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An Affirmative Answer to Quasi-Contractions' Open Problem under Some Local Constraints in JS−metric Spaces

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Abstract. In this work, we first present JS-Pompeiu-Hausdorff metric in JS -metric spaces and then introduce well-behaved quasi-contraction in order to find an affirmative answer to quasi-contractions' open problem under some local constraints in JS-metric spaces. In the literature, this problem solved when the constant modules $\alpha \in [0, 1/2]$ and when $\alpha \in (1/2, 1]$, finding conditions by which the set of all fixed points be non-empty, has remained open yet. Moreover, we support our result by a notable example. Finally, by taking into account the approximate strict fixed point property we present some worthwhile open problems in these spaces.

Keywords: iterative fixed point, strict fixed point, JS-Pompeiu-Hausdorff metric, well-behaved quasi-contraction.

AMS Subject Classification: 47H10; 37C25.

1 Introduction and preliminaries

Let (M, d) be a metric space. A map $T : M \to M$ is known as quasi-contraction if there exists a constant $\alpha \in (0, 1)$ such that for every $x, y \in X$,

$$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

This notion was introduced and studied first by Ćirić [3] as one of the most general contractive type mappings. A well-known Ćirić's result is that a quasi-contraction T possesses a unique fixed point.

Investigating the literature related to the results in which quasi-contractions plays the crucial role, one can find out that the following problem presented by Amini Harandi [1] has been remaind open yet:

Problem 1. Let (M, d) be a complete metric space and $T : M \to P_{cb}(M)$ be a set-valued quasi-contraction for some $\alpha \in (\frac{1}{2}, 1)$. Does T has a fixed point?

Very recently, Khojasteh et al. [10] presented a necessary and sufficient condition for quasi-contractions to have at least a strict fixed point (see [10, Theorem 4.2] for more detail). In spite of the fact that many studies have been accomplished to find at least a fixed point for multi valued version of quasi-contraction mappings, all these efforts either have conduced to failure or the authors forced to change the original version into a lighter one. It is worth mentioning that no one has found a weaker condition by which any quasi-contraction satisfies $\mathcal{F}(T)\backslash S\mathcal{F}(T) \neq \emptyset$ yet (see [4, 5, 11] for example).

In 2015 Jleli and Samet [8], introduced JS-metric spaces that recover a large class of topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces [6, 12].

Catching a glimpse to the literature included JS-metirc space, one can notice that Hausdorff metric induced by a JS-metric space, has not been considered yet. It turned out that we encouraged to put forward the multi-valued quasi-contraction and solving the Problem 1 in such spaces.

Let $\varDelta$ be a nonempty set and $\varTheta : \varDelta \times \varDelta \to [0, +\infty]$ be a given mapping. For every $u \in \varDelta$, consider the following set

$$\mathcal{M}(\varTheta, \varDelta, u) = \{\{u_n\} \subset \varDelta : \lim_{n\to\infty} \varTheta(u_n, u) = 0\}.$$

Definition 1. [8] We say that $\varTheta$ is a JS-metric on $\varDelta$ if it satisfies the following conditions:

(a_1) for each pair $(u, v) \in \varDelta \times \varDelta$, we have $\varTheta(u, v) = 0$ implies $u = v$;

(a_2) for each pair $(u, v) \in \varDelta \times \varDelta$, we have $\varTheta(u, v) = \varTheta(v, u)$;

(a_3) there exists $\kappa > 0$ such that

$$\text{if } (u, v) \in \varDelta \times \varDelta, \{u_n\} \in \mathcal{M}(\varTheta, \varDelta, u), \text{ then } \varTheta(u, v) \leq \kappa \limsup_{n\to\infty} \varTheta(u_n, v).$$

In this case, we say the pair $(\varDelta, \varTheta)$ is a JS-metric space or for short JS-metric space.

In the following, we list some topological properties selected from [8].

Let (Δ, Θ) be a JS-metric space and let Let $\{u_n\}$ be a sequence in Δ.

