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One–Parametric Semigroups of Diffeomorphisms and One–Sided Vector Fields

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Abstract. The fractional generalization of dynamical systems is considered. For this purpose, the concepts of fractional phase semi-flow, fractional autonomous system and the generalized exponent of the vector field are introduced. Two examples of their application are explained in detail.
Keywords: dynamical system, phase flow, diffeomorphism.

AMS Subject Classification: 37E35; 26A33; 34C05.

1 INTRODUCTION

Dynamical systems (DSs) are among the basic tools for investigating evolutionary processes. The classical DS definition contains the concept of the finite-dimensional manifold as the phase space and the one-parametric group of diffeomorphisms as the phase flow [4]. Later on, the concept of the DS underwent changes. DSs have been generalized to infinite-dimensional systems; Poisson brackets formalism led to the simplectic manifolds; application of the discrete time resulted in the cascades, and so on [3]. Essentially new types of DSs are being generated by the new mathematical tools and physical ideas. One of the latest ideas concerns the concept of fractional calculus.

Derivatives and integrals of fractional order [1, 8, 9, 20, 25] have found many applications in recent studies in general physics [7, 16, 21, 28] and mechanics [2, 5, 6, 10, 14, 22, 23]. The interest in fractional analysis has been growing continually during the last few years. Fractional analysis is widely used in kinetic theories [11, 26, 29], statistical mechanics [13, 27], quantum mechanics [12, 17], dynamics in complex media [18, 15, 24], and many others. In mechanics much more attention has been given to continuous DSs describing diffusion and wave processes [21]. At the same time, some solutions of oscillator-type equations have been obtained [7, 28]. It seems that time has come for a wide generalization of the basic conception of DSs.

The classical DS is characterized by determinism, finite dimensionality and differentiation. Phase flow is the mathematical model of a determined process. Formalization of the conditions of finite dimensionality and differentiation implies the necessity for the phase space to be a differentiable manifold of finite dimensionality, and the phase space must be a one-parametric group of diffeomorphisms of this manifold.

In this paper, the fractional generalization of finite-dimensional DSs is considered. For this purpose, we introduce the concepts of fractional phase semi-flow, fractional DSs, the generalized exponent of the vector field.

2 Fractional Autonomous Systems

Definition 1. The phase semi-flow $(\mathcal{M}, \{g^{t+}\})$ is a pair that consists of the manifold $\mathcal{M}$ and the one-parametric semi-group $\{g^{t+}\}$ of its transformations.

Let the vector field $\xi^i = \xi^i(x^1, \ldots, x^n)$ be given in the space domain $\Omega = (x^1, \ldots, x^n)$. Each field of this kind is related to an autonomous system of integro-differential equations of the kind

$$\frac{d^\alpha x^i(t)}{dt_\pm^\alpha} = \xi^i(x^1(t), \ldots, x^n(t)), \quad i = 1, \ldots, n, \tag{2.1}$$

where $\dfrac{d^\alpha x^i(t)}{dt_\pm^\alpha}$ is the right $(+)$ or left $(-)$ fractional derivative of the order α, *i.e.* an integro-differential singular operator, a fractional generalization of the derivative (see Appendix). Depending on the type of the derivative $(\pm)$, the vector field is right- or left-sided. Below, if not indicated otherwise, right-sided vector fields will be considered. For $\alpha = 1$, the autonomous system of integro-differential equations (2.1) turns into a classical autonomous system.

Definition 2. A fractional dynamical system (FDS) comprises a phase semi-flow and the corresponding fractional autonomous system.

Solution of autonomous system (2.1), $x^i = x^i(t)$ is an integral curve of the vector field ξ^i. Let us through

$$F_t^{i(\alpha)}(x_0^1, \ldots, x_0^n) = x^i = x^i(t, x_0^1, \ldots, x_0^n) \tag{2.2}$$

denote the integral curve of the field ξ^i with the initial condition $x^i(0) = x_0^i$. Formula (2.2) gives the mapping

$$F_t^{i(\alpha)} : (x_0^1, \ldots, x_0^n) \to (x^1(t, x_0^1, \ldots, x_0^n), \ldots, x^n(t, x_0^1, \ldots, x_0^n))$$

of the initial space area Ω into itself, depending on parameter t (a shift by t along the integral curve).

