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ON CALCULATION OF LINEAR RESOURCE PLANNING MODELS FOR OPTIMAL PROJECT SCHEDULING

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Abstract. Recent author's papers have shown new opportunities resulting from the treatment of resource planning in project scheduling as the optimization problem for a hybrid system. This approach gives the possibility to work out the optimum resource sharing in an iteration process of branch-and-bound type. The present paper concentrates on the most standard case of the problem in question for which all the relationships may be represented in the linear form. Two exact finite methods are proposed. The first method is obtained using the piecewise-linear form of Bellman function, the second evolves from the decomposition approach for dynamic linear programming problem.

Key words: project, scheduling, work, capacity, control, optimum, decomposition, Bellman function.

1 Introduction

The question of internal resources planning for project scheduling is not incorporated in the classical models of PERT/CPM and in most of later improvements of them [5]. The problem still attracts attention [1], but the results are not promising: for all general problem formulations only heuristics were proposed; the cases when an exact solution may be calculated with a regular method have little significance.

A new form of the model proposed by the author [8] gives the chance to calculate the optimum resources sharing between parallel works with the two-level optimization method. As most optimization techniques the proposed method results in an iteration algorithm not having the property of finite convergence.

The present paper focuses on the most widely used case when the resources usage may be represented with linear relationships. It means that each work intensity is proportional to its share of capacities that may alter from one stage to another. It is supposed that the use of materials is proportional to works rates; the difference between linear and non-linear models is displayed in this aspect only. Linear form of the model enables to incorporate in it some additional restrictions, such as restrictions on terms of some works (initiation and termination times and duration), admits conditions of non-strict precedence of works etc. However, here we use explicitly only the basic model formulated in [8] and present it in a slightly different form. Most results formulated here stay valid for the other linear resource planning models.

We assume that the project consists of n jobs and N_R capacity types; let I_{Rj} be the set of jobs using the j type. The qualitative state d of the project is a vector, d_i being the state of the i-th job with three possible values:

- 0 ("not started"),
- 1 ("in performance"),
- 2 ("finished").

Let $I_l(d) = \{d_i = l,\, i \in \{1, \dots, n\}\}$, where $l = 0, 1, 2$. Obviously $I_0(d) \cup I_1(d) \cup I_2(d) = \{1, \dots, n\}$. Every such a triple $(I_0(d),\, I_1(d),\, I_2(d))$ defines the unique d. For the amount of i-th job we use the notation x_i to denote its current value and x_{Ti} for its final value. For each i-th job we introduce the set of strictly preceding jobs P_i, so that $d_i = 0$, if $d_j < 2$ for some $j \in P_i$. The period of the project fulfillment is divided into the succession of stages by the events of jobs termination and subsequent jobs origination. It well known that the optimum resource sharing is constant between two successive events, thus it is sufficient to consider the project state only at the moments of these events.

It is convenient [8] to split each instant of a job termination into two parts: the stage termination (the job is being done the last time and subsequent jobs are not begun) and the beginning of the next stage (the job is terminated and subsequent jobs are started). The set of all possible values of quantitative state vector X for a given d is defined as

$$X(d) = \{x \mid x_i = 0, \text{ if } d_i = 0;\ 0 \le x_i \le x_{Ti}, \text{ if } d_i = 1;\ \ x_i = x_{Ti}, \text{ if } d_i = 2\}.$$

All possible qualitative states form the set A_D defined by the following conditions

$$d_i = \begin{cases} 0, & \text{if for some } j \in P_i \quad d_j < 2, \\ l,\ l \in \{0, 1, 2\}, & \text{otherwise.} \end{cases}$$

The initial value d^0 is given by $d_i^0 = 1$ if $P_i = \emptyset$, and $d_i^0 = 0$, otherwise.

Let $d \in A_D$. The next state d' must satisfy the following conditions:

$$I_T \ne \emptyset,\ I_T \subseteq I_1(d),\ I_2(d') = I_2(d) \cup I_T,\ I_{01} = \{i \in I_0(d) \mid P_i \subseteq I_2(d')\}, \quad (1.1)$$
$$I_1(d') = (I_1(d) \setminus I_T) \cup I_{01}, \quad I_0(d') = I_0(d) \setminus I_{01}.$$

We denote by $D_+(d)$ the set of all states d' satisfying conditions (1.1) and by $D_+(d, I_T)$ the set of states d' satisfying (1.1) for a given $I_T \subset I_1(d)$.