- We say that $\{u_n\}$ Θ-converges to u if $\{u_n\} \in \mathcal{M}(\Theta, \Delta, u)$,
- if (u_n) Θ-converges to u and Θ-converges to v then $u = v$,
- $\{u_n\}$ is Θ-Cauchy sequence if $\lim\limits_{m,n\to\infty} \Theta(u_n, u_{n+m}) = 0$,
- (Δ, Θ) is said to be Θ-complete if every Cauchy sequence in Δ is convergent to some element in Δ.

In this research, after introducing JS-Pompeiu-Hausdorff metric and proving the Nadler fixed point theorem, we presented well-behaved quasi-contraction multi valued mapping in JS-metric space and found at least a fixed point under some local constraints when $\alpha \in (0, 1)$. Supporting the results, we expressed a notable instance to show that the family of well-behaved quasi-contractions are. Finally, we terminated our results by introducing some open questions related to strict fixed point of quasi-contractions and its generalization in JS-metric spaces.

2 JS-Pompeiu-Hausdorff metric spaces

Let (Δ, ϱ) be a metric space, and let $P_{cb}(\Delta)$ denote the classes of all non empty, closed and bounded subsets of Δ. Let $T : \Delta \to P_{cb}(\Delta)$ be a multi-valued mapping on Δ. A point $u \in \Delta$ is called a fixed point of T if $u \in Tu$. Set $\digamma(T) = \{u \in \Delta : u \in Tu\}$. An element $u \in \Delta$ is said to be an **strict fixed point**(endpoint) of T, if $Tu = \{u\}$. Denote $S\digamma(T) = \{u \in \Delta : Tu = \{u\}\}$, the set of all strict fixed point of T. Note that $S\digamma(T) \subseteq \digamma(T)$.

For more details about the progress of fixed point theory in the set of multi-valued mapping, we refer the authors to study the notations and results from [7].

Let H be the Pompeiu-Hausdorff [2] metric on $P_{cb}(\Delta)$ induced by ϱ, that is,

$$\mathcal{H}(A, B) := \max\Big\{\sup_{u\in B} \varrho(u, A), \sup_{u\in A} \varrho(u, B)\Big\}, \quad A, B \in P_{cb}(\Delta),$$

in which, $\varrho(u, A) = \inf\{\varrho(u, v) : v \in A\}$.

Definition 2. Let (Δ, Θ) be a JS-metric space. Let $\mathcal{H}_\Theta$ define as follows:

$$\mathcal{H}_\Theta(A, B) := \max\Big\{\sup_{u\in B} \Theta^*(u, A), \sup_{u\in A} \Theta^*(u, B)\Big\}, \quad A, B \in P_{cb}(\Delta),$$

in which, $\Theta^*(u, A) = \inf\{\Theta(u, v) : v \in A\}$.

It is worth mentioning that whether $\mathcal{H}_\Theta$ is JS-metric on $P_{cb}(\Delta)$ or not. In the following lemma, we show that $(P_{cb}(\Delta), \mathcal{H}_\Theta)$ is a JS-metric space.

Note that

$$\mathcal{M}(\mathcal{H}_\Theta, P_{cb}(\Delta), A) = \{A_n \in P_{cb}(\Delta) : \lim_{n\to\infty} \mathcal{H}_\Theta(A_n, A) = 0\}.$$

In this section, we add a property to JS-metric spaces by which we are also able to relax the triangle inequality and present some new fixed point theorems in the set of multi-valued mappings.

Let Θ be a JS-metric space which satisfies the following additional condition:

For each sequences $\{u_n\}$, $\{v_n\}$

$$\Theta(u_n, v_n) \to 0 \quad \text{implies} \quad \limsup_{n\to\infty} \Theta^*(u_n, A) = \limsup_{n\to\infty} \Theta^*(v_n, A), \tag{a_4}$$

in which $\Theta^*(u, A) = \inf\{\Theta(u, v) : v \in A\}$, for all $u \in \Delta$ and $A \subseteq \Delta$.

In the rest of this paper, we consider JS-metric spaces satisfies in the (a_4).

The following Lemma plays the crucial role throughout this paper.