Like the classical DS, the non-autonomous FDS is not more general than the autonomous one. Upon introducing the new function $x^{n+1} = t$, the non-autonomous FDS is reduced to an autonomous one.

The main theorem of the FDS serves as the generalization of the main theorem of the classical DSs [4].

The Main Theorem of the FDS. *Let $\xi^i(x)$ be a real, continuous function limited in the area Ω and meeting the Lipschitz condition*

$$|\xi^i(x^1,\dots,x_a^j,\dots,x^n) - \xi^i(x^1,\dots,x_b^j,\dots,x^n)| \le A|x_a^j - x_b^j|\,.$$

In this case, there exists the area $D \subset \Omega$ in which the solution of equation (2.1) with the initial condition $x^i(0) = x_0^i$ exists continuously and uniquely.

From the theorem of existence and uniqueness of the solution of system (2.1) it follows that the mapping $F_t^{(\alpha)}$ has been defined for small t values in the environment of this point $(x_0^1,\dots,x_0^n)$ and locally is a diffeomorphism.

Example 1. Let us consider one of the simplest examples of FDS: the fractional generalization of the law of unrestricted reproduction of microorganisms in a nutritive medium. The equation of motion is

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lambda x\,, \quad \alpha \le 1\,, \quad \lambda > 0\,, \quad x(0) = x_0\,. \tag{2.3}$$

When $\alpha \to 1$, we get the classical reproduction law (Malthus)[19]. Solution of equation (2.3) is

$$x(t) = x_0 \exp(\lambda^{1/\alpha} t)\,.$$

The diffeomorphism $F_t^{(\alpha)}$ here is an exponential function:

$$F_t^{(\alpha)}: \quad x_0 \to x_0 \exp(\lambda^{1/\alpha} t)\,.$$

The phase trajectories are straight lines $\dot{x}(t) = \lambda^{1/\alpha} x$, with a slope to the axis depending on the fractional parameter α (see Fig. 1). If F_0 is an identical

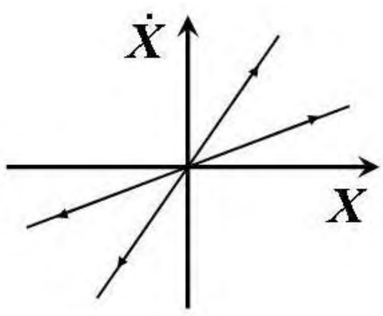


Figure 1. The vector field $\xi = (x, \lambda^{1/\alpha} x)$ in Cartesian coordinates for various initial conditions.

mapping and the diffeomorphisms satisfy the condition

$$F_{t^{1/\alpha}} \circ F_{s^{1/\alpha}} = F_{(t+s)^{1/\alpha}} \tag{2.4}$$

valid for both parts at the corresponding values of parameters $t, s, t+s$, then diffeomorphisms F_t comprise a local semi-group. For diffeomorphisms F_t to be a group, besides the condition (2.4) and existence of inverse mapping, one more condition should be met: $(F_t^\alpha)^{-1} = F_{-t}^\alpha$, which is not the case here. Here,

$$(F_t^\alpha)^{-1} = F_{t^*}^\alpha, \quad t^* = (-1)^{1/\alpha} t. \tag{2.5}$$

Thus, to each vector field ξ^i a local one-parametrical semigroup of diffeomorphisms $F_t^{i(\alpha)}$ is related. At small values of the parameter t, the explicit form of the mapping $F_t^{i(\alpha)}$ is given as follows:

$$x^i(t, x_0^1, \dots, x_0^n) = x_0^i + \frac{t^\alpha}{\Gamma(1+\alpha)} \xi^i(x_0^1, \dots, x_0^n) + o(t),$$

where $\Gamma(x)$ is the Euler gamma-function. With the same accuracy, the Jacobi matrix of the mapping $F_t^{i(\alpha)}$ is expressed as

$$\frac{\partial x^i(t)}{\partial x_0^j} = \delta_j^i + \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{\partial \xi^i}{\partial x_0^j} + o(t).$$

