic mean of two functions namely, $\frac{mf(x_n)\big(f'(x_n)-2\gamma f(x_n)\big)}{\big(f'(x_n)-\gamma f(x_n)\big)^2}$ and $\frac{f(x_n)\big(f'(x_n)-\gamma f(x_n)\big)^2}{\big(f'(x_n)^2-f(x_n)f''(x_n)\big)\big(f'(x_n)-2\gamma f(x_n)\big)}$.

If we take the harmonic mean in (2.8) instead of the arithmetic one, we get

$$\begin{aligned} x_{n+1} &= x_n \\ &- \frac{2mf(x_n)\big(f'(x_n) - 2\gamma f(x_n)\big)\big(f'(x_n) - \gamma f(x_n)\big)^2}{m\big(f'(x_n)-2\gamma f(x_n)\big)^2\big(f'(x_n)^2-f(x_n)f''(x_n)\big)+\big(f'(x_n)-\gamma f(x_n)\big)^4}. \end{aligned} \tag{2.9}$$

This is a modification over the well-known cubically convergent Halley's method. The first most striking feature of this contribution is that we have developed families of Rall's, Schröder's, super-Halley and Halley's methods for the first time which will converge even though the guess is far from the desired root or the derivative is small in the vicinity of the root and have the same error equations as those of their original methods respectively. It is also interesting to note that for $\gamma = 0$, these formulas reduce to Rall's [12], Schröder's [13], super-Halley and Halley's [5] methods for multiple roots respectively.

3 Optimal families of multi-point methods

The main practical difficulty associated with the recently developed methods is that they require lengthy computation of second-order derivatives that reduce the efficiency and accuracy of the methods. Therefore, second-order derivative free methods are still needed. In the past and recent years, many multi-point iterative methods have been proposed for solving nonlinear equations that improve local convergence order of the classical modified Newton's method (Rall's method), see [8], [7], [15], [19], [16] and the references cited therein.

Therefore, obtaining new optimal methods of fourth order, not requiring the computation of second-order derivative, is a very important and interesting task from a practical point of view, because their corresponding efficiency index [10] is 1.587. Motivated in this direction, we develop many new interesting fourth-order optimal families of Jarratt's type methods free from second-order derivatives.

Let us consider a Newton-like iterate $w_n = x_n - \frac{2m}{m+2}v(x_n)$, where $v \equiv v(x_n) = \frac{f(x_n)\{f'(x_n)-2\gamma f(x_n)\}}{\{f'(x_n)-\gamma f(x_n)\}^2}$ and expand $f'\left(x_n - \frac{2m}{m+2}v\right)$ about a point $x = x_n$ by Taylor's series expansion as follows:

$$f'(w_n) = f'(x_n) - \frac{2m}{m+2}vf''(x_n) + O\left(\frac{2m}{m+2}v\right)^2.$$

Therefore, one obtains another approximation for $f''(x_n)$ as follows:

$$f''(x_n) \approx \frac{f'(x_n) - f'(w_n)}{2mv/(m+2)},$$

where v is as defined earlier. Using this approximation of $f''(x_n)$ in the expression (2.8), we have

$$\begin{aligned} x_{n+1}=x_n-\frac{1}{2}\Big[&2mvf(x_n)(f'(x_n)-\gamma f(x_n))^2/\big((f'(x_n)-2\gamma f(x_n))\big(2mvf'(x_n)^2 \\ &-(m+2)f(x_n)(f'(x_n)-f'(w_n))\big)\big)+\frac{mf(x_n)(f'(x_n)-2\gamma f(x_n))}{(f'(x_n)-\gamma f(x_n))^2}\Big]. \end{aligned}$$

We introduce some disposable parameters and after simplifications, we obtain

$$\begin{aligned} x_{n+1} =&x_n \\ &-\frac{mf(x_n)\left[2va_5\lambda_1^4+\left(2a_6mvf'(x_n)^2+a_7(m+2)f(x_n)\beta\right)\lambda_2^2\right]}{2\left((m+2)f(x_n)\beta+2a_8mvf'(x_n)^2\right)\lambda_2\lambda_1^2}, \end{aligned} \tag{3.1}$$

where $\lambda_1 = f'(x_n)-\gamma f(x_n)$, $\lambda_2 = f'(x_n)-2\gamma f(x_n)$, $\beta = f'(w_n)-f'(x_n)$, while a_5, a_6, a_7 and a_8 are disposable parameters such that the order of convergence reaches at the optimal (according to Kung-Traub conjecture [6]) level four without using any more functional evaluations. Theorem 1 indicates that under what choices on the disposable parameters in (3.1), the order of convergence will reach at the optimal level four.

4 Convergence analysis

Theorem 1. *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function defined on an open interval D, enclosing a multiple zero of $f(x)$, say $x = r_m$ with multiplicity $m \geq 1$. Then, for $\gamma \in \mathbb{R}$, iteration schemes defined by formulas:*

(i) Schemes (2.5) *and* (2.7) *have quadratic order of convergence.*

(ii) Schemes (2.8) *and* (2.9) *have cubic order of convergence.*

(iii) Scheme (3.1) *has optimal fourth-order of convergence if*

$$\begin{cases} a_5 = \frac{1}{4}(2+m)\left(4\mu + m(3\mu - 1)\right), \\ a_6 = -\dfrac{(2+m)\left(-4+m+4\mu+m\mu\right)}{4m}, \\ a_7 = -\,(m-2), \quad a_8 = -(2+m)\left(-1+\mu\right)/(2m), \end{cases} \tag{4.1}$$

respectively, where $\mu = \left(\frac{m}{2+m}\right)^m$.

Proof. Let $x = r_m$ be a multiple zero of $f(x)$ and $e_n = x_n - r_m$ be the error at the n-th iteration. Expanding $f(x_n)$ and $f'(x_n)$ about $x = r_m$ by the Taylor's series expansion (with the help of computer algebra software *Mathematica* 9), we have

$$f(x_n) = \frac{1}{m!} f^{(m)}(r_m) e_n^m \left[1 + e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + e_n^4 c_4 + O(e_n^5)\right], \tag{4.2}$$

$$\begin{aligned} f'(x_n) =& \frac{1}{m!} f^{(m)}(r_m) e_n^{m-1} \big[m + (m+1)e_n c_1 + (m+2)e_n^2 c_2 \\ &+ (m+3)e_n^3 c_3 + (m+4)e_n^4 c_4 + O(e_n^5)\big], \end{aligned} \tag{4.3}$$

$$\begin{aligned} f''(x_n) =& \frac{1}{m!} f^{(m)}(r_m) e_n^{m-2} \big[-m + m^2 + m(1+m)c_1 e_n \\ &+ (2+3m+m^2)c_2 e_n^2 + (6+5m+m^2)c_3 e_n^3 + O(e_n^4)\big], \end{aligned} \tag{4.4}$$

respectively. Making use of equations (4.2)–(4.4) in iterative schemes (2.5) and (2.7) and after some simplifications, one can have the following error equations

$$\begin{aligned} e_{n+1} =& \frac{c_1}{m} e_n^2 + O(e_n^3), \quad \text{(same as Rall's method)}, \\ e_{n+1} =& -\frac{c_1}{m} e_n^2 + O(e_n^3), \quad \text{(same as Schröder's method)}, \end{aligned}$$

respectively. This proves the quadratic convergence of iterative schemes (2.5) and (2.7).