Lemma 1. *Let (Δ, Θ) be a JS-metric space satisfies in (a_4) and let $\{u_n\} \subset \Delta$ be any sequence which converges to $u \in \Delta$. Then, for each $v \in \Delta$ we have*

$$\limsup_{n\to\infty} \Theta(u_n, v) = \Theta(u, v).$$

Proof. By the assumption, (Δ, Θ) satisfies in (a_4) and $\lim_{n\to\infty} \Theta(u_n, u) = 0$. Taking into account $\{v_n = u\}$ and $A = \{v\}$ for each $n \in N$ in (a_4), we conclude that

$$\limsup_{n\to\infty} \Theta(u_n, v) = \limsup_{n\to\infty} \Theta(v_n, v) = \Theta(u, v).$$

□

Note that, our added property to JS-metric space makes it more natural toward the limit process.

Lemma 2. *Let (Δ, Θ) be a JS-metric space. Then $(P_{cb}(\Delta), \mathcal{H}_\Theta)$ is also a JS-metric space.*

Proof. We have to prove that all conditions of Definition 1 are satisfied. To prove (a_1), suppose that $A, B \in P_{cb}(\Delta)$ and $\mathcal{H}_\Theta(A, B) = 0$. Then,

$$\sup_{u\in A} \Theta^*(u, B) = \sup_{u\in B} \Theta^*(u, A) = 0.$$

Let $u \in A$ is arbitrary, then $\Theta^*(u, B) = 0$. So for each $n \in \mathbb{N}$, there exists a sequence $\{b_n\}$ in B such that $\{b_n\}$ is Θ-converges to u. Thus, $u \in \bar{B} = B$. Therefore, $A \subset B$. The other case has the same argument. Hence, $A = B$. Thus, (a_1) is proved. Proving (a_2) is transparent so we pass it by. To prove (a_3), we have to show that there exists $\kappa > 0$ such that if $(A, B) \in P_{cb}(\Delta) \times P_{cb}(\Delta)$ and the sequence $\{A_n\} \in \mathcal{M}(\mathcal{H}_\Theta, P_{cb}(\Delta), A)$, then

$$\mathcal{H}_\Theta(A, B) \leq \kappa \limsup_{n\to\infty} \mathcal{H}_\Theta(A_n, B).$$

Suppose that $u \in B, v \in A$ are arbitrary. Since $\lim_{n\to\infty} \mathcal{H}_\Theta(A_n, A) = 0$, for each $n \in \mathbb{N}$, there exists $k_n > 0$ such that for all $n \geq k_n$ we have

$$\mathcal{H}_\Theta(A_n, A) < \frac{1}{n}. \tag{2.1}$$

Let n be arbitrary and fixed. Then by the definition of $\mathcal{H}_\Theta$ for each $\epsilon > 0$, there exists $v_\epsilon \in A_n$ such that

$$\Theta(v, v_\epsilon) < \mathcal{H}_\Theta(A_n, A) + \epsilon. \tag{2.2}$$

Considering (2.1) and (2.2) together, one can easily derive that for all $n \geq k_n$,

$$\Theta(v, v_\epsilon) < \frac{1}{n} + \epsilon.$$

Thus, we can choose a subsequence $\{v_m\} \subset A_n$ such that $\{v_m\} \in \mathcal{M}(\Theta, \Delta, v)$. By (a_3), there exists $\kappa > 0$ such that

$$\Theta(u, v) \leq \kappa \limsup_{m\to\infty} \Theta(u, v_m).$$

Also, we have

$$\Theta^*(u, A) \leq \Theta(u, v) \leq \kappa \limsup_{m\to\infty} \Theta(u, v_m),$$

so

$$\frac{\Theta^*(u, A)}{\kappa} \leq \inf_{k\geq 1} (\sup_{m\geq k} (\Theta(u, v_m))) \leq \sup_{m\geq k} (\Theta(u, v_m)). \tag{2.3}$$