As follows from the (2.5), for the inverse mapping

$$x_0^i(t, x^1, \dots, x^n) = x^i - \frac{t^\alpha}{\Gamma(1+\alpha)} \xi^i(x^1, \dots, x^n) + o(t)$$

the corresponding Jacobi matrix is

$$\frac{\partial x_0^i}{\partial x^j} = \delta_j^i - \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{\partial \xi^i}{\partial x^j} + o(t).$$

Vice versa, if we have a one-parameter local semi-group or even a group of diffeomorphisms $F_t^i = (F_t^1, \dots, F_t^n)$, it allows an unambiguous reconstruction of the vector field

$$\xi^i = \frac{d^\alpha}{dt^\alpha} F_t^i|_{t=0}, \quad i = 1, \dots, n. \tag{2.6}$$

From the correlation (2.6), specifically, it follows that different vector fields may have one and the same integral curve. To elucidate the difference between the classical and the fractional DS, let us take a simple group of diffeomorphisms and find the corresponding vector fields.

Example 2. . We will consider, in a plane with coordinates (x, y), a one-parameter group or rotations at the angle t around the origin of coordinates. Then, the mapping F_t^i may be written as

$$x(t) = x_0 \cos t - y_0 \sin t, \quad y(t) = x_0 \sin t + y_0 \cos t.$$

From the property of the fractional derivative (see Appendix), it follows that the fractional derivative of the mapping F_t^i is

$$\frac{d^\alpha x(0)}{dt^\alpha} = x_0 \cos\left(\frac{\pi\alpha}{2}\right) - y_0 \sin\left(\frac{\pi\alpha}{2}\right),$$
$$\frac{d^\alpha y(0)}{dt^\alpha} = x_0 \sin\left(\frac{\pi\alpha}{2}\right) + y_0 \cos\left(\frac{\pi\alpha}{2}\right).$$

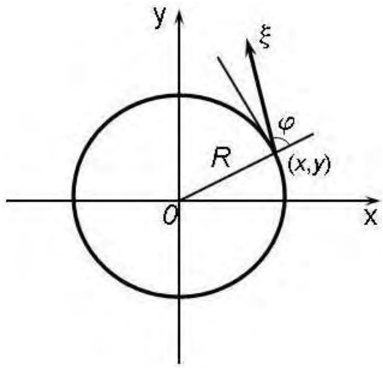


Figure 2. The vector field $\xi = (bx - ay, ax + by)$ in Cartesian coordinates ($a = \sin\varphi$, $b = \cos\varphi$, $\varphi = \alpha\pi/2$, $|\xi| = R$).

Thus, the field of velocities ξ^i, $i = 1, 2$ in Cartesian coordinates (x, y) has the form

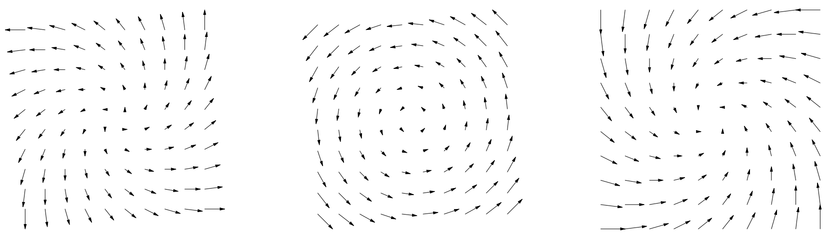
$$\xi(x, y) = (bx - ay, ax + by)\,, \quad |\xi| = R\,,$$

where $a = \sin\frac{\pi\alpha}{2}$, $b = \cos\frac{\pi\alpha}{2}$, $R = \sqrt{x_0^2 + y_0^2}$. The integral curves of this field, like in the classical case $\alpha = 1$, are circles $x^2 + y^2 = R^2 = const$ (Fig. 2)[1]. But in contrast to the classical case $\alpha = 1$, the vector field along these integral curves is an *isocline* field. Indeed, the scalar product

$$\langle \xi, r \rangle = \cos\left(\pi\alpha/2\right) R^2 = const\,,$$

whence it follows that the vector field is tangent when $\alpha = 2n + 1$ and turns into a field of normals when $\alpha = 2n, n \in \mathbb{Z}$.