Further, making use of equations (4.2)–(4.4) in iterative schemes (2.8) and (2.9) and after some simplifications, one can have the following error equations

$$\begin{aligned} e_{n+1} =& \left(\frac{(m-1)c_1^2 - 2mc_2}{2m^2}\right) e_n^3 + O(e_n^4), \quad \text{(same as super-Halley method)}, \\ e_{n+1} =& \left(\frac{(m+1)c_1^2 - 2mc_2}{2m^2}\right) e_n^3 + O(e_n^4), \quad \text{(same as Halley's method)}, \end{aligned}$$

respectively. This proves the cubic convergence of iterative schemes (2.8) and (2.9).

Using equations (4.2) and (4.3), one gets

$$v = \frac{f(x_n)\Big(f'(x_n) - 2\gamma f(x_n)\Big)}{\Big(f'(x_n) - \gamma f(x_n)\Big)^2} = \frac{e_n}{m} - \frac{c_1 e_n^2}{m^2} + \frac{\left((1+m)c_1^2 - 2mc_2 - \gamma^2\right) e_n^3}{m^3} + O(e_n^4),$$

and

$$f'\left(x_n - \frac{2m}{m+2}v\right) = f^{(m)}(r_m)e_n^{m-1}\Big[\frac{\mu(2+m)}{m!} + \frac{\mu(2m+3m^2+m^3-4)c_1 e_n}{m^2 m!} + \frac{\mu\left(4(2-m)c_1^2 + m\left(2(m+m^2-2)\gamma^2 + b_0 c_2\right)\right) e_n^2}{m^4 m!} + O(e_n^3)\Big], \tag{4.5}$$

where $b_0 = m(4m + 4m^2 + m^3 - 8)$. Substituting (4.1)–(4.3) and (4.5) in iterative scheme (3.1) and after some simplification, one can have the following error equation

$$e_{n+1} = \frac{\left[2\mu b_1 c_1^3 - 3m^2(2+m)^2 c_1\left((1-\mu)\gamma^2 + 2m\mu c_2\right) + 6m^5\mu c_3\right] e_n^4}{6\mu m^4(2+m)^2} + O(e_n^5),$$

where $b_1 = (2+m)^2(2m + 2m^2 + m^3 - 2)$. This completes the proof of the Theorem 1. It is noteworthy that all our methods are also working for simple roots if you simply taking the value of $m = 1$ with same order. □

5 Extraneous fixed points

Multipoint iterative methods [6] solving a generic nonlinear equation of the form $f(x) = 0$ can be represented by a discrete dynamical system

$$x_{n+1} = R_f(x_n),$$

where R_f is the iteration function whose fixed points are zeros of $f(x)$ under consideration. The iteration function R_f, however, might possess other fixed points that are not zeros of f. Such fixed points different from zeros of f are called the *extraneous fixed points* [3, 17] of the iteration function R_f. Extraneous fixed points may form attractive, indifferent, repulsive cycles or periodic orbits to display chaotic dynamics behind the basin of attraction under investigation. The existence of such extraneous fixed points would affect the global iteration dynamics, which was demonstrated via König functions by Vrscay and Gilbert [17]. Especially the presence of attractive cycles induced by the extraneous fixed points of R_f may alter the basin of attractions due to the trapped sequence $\{x_n\}$. Even in the case of repulsive or indifferent fixed points, an initial value x_0 chosen near a desired root may converge to another unwanted remote root. Indeed, these aspects of the Schröder functions [17] were observed in an application to the family of functions $\{f_k(x) = x^k - 1, k \geq 2\}$. Such dynamical aspects motivate our investigation of the extraneous fixed points that may affect the basins of attraction for the proposed methods (3.1).

For notational convenience, we first denote $h = \frac{f(x_n)}{f'(x_n)}$ and $t = \frac{f'(w_n)}{f'(x_n)}$. Then with $v = \frac{h(1-2\gamma h)}{(1-\gamma h)^2}$, iterative methods (3.1) can be written as

$$\begin{aligned} w_n =& x_n - \frac{2m}{m+2} v(x_n), \\ x_{n+1} =& x_n - h\frac{m}{2}\left[\frac{\big(a_7(t-1)(m+2)h + 2a_6 mv\big)(1-2h\gamma)^2 + 2a_5 v(1-h\gamma)^4}{\big((t-1)(m+2)h + 2a_8 mv\big)(1-2h\gamma)(1-h\gamma)^2}\right]. \end{aligned}$$

The above equation can be represented in the form of a weighted Newtonian discrete dynamical system:

$$x_{n+1} = R_f(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n),$$

where $H_f(x_n) = \frac{m}{2}\frac{\big(a_7(t-1)(m+2)h+2a_6 mv\big)(1-2h\gamma)^2+2a_5 v(1-h\gamma)^4}{\big((t-1)(m+2)h+2a_8 mv\big)(1-2h\gamma)(1-h\gamma)^2}$ with coefficients a_5, a_6, a_7, a_8 given by (4.1). It is clearly the form of $H_f(x_n)$ to characterize a variety of iterative methods. The zero r_m of $f(x)$ is obviously a fixed point of R_f. The points $\xi \neq r_m$ for which $H_f(\xi) = 0$ are extraneous fixed points of R_f.

Let $H(z)$ represent a $H_f(z)$ when $f(z)$ is a finite-order rational function of z. Then it would be of great interest for us to investigate the complex dynamics of the rational iterative map R_p of the form

$$z_{n+1} = R_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} H(z_n), \tag{5.1}$$

in connection with the basins of attraction for a variety of polynomials $p(z_n)$. Clearly, $R_p(z)$ represents the classical Newton's method with weighing function $H(z)$ and may possess its fixed points as zeros of $p(z)$ or extraneous fixed points associated with $H(z)$.

Indeed, if we take $f(z) = p(z)$, the same polynomial as given in (5.1), then we get the usual rational iterative map $\mathcal{R}_p$ of the form

$$z_{n+1} = \mathcal{R}_p(z_n) = z_n - \frac{p(z_n)}{p'(z_n)} \mathcal{H}(z_n). \tag{5.2}$$

The complex dynamics of (5.2) along with its basins of attraction will be described later in the first part of Section 7.

We now turn to a different dynamics originated from the extraneous fixed points of iterative map (5.1). We are interested also in the investigation of unified dynamics associated with these extraneous fixed points. To this end, we apply a simple quadratic polynomial raised to the power of multiplicity m, i.e., $f(z) = (z^2 - 1)^m$ to $H_f(x_n)$, simple-root cases of which were introduced by Cayley [2] and Vrscay et al. [17] in dynamical studies of the Schröder and

König functions for a family of functions $f_k(z) = z^k - 1,\ k \in \mathbb{N}$ to minimize perturbations of the Julia set boundaries.