For all $k \in \mathbb{N}$, there exists $m_k \geq k > 0$ such that

$$\sup_{m\geq k} (\Theta(u, v_m)) < \Theta(u, v_{m_k}) + \epsilon. \tag{2.4}$$

Combining (2.3) and (2.4), one can conclude that

$$\frac{\Theta^*(u, A)}{\kappa} \leq \Theta(u, v_{m_k}) + \epsilon \leq \Theta^*(u, A_n) + \epsilon \leq \sup_{u\in B} \Theta^*(u, A_n) + \epsilon.$$

Letting ϵ approach to zero, one can obtain the following:

$$\Theta^*(u, A) \leq \kappa \sup_{u\in B} \Theta^*(u, A_n).$$

Therefore,

$$\sup_{u\in B} \Theta^*(u, A) \leq \kappa \sup_{u\in B} \Theta^*(u, A_n). \tag{2.5}$$

Taking lim sup on both sides of (2.5), we have

$$\sup_{u\in B} \Theta^*(u, A) \leq \kappa \limsup_{n\to\infty} (\sup_{u\in B} \Theta^*(u, A_n)). \tag{2.6}$$

Thus, by the definition of $\mathcal{H}_\Theta$ and applying (2.6), we have

$$\mathcal{H}_\Theta(A, B) \leq \kappa \limsup_{n\to\infty} \mathcal{H}_\Theta(A_n, B).$$

So (a_3) is proved. □

Lemma 3. *Let (Δ, Θ) is a JS-metric space satisfies in (a_4), and let $\{u_n\}$ be a sequence which Θ-converges to u. Then*

$$\limsup_{n\to\infty} \Theta^*(u_n, A) = \Theta^*(u, A).$$

for each $A \subseteq \Delta$.

Proof. The proof is straightforward by additional assumption (a_4) and the fact that $\{u_n\}$ converges to u in (Δ, Θ). □

3 Main result

In this section, we present our main result. The following notation is needed throughout this research.

For every $\{u_n\} \subset \varDelta$ denote

$$\delta(\Theta, T, u_n) = \sup\{\Theta(u_{i+1}, u_{j+1}) : u_{i+1} \in Tu_i, u_{j+1} \in Tu_j \textit{ and } i, j \geq n\}.$$

Theorem 1. *Let $(\varDelta, \Theta)$ is a Θ-complete JS-metric space and let $T : \varDelta \to P_{cb}(\varDelta)$ is a multi-valued mapping and there exists $\alpha \in [0,1]$ such that for all $u, v \in \varDelta$*

$$\mathcal{H}_\Theta(Tu, Tv) \leq \alpha\Theta(u, v).$$

Let there exists $u_0 \in \varDelta$ such that $\delta(\Theta, T, u_0) < \infty$, then T has a fixed point.

Proof. Let $u_0 \in \varDelta$ be arbitrary and let $u_1 \in Tu_0$. If $u_1 = u_0$ then u_0 is the fixed point of T. So let $u_0 \neq u_1$. Then, for $\epsilon = \left(\frac{1}{\sqrt{\alpha}} - 1\right)\mathcal{H}_\Theta(Tu_1, Tu_0)$, there exists $u_2 \in Tu_1$ such that

$$\Theta(u_1, u_2) < \mathcal{H}_\Theta(Tu_1, Tu_0) + \epsilon = \frac{1}{\sqrt{\alpha}}\mathcal{H}_\Theta(Tu_1, Tu_0) \leq \sqrt{\alpha}\Theta(u_1, u_0).$$

Thus, for chosen $u_n \in Tu_{n-1}$, there exist $u_{n+1} \in Tu_n$ such that

$$\Theta(u_{n+1}, u_n) \leq \sqrt{\alpha}\Theta(u_n, u_{n-1}).$$

Therefore, taking $h = \sqrt{\alpha}$, one concludes that

$$\begin{aligned} \Theta(u_{n+1}, u_n) &\leq h\Theta(u_n, u_{n-1}) \leq h^2\Theta(u_{n-1}, u_{n-2}) \\ &\leq \cdots \leq h^n D(u_0, u_1) \leq h^n\delta(\Theta, T, u_0). \end{aligned} \tag{3.1}$$