Since the fractional derivative has been determined for all $\alpha \in \mathbb{R}$ (see Appendix), at $\alpha < 0$ the family of circles consists of integral curves of a corresponding autonomous system of integral equations. Depending on α, the vector field $\xi^\alpha = \xi^{\alpha \bmod (4)}$ is periodical (see Fig. 3).



a) b) c)

Figure 3. The general view of the vector field $\xi = (bx - ay, ax + by)$ for various values of the parameter $\alpha = (0.5, 1, 1.5)$ for the corresponding a), b) and c) .

[1] The Mathematica 4.1 package has been used for drawing Fig. 2, 3. License ♯: L2990-7548

3 The Generalized Exponent of the Vector Field

The one-parameter semi-group of diffeomorphisms $F_t^{(\alpha)}(x)$ corresponding to the right-side vector field $\xi(x)$ acts on the functions $f = f(x)$ according to the rule

$$(F_t^{(\alpha)} f)(x) := f(F_t^{(\alpha)}(x)) . \tag{3.1}$$

Let us consider, *e.g.* on a straight line, a one-parameter semigroup of right shifts $F_t^{(\alpha)}(x) = x + t$. The vector field ξ here is constant. The transformations (3.1) may be expressed as $F_t^{(\alpha)} f(x) = f(x + t)$.

For the Lebesgue measurable function $f(x)$, expression (3.1) at the small values of t may be presented as a Taylor series of fractional powers:

$$F_t^{(\alpha)} f(x) = f(x+t) = f(x) + \frac{D_{a+}^{\alpha} f(0)}{\Gamma(1+\alpha)} t^{\alpha} + \frac{D_{a+}^{1+\alpha} f(0)}{\Gamma(2+\alpha)} t^{1+\alpha} + \ldots = (1 + E_{-\alpha}^{t\partial_x}) f(x) ,$$

where E_{α}^{x} is a generalized exponent function (see Appendix).

In the general case, it is expedient to introduce the following definition:

Definition 3. The generalized exponent of the vector field ξ is the operator

$$1 + E_{-\alpha}^{t\partial_\xi} = 1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} \partial_\xi^{\alpha} + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} \partial_\xi^{1+\alpha} + \ldots = 1 + \sum_{n=0}^{\infty} \frac{t^{n+\alpha}}{\Gamma(1+n+\alpha)} \partial_\xi^{n+\alpha} ,$$

where $\partial_\xi^{\alpha} = \xi^{(\alpha)} \partial_x$ is a fractional directional derivative of the field ξ.

This definition allows us to generalize the classical shifting operator and use it not only in classical, but also in quantum mechanics [17].

4 Linear Vector Fields

Let $X = \left(X_k^i\right)$ be a real (or complex) matrix of the rank n. We shall construct a right-side vector field T_{X+} or a left-side vector field T_{X-} in space $\mathbb{R}^n$ (or $\mathbb{C}^n$), mapping its values in the point $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$) to equal $T_{X+}(x) = -Xx$ ($T_{X-}(x) = Xx$).

Theorem 1. *The integral curve $x_{\pm}(t)$ of the right-side vector field $T_{X+}(x)$ or of the left-side vector field $T_{X-}(x)$*

$$\frac{d^{\alpha} x}{dt_{\pm}^{\alpha}} = (\mp X)^{\alpha} x , \quad x(0) = x_0 \tag{4.1}$$

has the following form:

$$x_{\pm}(t) = \exp(\mp tX) \cdot x_0 , \tag{4.2}$$

the power of the matrix as usual being $A^{\alpha} \equiv \exp(\alpha \log A)$. The proof follows from the direct substitution of (4.2) *into expression* (4.1) *and from the property of the fractional derivative (see Appendix).*

5 Conclusions and Discussion

Thus, we have that: a) FDS is the fractional generalization of the classical DS; b) the analog of the tangent vector field for the integral curve is the field of isoclines; c) the generalized exponent of the vector field $1+E_{-\alpha}^{t\partial_\xi}$ is the fractional generalization of the shifting operator; d) the integral curve of the right-side linear vector field has the form of the exponent mapping.