Hence in this section we will exclusively discuss the complex dynamics of (5.1) associated with its extraneous fixed points. To this end, we first choose $\gamma = 1$ and denote (3.1) by OM_4 to solve a variety of nonlinear equations which are mentioned in the following examples 1–3. We write $H(z)$ in the form of

$$H(z) = A(z) \cdot \frac{F(z)}{D(z)}, \tag{5.3}$$

where $A(z)$ is a function of z whose roots may contain $z = 0$(which causes infinity to h) or $z = \pm 1$ (which satisfy $(z^2-1)^m$) independently of the extraneous fixed points of H; $F(z)$ and $D(z)$ are polynomials having no common factors; $F(z)$ may indeed contain the extraneous fixed points of H.

In order to compare the dynamics behavior of Method OM_4 related to extraneous fixed points, let us now employ the existing four optimal methods with quartic convergence. We conveniently denote these existing optimal fourth-order methods by SM_4, ZM_4, SSM_4 and LM_4, which are proposed by Soleymani et al. [16] with their best expression (18), proposed by Zhou et al. [19] with their expression (11), proposed by Sharma and Sharma [15], and proposed by Li et al. [8] with their expression (75), respectively.

By similarly following the development procedure of $H(z)$ for Method OM_4, we find the corresponding $H(z)$ for existing methods LM_4, SSM_4, ZM_4, SM_4 as follows:

$$H(z) = \begin{cases} \frac{(4-m^2)\delta\sigma^{m-1}+m\lambda}{(2+m)\delta\sigma^{m-1}-\lambda}, & \text{for } LM_4, \\ \frac{(m+2)\rho^2-2(m-1)(m+2)\delta\rho\sigma^{m-1}+(8-4m+m^3)\delta^2\sigma^{2m-2}}{\delta^2\sigma^{2m-2}}, & \text{for } SSM_4, \\ \frac{(m^3+6m^2+8m+8)z^{4m}-2m(m+3)\phi z^{2m}+m\phi^2}{z^{4m}}, & \text{for } ZM_4, \\ \frac{(m+2)^{2m-1}\delta\mu\sigma^{m-1}z^{2m}}{\eta_1\mu^2 z^{4m}-2\eta_2\delta\mu\sigma^{m-1}z^{2m}+m^4\delta^2\sigma^{2m-2}}, & \text{for } SM_4, \end{cases} \tag{5.4}$$

with $\delta = 1+(1+m)z^2, \sigma = -1+(1+m)^2z^2, \rho = (m+2)^{2m}z^{2m}\mu$, $\lambda = m^m(m+2)^m z^{2m}, \mu = (\frac{m}{m+2})^m$, $\phi = m^{1-m}(m+2)^{1-m}\delta\sigma^{m-1}$, $\eta_1 = (m-2)(m+2)^{4m+1}$, $\eta_2 = (m+2)^{2m-1}(m^4+2m^3-2m^2-4m-8)$.

Since $H(z)$ in (5.3) or (5.4) defines a high-order rational function as the multiplicity m increases, it is convenient to study the typical cases of $m \in \{2,3,4,5\}$ for locating the corresponding extraneous fixed points. In fact, Tables 1 and 2 list $H(z)$ and the extraneous fixed points ξ, respectively for the values of $2 \le m \le 5$. The numerator $F(z)$ or denominator $D(z)$ is shortened if its expression is so lengthy.

We further note that the stability of the extraneous fixed points of the listed methods in Table 2 varies in a variety of ways. The last three columns of Table 2 respectively indicate a number of attractive, indifferent, and repulsive extraneous fixed points. Interestingly, it is straightforward to show that the extraneous fixed points of Method SM_4 are all found to be indifferent due to its inherent structure of $H(z)$. In the latter part of Section 7 complex dynamics will be discussed along with chaotic behavior of rational iterative maps (5.1) when applied to various polynomials $p(z)$, based on visual description of their

Table 1. $A(z), F(z)$ and $D(z)$ of various fourth-order methods for typical cases

Method	m	$A(z)$	$F(z)$
OM_4	2	$z/8$	$(z^2-4z-1)^2(1+16z+156z^2+\cdots+z^8)$
	3	$3z$	$169957+2185776z+3231732z^2+\cdots+25z^{24}$
	4	z	$21295541+494285440z+\cdots-459z^{32}$
	5	$5z$	$1821421328553+67008048468600z+\cdots+2960433z^{40}$
LM_4	2	z^4	32
	3	3	$-1+28z^2-128z^4+1001z^6$
	4	4	$1-70z^2+1500z^4-6250z^6+32467z^8$
	5	5	$-3+414z^2-20736z^4+419904z^6-1679616z^8+7282537z^{10}$
SSM_4	2	2	$(1-12z^2+46z^4-60z^6+1049z^8)$
	3	3	$23-1288z^2+23920z^4+\cdots+11950373z^{12}$
	4	4	$7-980z^2+55300z^4+\cdots+8641095007z^{16}$
	5	5	$113-31188z^2+3714084z^4+\cdots+1145335160540383z^{20}$
ZM_4	2	1	$(1+2z-4z^2-2z^3+19z^4)(1-2z-4z^2+2z^3+19z^4)$
	3	1	$1-56z^2+1040z^4+\cdots+190651z^{12}$
	4	1	$1-140z^2+7900z^4+\cdots+538664425z^{16}$
	5	1	$1-276z^2+32868z^4+\cdots+5065625942791z^{20}$
SM_4	2	z^4	32
	3	z^6	$1200(-1+4z)^2(1+4z)^2(1+4z^2)$
	4	z^8	$13824(-1+5z)^3(1+5z)^3(1+5z^2)$
	5	z^{10}	$960400(-1+6z)^4(1+6z)^4(1+6z^2)$
			$D(z)$
OM_4	2	$z/8$	$(z^2-2z-1)(3+6z-32z^2+\cdots+6z^9)$
	3	$3z$	$2(z^2-6z-1)^2(z^2-3z-1)(14161+97032z+\cdots-675z^{18})$
	4	z	$2\tau_1(166375+2531064z+\cdots+1593z^{26})$
	5	$5z$	$4\tau_2(15178486401+406610605840z+\cdots-19032727z^{34})$
LM_4	2	z^4	$11z^4+6z^2-1$
	3	3	$2(1-28z^2+128z^4+349z^6)$
	4	4	$-1+70z^2-1500z^4+6250z^6+22829z^8$
	5	5	$2(1-138z^2+6912z^4-139968z^6+559872z^8+2574571z^{10})$
SSM_4	2	2	$(-1+3z)^2(1+3z)^2(1+3z^2)^2$
	3	3	$8(-1+4z)^4(1+4z)^4(1+4z^2)^2$
	4	4	$(-1+5z)^6(1+5z)^6(1+5z^2)^2$
	5	5	$8(-1+6z)^8(1+6z)^8(1+6z^2)^2$
ZM_4	2	1	$128z^8$
	3	1	$45000z^{12}$
	4	1	$95551488z^{16}$
	5	1	$720600125000z^{20}$
SM_4	2	z^4	$-1+6z^2+11z^4$
	3	z^6	$3-168z^2+3120z^4+\cdots+288703z^{12}$
	4	z^8	$1-140z^2+7900z^4+\cdots+193295977z^{16}$
	5	z^{10}	$1-276z^2+32868z^4+\cdots+1400659421421z^{20}$

We denote $\tau_1=(z^2-8z-1)^2(z^2-4z-1)$ and $\tau_2=(z^2-10z-1)^2(z^2-5z-1)$.