Since $\delta(\Theta, T, u_0) < \infty$ and $0 < h < 1$ we have

$$\lim_{n\to\infty} \Theta(u_{n+1}, u_n) = 0.$$

Considering (3.1), it yields that

$$\delta(\Theta, T, u_n) \leq h^n\delta(\Theta, T, u_0).$$

So for all $n, m \in \mathbb{N}$, we have

$$\Theta(u_n, u_{n+m}) \leq \delta(\Theta, T, u_n) \leq h^n\delta(\Theta, T, u_0),$$

thus

$$\lim_{n,m\to\infty} \Theta(u_n, u_{n+m}) = 0.$$

Therefore, $\{u_n\}$ is a Θ-Cauchy sequence and $\varDelta$ is Θ-complete, thus it converges to some $z \in \varDelta$.

Now we show that $z \in Tz$. We have

$$\limsup_{n\to\infty} \Theta^*(u_{n+1}, Tz) \le \limsup_{n\to\infty} \mathcal{H}_\Theta(Tu_n, Tz) \le \limsup_{n\to\infty} \alpha\Theta(u_n, z) = 0.$$

Thus, $\lim_{n\to\infty} \Theta^*(u_{n+1}, Tz) = 0$. Applying Lemma 3, we have

$$\Theta^*(z, Tz) = \limsup_{n\to\infty} \Theta^*(u_{n+1}, Tz) = 0.$$

It means that $z \in Tz$ and the proof is completed. □

Let (Δ, ϱ) be a metric space. A multi-valued mapping $T : \Delta \to P_{cb}(\Delta)$ is said to be quasi-contraction if there exists $\alpha \in (0,1)$ such that

$$\mathcal{H}(Tu, Tv) \le \alpha \max\{\varrho(u,v), \varrho(u,Tu), \varrho(v,Tv), \varrho(u,Tv), \varrho(v,Tu)\}$$

for all $u, v \in \Delta$.

4 Well-behaved quasi-contraction

Investigating the literature related to the results in which quasi-contractions plays the crucial role, we find out that the following problem presented by Amini Harandi [1] has been remaind open yet:

Problem 2. Let (Δ, ϱ) be a complete metric space and $T : \Delta \to P_{cb}(\Delta)$ be a set-valued quasi-contraction for some $\alpha \in (\frac{1}{2}, 1)$. Does T has a fixed point?

Very recently, Khojasteh et al. [10] presented a necessary and sufficient condition for quasi-contractions to have at least a strict fixed point (see [10, Theorem 4.2] for more detail). It is worth mentioning that no one has found a weaker condition by which any quasi-contraction satisfies $\digamma(T) \backslash S\digamma(T) \neq \emptyset$ yet (see [4] for example).

In the following, we present an affirmative answer to this open problem under some local constraint in JS-metric spaces.

Definition 3. Let (Δ, Θ) is a JS-metric space a let $T : \Delta \to P_{cb}(\Delta)$ be a multi-valued mapping. We say that a sequence $\{u_n\}$ based at u_0 is iterative if $u_0 \in \Delta$ and for all $n \in \mathbb{N}$, $u_n \in Tu_{n-1}$.

Definition 4. For every $u, v \in \Delta$ let

$$N_\Theta(u,v) := \max\Big\{\Theta(u,v), \Theta^*(u,Tu), \Theta^*(v,Tv), \Theta^*(u,Tv), \Theta^*(v,Tu)\Big\},$$

we say that T is well-behaved quasi-contraction if there is $0 < \alpha < 1$ such that

$$\mathcal{H}_\Theta(Tu, Tv) \le \alpha N_\Theta(u,v)$$

and for each iterative sequence $\{u_n\}$ such that $u_n \neq u_{n-1}$, there exists $\{s_n\} \subset [1, +\infty)$ such that

$$a)\ \Theta^*(u_{n-1}, Tu_n) \le s_n \Theta(u_{n-1}, u_n), \quad b)\ \limsup_{n\to\infty} s_n < 1/\sqrt{\alpha}.$$

Theorem 2. *Let (Δ,Θ) is a Θ-complete JS-metric space and let $T:\Delta\to P_{cb}(\Delta)$ is a well-behaved quasi-contraction. Moreover, suppose that there exists $u_0\in\Delta$ such that $\delta(\Theta,T,u_0)<\infty$. Then T has at least a fixed point in Δ.*