From the mathematical point of view, a sufficient condition for the existence of a fractional derivative, e.g. its Riemann–Liouville form, is belonging to the class of continuous functions (see *e.g.* [25]). In other words, a continuous but non-differentiable function can never be the solution of the classical DS, while it can be for a FDS. The physical contents of the FDS is in the process of construction, but even now we can note that FDSs are a generalization of the classical DSs when the classical concept of velocity does not work.

Some notes on the introduction of the FDS. The core object is the fractional derivative. However, the latter may be not only in the Caputo–Weyl form, but also in that of Riemann–Liouville. How much the FDS properties depend on the form of its introduction? The basic criterion while introducing the FDS is the physical correspondence between the fractional and the classical DSs systems: when the order of the fractional derivative $\alpha \to 1$, a FDS must turn into a classical DS: $FDS \xrightarrow[\alpha\to 1]{} DS$. In the case of the Riemann–Liouville form, this criterion is satisfied. However, another problem arises: when $0 < \alpha < 1$, there is no concept of velocity, and the phase trajectory does not exist.

The complex FDS, when $x^i(t), \xi^i(x) \in \mathbb{C}$, is equivalent to the real FDS (2.1) with $2n$ unknown functions

$$\frac{d^\alpha}{dt_\pm^\alpha}(\operatorname{Re} x^i(t)) = \operatorname{Re} \xi^i(x)\,, \qquad \frac{d^\alpha}{dt_\pm^\alpha}(\operatorname{Im} x^i(t)) = \operatorname{Im} \xi^i(x)\,.$$

A coherent union of two real FDSs, differing from their direct sum, should take place in the case of analytical vector fields ξ^i; however, no result has been so far obtained in this direction. We may show that

$$e^{\frac{d}{dt}} x(t) = \frac{1}{2\pi i} \int_C \Gamma(-\alpha) D_t^\alpha x(t)\, d\alpha\,,$$

where $\Gamma(z)$ is the Euler gamma-function, and the closed curve C has no peculiarities of the phase trajectory $x(t)$, the order of the fractional derivative α is a complex number. This identity may be considered as a certain hint to the existence of such a nontrivial interrelation.

Another important field of FDS is dimensional reduction: for $\alpha \to 0$, part of the dynamical equations turn into constraints. We obtain a unique tool: the phase semi-flow of rank n under continuous limit transition $\alpha \to 0$ turns into a phase semi-flow of the rank $n-1$ and into certain constraints. The correlation among the classical DSs of different dimensions deserves a detailed analysis.

FDSs belong to the class of continuous DSs. The relation of the FDS with ergodic theory and topological DSs is absolutely unclear. This paper, of course, does not exhaust all the properties of the FDS. Rather, it opens the door to a realm worthy of far more extensive studies.

6 Appendix

The left Riemann– Liouville fractional derivative

$$ {}^{RL}_{\ a}D^{\alpha}_{t+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}\frac{f(\tau)\,d\tau}{(t-\tau)^{1+\alpha-n}} . $$

The right fractional derivative

$$ {}^{RL}_{\ b}D^{\alpha}_{t-}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^{n}\int_{t}^{b}\frac{f(\tau)\,d\tau}{(\tau-t)^{1+\alpha-n}} , $$

where $n = [\alpha] + 1$ and $\alpha > 0$.

The corresponding left Caputo's fractional derivative is defined as follows:

$$ {}_{a}D^{\alpha}_{t+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\frac{d\tau}{(t-\tau)^{1+\alpha-n}}\left(\frac{df(\tau)}{d\tau}\right)^{n} , $$

and the right Caputo's fractional derivative is defined as

$$ {}_{b}D^{\alpha}_{t-}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}\frac{d\tau}{(\tau-t)^{1+\alpha-n}}\left(-\frac{df(\tau)}{d\tau}\right)^{n} , $$

where α represents the order of the derivative: $n-1 < \alpha < n$ and $\alpha > 0$.