Table 2. Extraneous fixed points ξ with stability check for selected cases with $2 \le m \le 5$

Method	m	ξ	$\lvert R'_p(\xi)\rvert$ < 1	1	> 1
OM_4	2	$\begin{pmatrix} -0.236068(double),\ 4.23607(double), \\ -0.383297 \pm 0.0513541i, -0.0566893 \pm 0.080201i, \\ 2.56294 \pm 0.343382i, 5.87705 \pm 8.31454i \end{pmatrix}$	4	0	8
	3	$\begin{pmatrix} -0.391184, 0.140217, 2.94977, 45.2356, 3.50113, \\ -0.297839 \pm 0.0130815i, -0.293747 \pm 0.305109i, \\ -0.268193 \pm 0.143408i, -0.250007 \pm 0.00452843i, \\ 0.277263 \pm 0.220113i, 4.58968 \pm 2.54771i, \\ 4.94834 \pm 1.33429i, 5.59758 \pm 1.67447i, \\ -20.5532, 6.25575 \pm 2.46539i \end{pmatrix}$	6	2	16
	4	$\begin{pmatrix} -0.362809, -0.19276, 0.0967611, 3.15177, 4.58622, \\ -0.312976 \pm 0.28064i, -0.230815 \pm 0.0119751i, \\ -0.214362 \pm 0.101119i, -0.201943 \pm 0.132782i, \\ -0.197308 \pm 0.00544175i, 0.121553 \pm 0.0285604i, \\ 0.338073 \pm 0.229271i, 4.45881 \pm 3.87221i, \\ 6.2742 \pm 1.54592i, 7.00175 \pm 1.77984i, \\ 7.52664 \pm 2.0502i, 8.07606 \pm 2.4942i, \\ -10.0102, 47.6944, 8.92701 \pm 3.60226i \end{pmatrix}$	19	0	13
	5	$\begin{pmatrix} -0.308556, 0.0967611, 3.392475, 5.639286, 59.27608, \\ -0.339249 \pm 0.255539i, -0.188593 \pm 0.00943813i, \\ -0.1793994 \pm 0.0767181i, -0.165326 \pm 0.119991i, \\ -0.165285 \pm 0.0946482i, -0.1630094 \pm 0.00520757i, \\ -8.39052, -0.1580223 \pm 0.00209708i, \\ 0.1102260 \pm 0.04038561i, 0.3706432 \pm 0.2330410i, \\ 4.461571 \pm 4.844355i, 7.5954991 \pm 1.802703i, \\ 8.439970 \pm 2.005078i, 8.98932 \pm 2.188386i, \\ 9.45558 \pm 2.415336i, 9.936863 \pm 2.745957i, \\ 10.54741 \pm 3.327619i, 11.59901 \pm 4.835288i \end{pmatrix}$	21	2	17
LM_4	2	-	0	0	0
	3	$\pm 0.316011 \pm 0.237459i, \pm 0.202285$	3	2	1
	4	$\pm 0.363341 \pm 0.25042i, \pm 0.166502 \pm 0.0278872i$	6	0	2
	5	$\begin{pmatrix} \pm 0.132338, \pm 0.389278 \pm 0.253836i, \\ \pm 0.144764 \pm 0.0387259i \end{pmatrix}$	4	4	2
SSM_4	2	$\pm 0.339862 \pm 0.0827798i, \pm 0.293496 \pm 0.40767i$	4	2	2
	3	$\begin{pmatrix} \pm 0.338352 \pm 0.147719i, \pm 0.326521 \pm 0.364069i, \\ \pm 0.206224 \pm 0.00537434i \end{pmatrix}$	4	0	8
	4	$\begin{pmatrix} \pm 0.345731 \pm 0.181977i, \pm 0.340903 \pm 0.335071i \\ \pm 0.167445 \pm 0.0296351i, \pm 0.165964 \pm 0.0260222i, \end{pmatrix}$	8	0	8
	5	$\begin{pmatrix} \pm 0.350911 \pm 0.208856i, \pm 0.34791 \pm 0.307393i, \\ \pm 0.144897 \pm 0.0396455i, \pm 0.143987 \pm 0.0382427i, \\ \pm 0.13209 \pm 0.000533604i \end{pmatrix}$	6	2	12
ZM_4	2	$\pm 0.444075 \pm 0.351364i, \pm 0.391443 \pm 0.104439i$	0	0	8
	3	$\begin{pmatrix} \pm 0.422507 \pm 0.362487i, \pm 0.386867 \pm 0.145445i, \\ \pm 0.207974 \pm 0.00300314i \end{pmatrix}$	2	0	10
	4	$\begin{pmatrix} \pm 0.407913 \pm 0.354286i, \pm 0.385877 \pm 0.167914i, \\ \pm 0.168244 \pm 0.0287327i, \pm 0.16712 \pm 0.0260866i \end{pmatrix}$	0	0	16
	5	$\begin{pmatrix} \pm 0.397403 \pm 0.339909i, \pm 0.384887 \pm 0.184884i, \\ \pm 0.145509 \pm 0.03926i, \pm 0.144565 \pm 0.0378459i, \\ \pm 0.132544 \pm 0.000542715i \end{pmatrix}$	0	0	20
SM_4	2	$-$	0	0	0
	3	$\pm 1/4(double),\ \pm i/2$	0	6	0
	4	$\pm 1/5(triple),\ \pm i/\sqrt{5}$	0	8	0
	5	$\pm 1/6(quadruple),\ \pm i/\sqrt{6}$	0	10	0

basins of attraction along with comparison of their dynamic properties and characteristics.

6 Numerical illustrations

In this section, we shall check the effectiveness of our proposed methods and validity of the theoretical results. We employ the present methods namely, method (2.6), (2.8), (2.9) and (3.1) (for $\gamma = 1$) denoted by MRM_2, MHM_3, MSM_3 and OM_4 respectively to solve a variety of nonlinear equations which are mentioned in examples 1–3.

First of all, we compare our one-point methods with one-point methods which are available in the literature namely, the method of Rall's method [12] (RM_2), Halley's method [5] (HM_3), super-Halley method [5] (SM_3).

After that we compare our optimal method OM_4 with optimal methods proposed by Soleymani et al. [16], between them we will choose their best expression (18) which is denoted by (SM_4), and also choose expression (11) proposed by Zhou et al. [19], denoted by (ZM_4). In addition to this, we also compare our optimal method with the one which has been recently developed by Sharma and Sharma [15], denoted by (SSM_4). Moreover, we compare OM_4 with the schemes given by Li et al. [8], out of which we have chosen method (75), denoted by (LM_4).

Finally, we compare them with schemes proposed by Argyros et al. [1], out of them we consider expression (2.4) with ($H_1(\tau)$, $\gamma = -0.01$, $\alpha = 1$) and ($H_2(\tau)$, $\gamma = -0.01$, $\alpha = 0$), called by AM_1 and AM_2, respectively. Actually, the methods AM_1 and AM_2 are designed only for simple roots according to their paper. Hence we expect the convergence of simple-root finders AM_1 and AM_2 is of linear character when locating a multiple root. It is well-known that Newton's method of simple roots for multiple roots gives linear convergence. Such aspects of AM_1 and AM_2 are evidently indicated in Tables 3–5 with a larger number of iterations as well as relatively bigger errors for convergence than our proposed methods.