Proof. Let $u_0\in\Delta$ be arbitrary and let $u_1\in Tu_0$. If $u_1=u_0$ then u_0 is the fixed point of T. So let $u_0\neq u_1$. Then, for $\epsilon=\left(\frac{1}{\sqrt{\alpha}}-1\right)\mathcal{H}_\Theta(Tu_1,Tu_0)$, there exists $u_2\in Tu_1$ such that

$$\Theta(u_1,u_2)<\mathcal{H}_\Theta(Tu_1,Tu_0)+\epsilon=\frac{1}{\sqrt{\alpha}}\mathcal{H}_\Theta(Tu_1,Tu_0)\leq\sqrt{\alpha}N_\Theta(u_1,u_0).$$

Thus, for chosen $u_n\in Tu_{n-1}$, there exist $u_{n+1}\in Tu_n$ such that

$$\Theta(u_{n+1},u_n)\leq\sqrt{\alpha}N_\Theta(u_n,u_{n-1}).$$

Also, applying $a)$ and $b)$ in Definition 4, one can conclude that

$$\begin{aligned}\Theta(u_{n+1},u_n)&\leq\sqrt{\alpha}N_\Theta(u_n,u_{n-1})\\&=\sqrt{\alpha}\max\Big\{\Theta(u_n,u_{n-1}),\Theta^*(u_n,Tu_n),\Theta^*(u_{n-1},Tu_{n-1}),\\&\qquad\Theta^*(u_n,Tu_{n-1}),\Theta^*(u_{n-1},Tu_n)\Big\}\\&\leq\sqrt{\alpha}\max\Big\{\Theta(u_n,u_{n-1}),\Theta(u_n,u_{n+1}),\Theta^*(u_{n-1},Tu_n)\Big\}\\&\leq\sqrt{\alpha}\max\Big\{\Theta(u_n,u_{n-1}),\Theta(u_n,u_{n+1}),s_n\Theta(u_{n-1},u_n)\Big\}\\&=\sqrt{\alpha}s_nD(u_n,u_{n-1})\leq(\sqrt{\alpha}s_n)^2\Theta(u_{n-1},u_{n-2})\\&\leq\cdots\leq(\sqrt{\alpha}s_n)^nD(u_1,u_0)\leq(\sqrt{\alpha}s_n)^n\delta(\Theta,T,u_0).\end{aligned}\tag{4.1}$$

Since $\delta(\Theta,T,u_0)<\infty$ and $0<\sqrt{\alpha}<1$, $b)$ of Definition 4 shows that

$$\limsup_{n\to\infty}\sqrt[n]{(\sqrt{\alpha}s_n)^n}=\limsup_{n\to\infty}\sqrt{\alpha}s_n<1.$$

It means that $\sum_{n=1}^{\infty}(\sqrt{\alpha}s_n)^n<\infty$. Therefore, $\lim\limits_{n\to\infty}(\sqrt{\alpha}s_n)^n=0$. Thus, we have

$$\lim_{n\to\infty}\Theta(u_{n+1},u_n)=0.$$

Considering (4.1), it yields that

$$\delta(\Theta,T,u_n)\leq(\sqrt{\alpha}s_n)^n\delta(\Theta,T,u_0).$$

So for all $n,m\in\mathbb{N}$, we have

$$\Theta(u_n,u_{n+m})\leq\delta(\Theta,T,u_n)\leq(\sqrt{\alpha}s_n)^n\delta(\Theta,T,u_0).$$

Hence,

$$\lim_{n,m\to\infty}\Theta(u_n,u_{n+m})=0.$$

Therefore, $\{u_n\}$ is a Θ-Cauchy sequence and Δ is Θ-complete, thus it converges to some $z\in\Delta$.