The relationship between the Riemann–Liouville and Caputo fractional derivatives is $(0 < \alpha < 1)$:

$$ {}_{a}D^{\alpha}_{t+}f(t) = {}^{RL}_{\ a}D^{\alpha}_{t+}f(t) - \frac{1}{\Gamma(1-\alpha)}\frac{f(a)}{(t-a)^{\alpha}}, $$
$$ {}_{b}D^{\alpha}_{t-}f(t) = -{}^{RL}_{\ b}D^{\alpha}_{t-}f(t) + \frac{1}{\Gamma(1-\alpha)}\frac{f(b)}{(b-t)^{\alpha}}. $$

Thus, *e.g.* the left Riemann–Liouville fractional derivative of the constant function C equals to $C/[\Gamma(1-\alpha)(t-a)^{\alpha}]$, whereas the left Caputo fractional derivative equal to zero. From the other hand, many properties of the Caputo fractional derivatives are the same for the Riemann–Liouville fractional derivative, when $f(a) = f(b) = 0$:

$$ {}_{a}D^{-\alpha}_{t+}f(t) = {}_{a}I^{\alpha}_{t+}f(t), \quad ({}_{b}D^{-\alpha}_{t-}f(t) = {}_{b}I^{\alpha}_{t-}f(t)), \quad \alpha > 0, \tag{6.1} $$
$$ {}_{a}I^{\alpha}_{t+}f(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}\frac{f(\tau)\,d\tau}{(t-\tau)^{1-\alpha}}, \quad t > a, $$
$$ {}_{b}I^{\alpha}_{t-}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{b}\frac{f(\tau)\,d\tau}{(\tau-t)^{1-\alpha}}, \quad t < b, $$
$$ {}_{a}D^{\alpha}_{t+}f(t) = {}_{a}I^{-\alpha}_{t+}f(t), \quad ({}_{b}D^{\alpha}_{t-}f(t) = {}_{b}I^{-\alpha}_{t-}f(t)), \quad \alpha > 0, $$
$$ {}_{a}D^{\alpha}_{t+}\,{}_{a}D^{\beta}_{t+}f(t) = {}_{a}D^{\beta}_{t+}\,{}_{a}D^{\alpha}_{t+}f(t) = {}_{a}D^{\alpha+\beta}_{t+}f(t), $$
$$ {}_{a}I^{\alpha}_{t+}\,{}_{a}I^{\beta}_{t+}f(t) = {}_{a}I^{\beta}_{t+}\,{}_{a}I^{\alpha}_{t+}f(t) = {}_{a}I^{\alpha+\beta}_{t+}f(t), $$

The analog of Taylor expansion is valid:

$$f(t)=\sum_{j=0}^{n-1}\frac{{}_aD_{t+}^{\alpha+j}f(0)}{\Gamma(1+\alpha+j)}t^{\alpha+j}+R_n(t)\,,\quad n=[\operatorname{Re}\alpha]+1\,,$$

where $R_n(t)={}_aI_{t+}^{\alpha+n}\,{}_aD_{t+}^{\alpha+n}f(t)$. The derivatives of some functions:

$${}_{-\infty}D_{t+}^{\alpha}\sin\lambda t=\lambda^{\alpha}\sin\left(\lambda t+\frac{\pi\alpha}{2}\right),\quad {}_{-\infty}D_{t+}^{\alpha}\cos\lambda t=\lambda^{\alpha}\cos\left(\lambda t+\frac{\pi\alpha}{2}\right),$$

where $\lambda>0\,,\alpha>-1$. When $\alpha\le -1$, we have to use the property (6.1).

$${}_{-\infty}D_{t+}^{\alpha}\mathrm{e}^{\lambda t+\mu}=\lambda^{\alpha}\mathrm{e}^{\lambda t+\mu}\,,\quad \operatorname{Re}\lambda>0\,.$$

Some special functions: $E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\dfrac{z^n}{\Gamma(\alpha n+\beta)}$ is the Mittag-Leffler function, and $1+E_{\alpha}^{z}=1+\sum_{n=0}^{\infty}\dfrac{z^{n+\alpha}}{\Gamma(1+\alpha+n)}$ is the generalized exponential function. The fractional Caputo derivative in this paper is used ${}_{-\infty}D_{t+}^{\alpha}\equiv\dfrac{d^{\alpha}}{dt_{+}^{\alpha}}$

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