For better comparisons of our proposed methods, we have given three comparison tables in each example: one is corresponding to absolute error value of given nonlinear functions with the same total number of functional evaluations (TNFE =12); second one is with respect to number of iterations taken by each method to obtain the accuracy of root up to 35 significant digits and the last one is regarding computational order of convergence. TNFE in the case of modified Newton's method, it will consume 12 functional evaluations in 6 iterations because it takes 2 functional evaluations per full iteration on the other hand, third-order methods namely, HM_3, SM_3, MHM_3 and MSM_3 will consume 12 functional evaluations in 4 iterations because they required three functional evaluations per iteration, and same law for fourth-order methods. This means the absolute error in the function after consuming 12 functional evaluations presented in the Tables 1–3 for second, third and fourth order methods is $|f(x_6)|$, $|f(x_4)|$ and $|f(x_4)|$, respectively.

Further, we use the following formula to calculate the computational order of convergence (see [18])

$$\rho = \ln|(x_{n+1} - r_m)/(x_n - r_m)|/\ln|(x_n - r_m)/(x_{n-1} - r_m)|, \ \ n > 1.$$

All the computations have been performed using the programming package *Mathematica* 9 with multiple precision arithmetic. We use $\epsilon = 10^{-100}$ as a tolerance error and A $-h$ (or A $+h$) stands for $A \times 10^{-h}$ (or $A \times 10^{+h}$). The following stopping criteria are used for computer programs:
$(i)|x_{n+1} - x_n| < \epsilon$ and $(ii)|f(x_{n+1})| < \epsilon$.

Example 1. Consider the following 8×8 matrix

$$B = \begin{bmatrix} 5 & 8 & 0 & 2 & 6 & -6 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 18 & -1 & 1 & 13 & -9 & 0 & 0 \\ 3 & 6 & 0 & 4 & 6 & -6 & 0 & 0 \\ 4 & -14 & -2 & 0 & 11 & -6 & 0 & 0 \\ 6 & 18 & -2 & 1 & 13 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -19 \end{bmatrix}.$$

The corresponding characteristic polynomial of this matrix is as follows:

$$f_1(x) = x(x-1)^3(x-2)(x-3)(x-4)(x+19). \tag{6.1}$$

The above equation has one multiple root at $x = 1$ of multiplicity three. It can be seen that Newton's method and its variants do not necessarily converge to the required root that is nearest to the starting value. For example, the methods RM_2, ZM_4, SSM_4 and LM_4 with initial guess $x_0 = 0.3$ diverge from the required root. In addition, the method SSM_4 also divergent for the other initial guesses 1.5 and 1.8. However, newly proposed families of methods namely, (2.6), (2.8), (2.9) and (3.1) (for $\gamma = 1$) do not exhibit this type of behavior.

Example 2. Let us consider the another nonlinear equation, which is given by

$$f_2(x) = x^2 \sin 4x. \tag{6.2}$$

This function has an infinite number of zeros but our desired root is $r = 0$ of multiplicity three, which is correct up to 35 digits. It can be seen that Newton's method and it's variants do not necessarily converge to the root that is nearest to the starting value. For example, methods SSM_4 and LM_4 are divergent for the initial guesses -0.4 and 0.4. However, newly proposed families of methods namely, (2.6), (2.8), (2.9) and (3.1) (for $\gamma = 1$) do not exhibit this type of behavior.

Example 3. Consider the following nonlinear equation

$$f_3(x) = (e^{-x} + \sin x)^3. \tag{6.3}$$

We find that this function has a finite number of zeros but our desired root is given by $r =$3.1830630119333635919391869956363946 of multiplicity three. It can

Table 3. Test problem (6.1)

$f(x)$	x_0	RM_2	MRM_2	HM_3	MHM_3	SM_3	MSM_3	SM_4
Comparison of different iterative methods with the same TNFE=12								
$f_1(x)$	0.3	D	1.3(−136)	2.6(−33)	1.3(−53)	4.3(−113)	8.3(−264)	3(−76)
	0.4	9.1(−108)	2.1(−145)	8.0(−66)	1.7(−70)	3.7(−106)	7.6(−101)	3(−68)
	1.5	6.2(−163)	7.2(−140)	8.5(−75)	1.3(−82)	6.0(−157)	6.0(−208)	8.1(−4)
	1.8	3.1(−15)	3.3(−39)	1.3(-1)	4.6(−24)	3.4(−10)	2.5(−67)	9.3(+0)
Comparison of different iterative methods with respect to number of iterations								
$f_1(x)$	0.3	D	8	6	6	5	5	5
	0.4	8	8	6	6	5	5	6
	1.5	8	8	6	6	5	5	947
	1.8	11	9	332	7	7	6	892
Computational order of convergence of different iterative methods								
$f_1(x)$	0.3	D	2.0000	3.0000	3.0000	3.0000	3.0000	4.0000
	0.4	2.0000	2.0000	3.0000	3.0000	3.0000	3.0000	4.0000
	1.5	2.0000	2.0000	3.0000	3.0000	3.0000	3.0000	1.0000
	1.8	2.0000	2.0000	1.0000	3.0000	3.0000	3.0000	1.0000

$f(x)$	x_0	ZM_4	SSM_4	LM_4	AM_1	AM_2	OM_4
Comparison of different iterative methods with the same TNFE=12							
$f_1(x)$	0.3	D	D	D	5.0(−3)	1.4(−9)	6.2(−84)
	0.4	1.8(−68)	6.2(−68)	1.5(−67)	2.1(−8)	5.0(−8)	3.4(−151)
	1.5	7.9(+59)	D	2.1(−16)	1.9(−3)	4.3(−3)	3.6(−492)
	1.8	1.7(+21)	D	1.5(−252)	9.1(−25)*	2.6(−107)*	3.0(−213)
Comparison of different iterative methods with respect to number of iterations							
$f_1(x)$	0.3	D	D	D	265	6	6
	0.4	6	6	6	261	294	5
	1.5	33	D	7	265	298	4
	1.8	16	D	5	4*	4	5
Computational order of convergence of different iterative methods							
$f_1(x)$	0.3	D	D	D	1.0000	4.0000	4.0000
	0.4	4.0000	4.0000	4.0000	1.0000	1.0000	3.9996
	1.5	4.0000	D	4.0000	1.0000	1.0000	4.0000
	1.8	4.0000	D	4.0000	4.0000*	4.0000*	4.0000

TNFE=total number of functional evaluations. D: stands for divergence. *: stands for converge to an undesired root.