Now we show that $z \in Tz$. On the contrary, suppose that $\Theta^*(z, Tz) > 0$, then

$$\begin{aligned}
&\Theta^*(u_{n+1}, Tz) \leq \mathcal{H}_\Theta(Tu_n, Tz) \leq \alpha N_\Theta(u_n, z) \\
&= \alpha \max\Big\{\Theta(u_n, z), \Theta^*(u_n, Tu_n), \Theta^*(z, Tz), \Theta^*(u_n, Tz), \Theta^*(z, Tu_n)\Big\} \\
&\leq \alpha \max\Big\{\Theta(u_n, z), \Theta(u_n, u_{n+1}), \Theta^*(z, Tz), \Theta^*(u_n, Tz), \Theta(z, u_{n+1})\Big\}. \quad (4.2)
\end{aligned}$$

Considering Lemma 3 and taking the limit on both sides of (4.2), one can conclude that

$$\Theta^*(z, Tz) \leq \limsup_{n\to\infty} \Theta^*(u_{n+1}, Tz) \leq \limsup_{n\to\infty} N_\Theta(u_n, z) \leq \alpha\Theta^*(z, Tz) = 0.$$

Thus, it yields $\alpha > 1$ and this is a contradiction. So $z \in \overline{Tz} = Tz$ and this means that $z \in \Delta$ is the fixed point of T. $\square$

Theorem 3. *Let (Δ, ϱ) is a complete metric space and for some $\alpha \in (0,1)$ let $T : \Delta \to P_{cb}(\Delta)$ satisfies the following:*

$$\mathcal{H}(Tu, Tv) \leq \alpha \max\Big\{\varrho(u, v), \varrho(u, Tu), \varrho(v, Tv), \varrho(u, Tv), \varrho(v, Tu)\Big\}$$

for all $u, v \in \Delta$. Additionally, suppose that for each iterative sequence $\{u_n\}$ such that $u_n \neq u_{n-1}$, there exists $\{s_n\} \subset [1, +\infty)$ such that

$$a)\ \varrho(u_{n-1}, Tu_n) \leq s_n \varrho(u_{n-1}, u_n), \quad b)\ \limsup_{n\to\infty} s_n < \frac{1}{\sqrt{\alpha}}.$$

Then T has at least a fixed point in Δ.

Proof. Using the fact that (Δ, ϱ) is a complete metric space, one can conclude the desired result from Theorem 2. $\square$

The following example shows that Theorem 2 is meaning-full and the family of well-behaved quasi-contractions is not empty.

Example 1. Let $\Delta = \{0\} \cup [2, +\infty)$ and let

$$\begin{cases} \Theta(u, v) = u + v, & u \neq 0 \text{ and } v \neq 0, \\ \Theta(u, 0) = \Theta(0, u) = u/2, & \text{otherwise} \end{cases}$$

and let $T0 = \{0, 2\}$ and let $Tu = 1/(1 + u) + 2$ for each $u \geq 2$. We show that

$$\mathcal{H}_\Theta(Tu, Tv) \leq \frac{1}{4} N_\Theta(u, v),$$

and by Theorem 2, one can infer that T has a fixed point. To see this, let $u, v \geq 2$ and without loose of generality we can suppose that $u > v$. Thus,

$$\mathcal{H}_\Theta(Tu, Tv) = \frac{u + v + 2}{(u + 1)(v + 1)} \quad \text{and} \quad \mathcal{H}_\Theta(Tu, T0) = \frac{1}{2(u + 1)} + 1.$$

Also,

$$N_{\Theta}(u,v) = \max\left\{u+v, u+\frac{1}{1+v}+2\right\} \quad \text{and} \quad N_{\Theta}(u,0) = \frac{1}{u+1}+2+u.$$

Note that, since $u > v$ we have

$$\begin{aligned}\frac{\mathcal{H}_{\Theta}(Tu,Tv)}{N_{\Theta}(u,v)} &= \frac{u+v+2}{\max\{u+v, u+1/(v+1)+2\}(u+1)(v+1)} \\ &\leq \frac{1}{\max\{u+v, u+1/(v+1)+2\}} < \frac{1}{2v} \leq \frac{1}{4},\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{\mathcal{H}_{\Theta}(Tu,T0)}{N_{\Theta}(u,0)} &= \frac{1/(2(1+u))+1}{1/(1+u)+2+u} = \frac{2u+3}{2u^2+6u+6} \\ &\leq \max_{u\geq 2}\left\{\frac{2u+3}{2u^2+6u+6}\right\} = \frac{7}{26}.\end{aligned} \tag{4.4}$$