Table 4. Test Problem (6.2)

$f(x)$	x_0	RM_2	MRM_2	HM_3	MHM_3	SM_3	MSM_3	SM_4
Comparison of different iterative methods with the same TNFE=12								
$f_2(x)$	−0.4	4.6(−382)	4.6(−466)	4.0(−72)	3.7(−75)	1.0(−175)	1.6(−153)	6.9(−13)
	0.4	4.6(−382)	4.6(−466)	4.0(−72)	3.7(−75)	1.0(−175)	1.6(−153)	6.9(−13)
Comparison of different iterative methods with respect to number of iterations								
$f_2(x)$	−0.4	6	6	6	6	5	5	6
	0.4	6	6	6	6	5	5	7
Computational order of convergence of different iterative methods								
$f_2(x)$	−0.4	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	5.0000
	0.4	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	5.0000
		ZM_4	SSM_4	LM_4	AM_1	AM_2	OM_4	
Comparison of different iterative methods with the same TNFE=12								
$f_2(x)$	−0.4	1.8(−10)	D	D	5.0(−6)	1.2(−4)	3.4(−185)	
	0.4	1.8(−10)	D	D	5.0(−6)	1.2(−4)	3.4(−185)	
Comparison of different iterative methods with respect to number of iterations								
$f_2(x)$	−0.4	7	D	D	261	265	5	
	0.4	7	D	D	261	265	5	
Computational order of convergence of different iterative methods								
$f_2(x)$	−0.4	5.000	D	D	1.0000	1.0000	5.0000	
	0.4	5.000	D	D	1.0000	1.0000	5.0000	

be seen that Newton's method and it's variants do not necessarily converge to the root that is nearest to the starting value. For example, method SSM_4 is divergent for initial guess $x_0 = 2$, while SM_4, ZM_4 and LM_4 converge to the undesired root. Similarly, methods SM_4 and SSM_4 are divergent for initial guess $x_0 = 4.2$ while ZM_4 converges to the undesired root. However, newly proposed families of methods namely, (2.6), (2.8), (2.9) and (3.1) (for $\gamma = 1$) do not exhibit this type of behavior.

7 Basins of attraction

This section directly describes the dynamics of iterative map (5.2) based on visual display of their basins of attraction when $f(z)$ is applied to a complex polynomial $p(z)$. We here investigate the comparison of basins of attraction for the attained best five or six multiple-root finders. It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterized by an iterative method applied to a fixed polynomial $p(z) \in \mathbb{C}$, see e.g. [9, 14]. The aim herein is to use basins of attraction as another way for comparing the convergence of the iteration algorithms in a global sense.

From the dynamical point of view, we consider a rectangle $D = [-3,\ 3] \times [-3,\ 3] \in \mathbb{C}$ with a 400×400 grid, and we assign a color to each point $z_0 \in D$ according to the multiple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each

Table 5. Test problem (6.3)

$f(x)$	x_0	RM_2	MRM_2	HM_3	MHM_3	SM_3	MSM_3	SM_4
Comparison of different iterative methods with the same TNFE=12								
$f_3(x)$	2.0	6.5(−112)	8.1(−147)	2.3(−67)	4.2(−75)	3.1(−165)	2.1(−124)	9.0(−10)*
	4.2	1.3(−109	5.7(−177)	5.5(−69)	6.6(−76)	2.5(−142)	8.2(−151)	224
Comparison of different iterative methods with respect to number of iterations								
$f_3(x)$	2.0	8	8	6	6	5	5	7*
	4.2	8	8	6	6	5	5	D
Computational order of convergence of different iterative methods								
$f_3(x)$	2.0	2.0000	2.0000	3.0000	3.0000	3.0000	3.0000	4.0000*
	4.2	2.0000	2.0000	3.0000	3.0000	3.0000	3.0000	D

$f(x)$	x_0	ZM_4	SSM_4	LM_4	AM_1	AM_2	OM_4
Comparison of different iterative methods with the same TNFE=12							
$f_3(x)$	2.0	1.8(−30)*	2.3(+5581)*	9.0(+55)*	3.4(−4)	2.0(−5)	1.8(−446)
	4.2	D	D	8.7(−84)	1.8(−5)	1.1(−5)	8.0(−232)
Comparison of different iterative methods with respect to number of iterations							
$f_3(x)$	2.0	5*	2224	32	266	249	4
	4.2	D	D	5	266	249	5
Computational order of convergence of different iterative methods							
$f_3(x)$	2.0	4.2482*	1.4735	5.0000	1.0000	1.0000	3.7333
	4.2	D	D	4.0000	1.0000	1.0000	4.0000

$*$: stands for convergence to an undesired root.

method is considered to be 100. In this way, we distinguish the attraction basins by their colors for different methods. The lighter colour of basin of attraction means faster convergence to the required root, while darker one means slower convergence.

For the first two test problems, we have taken the polynomial functions.

Problem 1. Let $p_1(z) = (z^5 + 2z - 1)^3$, having multiple zeros $\{-0.945068 \pm 0.854518i, 0.701874 \pm 0.879697i,\ 0.486389\}$ with multiplicity three. It is straight forward to see from Figure 1 that our methods, namely MRM_2, MHM_3 and MSM_3 have the same basin of attraction as compared to the classical RM_2, HM_3 and SM_3, respectively. Further, we observe from Figure 2 that our method OM_4 has lesser number of divergent points in comparison to the methods namely, SM_4, ZM_4 and LM_4 and shows less chaotic behavior as compared to methods namely, SM_4, ZM_4, LM_4 and SSM_4 in the above mentioned region.

Problem 2. Let $p_2(z) = (z^4 + z)^3$, having multiple zeros $\{-1,\ 0.5 + 0.866025i,\ 0.5 - 0.866025i,\ 0\}$ with multiplicity three. We can easily say, after seeing the Figure 3 that our methods, namely MRM_2 and MHM_3 have the same basin of attraction as compared to the classical RM_2 and HM_3, respectively. Further, from Figure 4, we observe that our method OM_4 has lesser number of divergent points in comparison to the methods namely, SM_4, ZM_4 and LM_4, lighter and lager basins of attraction belong to our method OM_4 as compared to other methods namely, SM_4, ZM_4, LM_4 and SSM_4.

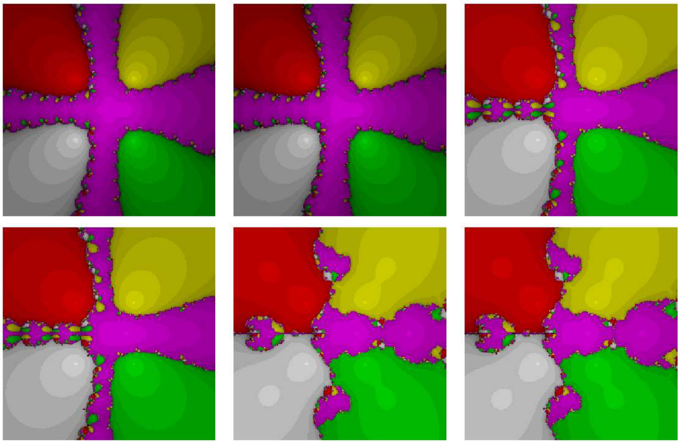

Figure 1. The basins of attraction for RM_2, MRM_2, HM_3, MHM_3, SM_3 and MSM_3, from left to right and top to bottom, respectively in Problem 1.