Hence, (4.3) and (4.4) lead us to take $\alpha = \frac{1}{4}$ and so

$$\mathcal{H}_{\Theta}(Tu,Tv) \leq \frac{1}{4}N_{\Theta}(u,v).$$

In addition, the only iterative sequence $\{u_n\}$ in which $u_n \neq u_{n-1}$ by the initial point $u_0 \in \mathit{\Delta}$ is $u_{n+1} = \dfrac{1}{1+u_n} + 2$ based at u_0 and since $\{u_n\}$ is converges to $\frac{1+\sqrt{13}}{2}$ and this is the fixed point of T. Taking

$$s_n = \frac{u_{n-1}+u_{n+1}}{u_{n-1}+u_n}$$

one can see that a) and b) of Definition 4 hold.

5 Open questions

In this section, we present some conjectures in the set of multi-valued mappings. It seems difficult to work on multi-valued contraction mappings by considering JS-Pompeiu-Hausdorff metric space because, triangle inequality possessed the results natural and well-behaved, in spite of, the substituted condition in JS-metric space which suffers from lacking the continuity.

Let $(\mathit{\Delta},\Theta)$ be a JS-metric space. A mapping $T : \mathit{\Delta} \to P_{cb}(\mathit{\Delta})$ has the approximately strict fixed point property (see [9]), if

$$\inf_{u\in\mathit{\Delta}} \sup_{v\in Tu} \Theta(u,v) = 0.$$

Let $T : \mathit{\Delta} \to \mathit{\Delta}$ be a single valued mapping. Then T has the approximately strict fixed point property if and only if T has the approximately fixed point property, i.e.,

$$\inf_{u\in\mathit{\Delta}} \Theta(u,Tu) = 0.$$

Problem 3. Let (Δ, Θ) be a Θ-complete JS-metric space. Suppose that $T : \Delta \to P_{cb}(\Delta)$ be a multi-valued mapping satisfied in

$$\mathcal{H}_\Theta(Tu, Tv) \leq \alpha N_\Theta(u, v)$$

for each $u, v \in \Delta$, where $\alpha \in [0, 1)$. Then under which well-posed constraints T has a unique strict fixed point if and only if T has the approximate strict fixed point property?

Problem 4. Let (Δ, Θ) be a Θ-complete JS-metric space. Suppose that $T : \Delta \to P_{cb}(\Delta)$ be a multi-valued mapping satisfied in

$$\mathcal{H}_\Theta(Tu, Tv) \leq \psi(N_\Theta(u, v))$$

for each $u, v \in \Delta$, where $\psi : [0, +\infty) \to [0, +\infty)$ is upper semi continuous, $\psi(t) < t$ for all $t > 0$, and satisfies $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then under which well-posed constraints T has a unique strict fixed point if and only if T has the approximate strict fixed point property.

For more details, we refer the authors to [9, 13].

6 Conclusions

Broadly translated our findings indicate that most of known fixed point achievements by taking into account triangle inequality or other its generalizations have analogous perception in which some struggle have been accomplished to obtain desired result. However, omitting the triangle inequality makes the results more difficult to prove, rehearsing fixed point results by considering JS-metric makes them worthwhile because, most of the spaces in natural phenomenon, in the most well-behaved conditions, suffer from lacking triangle inequality.

In this research, after introducing JS-Pompeiu-Hausdorff metric and proving the Nadler fixed point theorem, we presented well-behaved quasi-contraction multi valued mapping in JS-metric space and found at least a fixed point under some local constraints when $\alpha \in (0, 1)$. Supporting the results, we expressed a notable instance to show that the family of well-behaved quasi-contractions are. Finally, we terminated our results by introducing some open questions related to strict fixed point of quasi-contractions and its generalization in JS-metric spaces. Further studies need to be carried out in fixed point theory when the other relations substituted by triangular inequality.

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