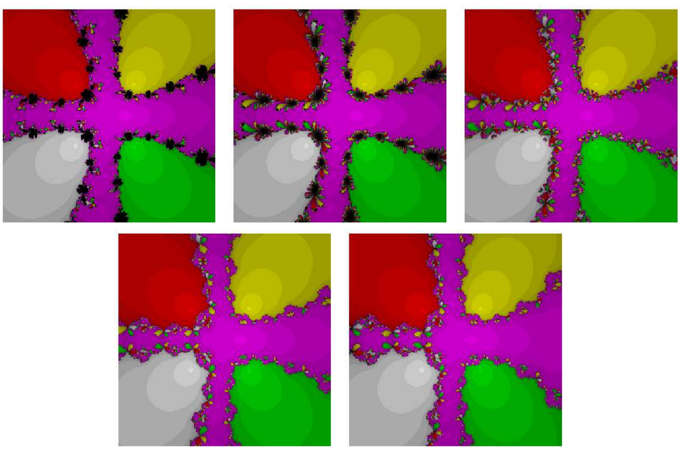

Figure 2. The basins of attraction for SM_4, ZM_4, SSM_4, LM_4 and OM_4, from left to right and top to bottom, respectively in Problem 1.

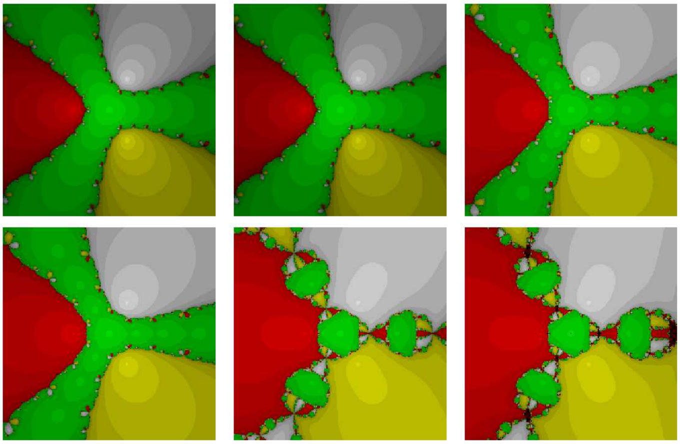

Figure 3. The basins of attraction for RM_2, MRM_2, HM_3, MHM_3, SM_3 and MSM_3, from left to right and top to bottom, respectively in Problem 2.

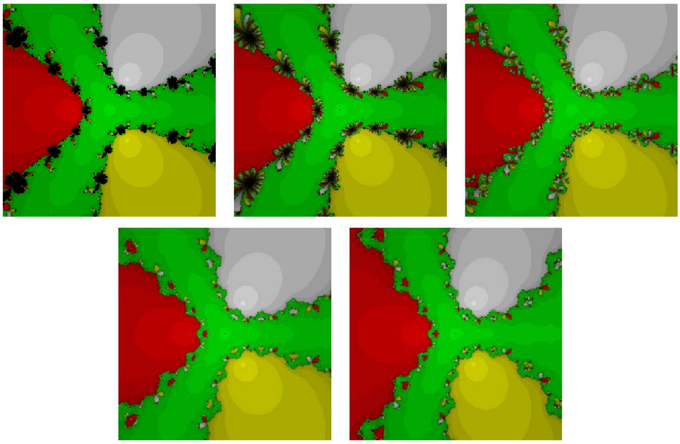

Figure 4. The basins of attraction for SM_4, ZM_4, SSM_4, LM_4 and OM_4, from left to right and top to bottom, respectively in Problem 2.

The last test problem is a non-polynomial function as follows.

Problem 3. Let $p_3(z) = \left(z^6 + \frac{1}{z}\right)^4$, having multiple zeros $\{-0.62349 \pm 0.781831i, 0.222521 \pm 0.974928i, -1, 0.900969 \pm 0.433884i\}$ with multiplicity four. Figure 5 demonstrates that our methods, namely MRM_2, MHM_3 and MSM_3 have the same basin of attraction as compared to the classical RM_2, HM_3 and SM_3, respectively. From Figure 6, we conclude that our method OM_4 has lesser number of divergent points in comparison to the methods namely, SM_4, ZM_4 and LM_4, and almost the same basin of attraction as compared to SSM_4 and LM_4.

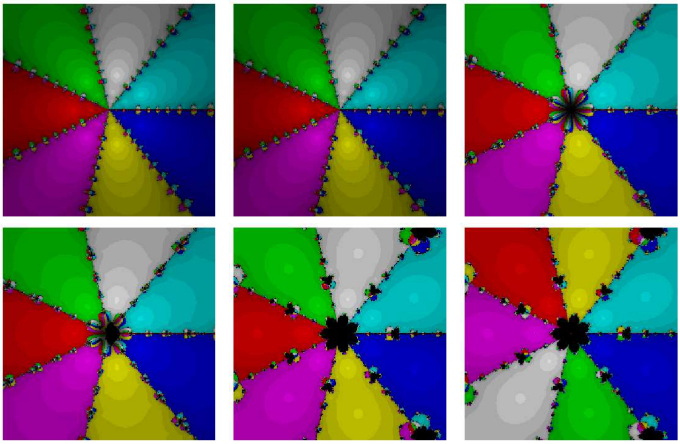

Figure 5. The basins of attraction for RM_2, MRM_2, HM_3, MHM_3, SM_3 and MSM_3, from left to right and top to bottom, respectively in Problem 3.

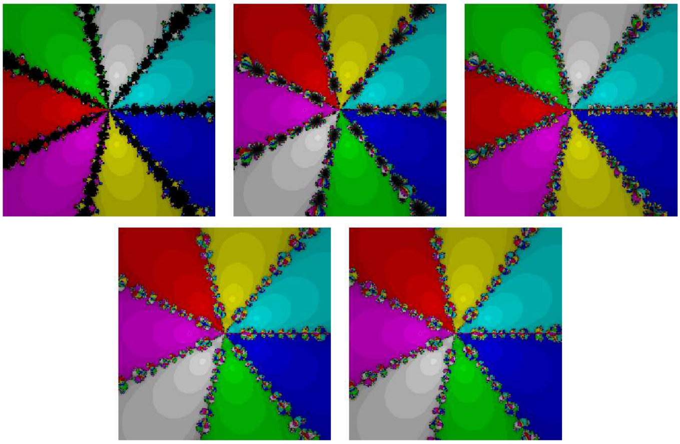

Figure 6. The basins of attraction for SM_4, ZM_4, SSM_4, LM_4 and OM_4, from left to right and top to bottom, respectively in Problem 3.

On the other hand, the second part deals with the dynamics of selected methods OM_4, LM_4, SSM_4, ZM_4 and SM_4 behind the extraneous fixed points ξ found from the roots of $H(z)$ whose construction is made by applying a simple quadratic polynomial $(z^2-1)^m$ to $f(z)$ in $H_f(x_n)$, using the rational iterative map (5.1). As mentioned in Section 5 about possible altering basins of attraction, we expect that the basins of attraction associated with extraneous fixed points would affect the Julia set basin boundaries. To visualize such basins of attraction with altered Julia set boundaries for specific examples, we employ four more examples described below in Problems 4–7. Indeed, Figures 7–10 well illustrate the affected Julia set boundaries.

Problem 4. Let $p_4(z) = \left(z^2-1\right)^2$, having multiple zeros $\{-1,\ 1\}$ with multiplicity two. From Figure 7, we conclude that our method OM_4 has larger and brighter basins of attraction to the root -1 as compared to the methods namely, LM_4, SSM_4, ZM_4 and SM_4.

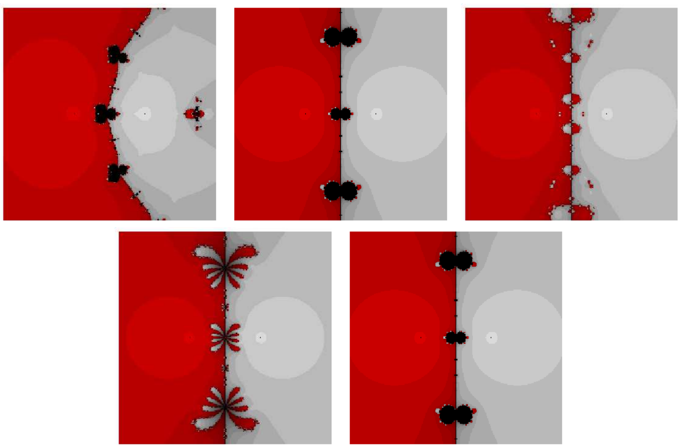

Figure 7. The basins of attraction for OM_4, LM_4, SSM_4, ZM_4 and SM_4, from left to right and top to bottom, respectively in Problem 4.

Problem 5. Let $p_5(z) = \left(z^2-1\right)^3$, having multiple zeros $\{-1,\ 1\}$ with multiplicity three. Figure 8 demonstrate that our method OM_4 has larger and brighter basins of attraction to the root -1 as compared to the methods namely, LM_4, SSM_4, ZM_4 and SM_4. Further, our method have very few divergent points in this particular region while SM_4 has many divergent points.

Problem 6. Let $p_6(z) = \left(z^3-z\right)^4$, having multiple zeros $\{-1,\ 1,\ 0\}$ with multiplicity four. If we inspect the basins of "0" more closely, SM_4 has the largest basin containing a huge number of divergent points and ZM_4 a medium basin containing a significant number of chaotic as well as divergent points. On the other hand, OM_4, LM_4 and SSM_4 has smaller basins containing less chaotic and less divergent points. It is, overall, straightforward in view of Figure 9 that

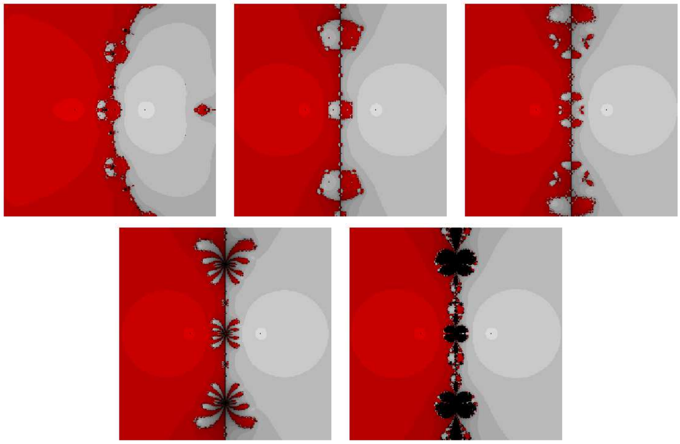

Figure 8. The basins of attraction for OM_4, LM_4, SSM_4, ZM_4 and SM_4, from left to right and top to bottom, respectively in Problem 5.

our method OM_4 has lesser chaotic behavior and larger basins of attraction as compared to other methods namely, LM_4, SSM_4, ZM_4 and SM_4. Further, our method has no divergent point in this particular region while SM_4 has many divergent points.

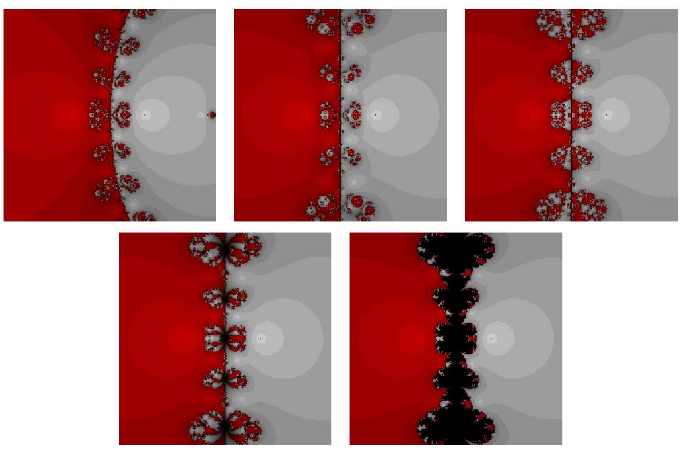

Figure 9. The basins of attraction for OM_4, LM_4, SSM_4, ZM_4 and SM_4, from left to right and top to bottom, respectively in Problem 6.

Problem 7. Let $p_7(z) = \left(z^4 - 1\right)^5$, having multiple zeros $\{\pm i,\ \pm 1\}$ with multiplicity five. By closely looking the basin colors of attractors $\pm i$, we find that SM_4 contains a large number of divergent points, while the rest of the listed methods contain much less divergent points. Overall, Figure 10 demonstrates that our method OM_4 has lesser number of divergent points in comparison to the methods namely, LM_4, SSM_4, ZM_4 and SM_4. Further, our method also shows lesser chaotic behavior in this figure as compared to other mentioned methods.

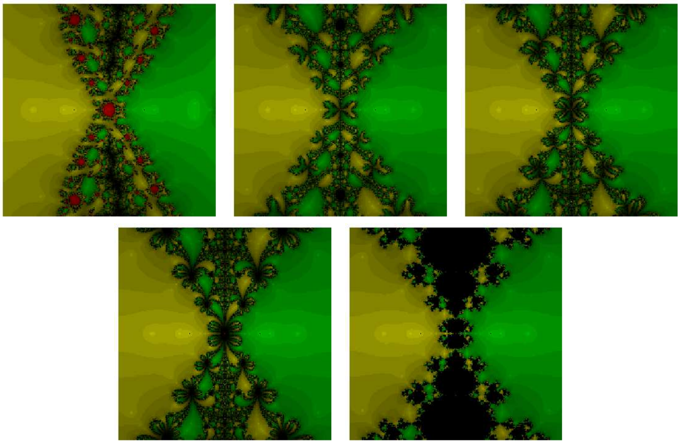

Figure 10. The basins of attraction for OM_4, LM_4, SSM_4, ZM_4 and SM_4, from left to right and top to bottom, respectively in Problem 7.

8 Conclusions

In this paper, we have proposed several formulas of second, third and fourth-order(optimal) methods respectively, for obtaining multiple roots of nonlinear equations numerically for the first time, which will converge to the required root even though the guess is far away from the desired root or the derivative is small in the vicinity of the root. Further, our proposed families of methods namely, (2.6), (2.8) and (2.9) have same error equations as those of classical Rall's method, super-Halley and Halley's method respectively, for multiple roots. These proposed methods offer a particular advantage for the cases where the traditional modified Newton's method and its variants of various order may not converge. Finally, we conclude that these methods are very effective in multi-precision environment and converge to a required root in a stable manner without divergence, oscillation or jumping problems. The dynamic study of the methods also supports the underlying theoretical aspects.

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