For a given d, $x \in X(d)$ the set of possible states (d', x'), $d' \in D_+(d)$, $x' \in X(d')$ given at the beginning of the next stage is defined by the following linear relationships involving the stage duration t and the amounts of jobs y_i:

$$0 < u_{\min i} t \le y_i \le u_{\max i} t, \;\; i \in I_1(d), \quad y_i = 0, \; i \notin I_1(d), \tag{1.2}$$
$$\sum_{i \in I_{Rj}} y_i \le u_{Rj} t; \quad t \ge 0,$$

and

$$x'_i = x_i + y_i, \;\; x'_i \le x_{Ti}, \; i = 1, \ldots, n,$$
$$I_T = \{i \mid x'_i = x_{Ti}\}, \quad d' = D_+(d, I_T).$$

2 Dynamic Programming Method for Linear Resource Planning Model

The optimum synthesis of the resources sharing is found with the dynamic programming method. The terminal state (x_T, d_T) of the project is $x_T = (x_{T1}, \ldots, x_{Tn})$, $d_T = (2, \ldots, 2)$.

To apply the method we define the Bellman function for states at the beginning of stages $W(x, d)$, expressing the minimum time for reaching the final state (x_T, d_T) from (x, d). We must set $W(x_T, d_T) = 0$. The Bellman equation is given as

$$W(x, d) = \min\{W(x, d, d') \mid d' \in D_+(d), \; P(x, d, d') \ne \emptyset\}, \quad x \in X(d),$$

where the polyhedron $P(x, d, d')$ is determined by restrictions (1.2) together with

$$x_i + y_i \le x_{Ti}, \text{ if } d_i = 1; \quad x_i + y_i = x_{Ti}, \text{ if } d_i = 1 \& d'_i = 2; \tag{2.1}$$

and

$$W(x, d, d') = \min\{t + W(x + y, d') \mid \; (y, t) \in P(x, d, d')\}.$$

Lemma 1. *If a function $W(x)$ defined on a convex compact set $X \subset R^n$ is continuous and piecewise-linear, i.e., there exists a finite set of linear functions $W_m(x) = C_{X0m} x + C_{0m}$, $m = 1, \ldots, M$, and $W(x) = W_{m(x)}(x)$ for every $x \in X$, then $X = P_1 \cup \ldots \cup P_L$, where $W(x) = W_{m(l)}(x)$ for each $x \in P_l$, $l = 1, \ldots, L$, and polyhedra P_l are determined by inequalities*

$$C_{Xlr} x + C_{lr} \le 0, r = 1, \ldots, n_l. \tag{2.2}$$

Proof. Let $i, j \in \{1, \ldots, L\}$, $i \ne j$, and let $X(i, j) = \{x \in X \mid W_i(x) < W_j(x)\}$. If $X(i, j) \ne \emptyset$, then $X(i, j)$ is an opened polyhedron and its closure $[X(i, j)] = \{x \in X \mid W_i(x) \le W_j(x)\}$ is a closed polyhedron. Let $J = \{m(1), \ldots, m(L)\}$ be an arbitrary permutation of $\{1, \ldots, M\}$ and

$$X(J) = \{x \in X \mid W_{m(s)}(x) < W_{m(s+1)}(x), \;\; s = 1, \ldots, M - 1\}.$$

Let $X(J) \neq \emptyset$. From the continuity of $W(x)$ we conclude that there exist an $m(J)$, such that $W(x) = W_{m(J)}(x)$ for any $x \in X(J)$, and hence $W(x) = W_{m(J)}(x)$ for any $x \in [X(J))]$. Each of such polyhedra may be treated as some $P_{l(J)}$ defined with a set of inequalities of the type (2.2):

$$(C_{X0m(s+1)} - C_{X0m(s)})x + (C_{0m(s+1)} - C_{0m(s)}) \leq 0, \ s, s+1 \in S(J). \quad (2.3)$$

The lemma is proved. □

Theorem 1. *For an arbitrary $d \in A_D$ function $W(x,d)$ is a continuous piecewise-linear function of x.*

Proof. Let us prove the continuity of $W(x,d)$. Let $d^0 \in A$, $x^0 \in X(d^0)$. All possible ways to finish the project beginning from (x^0, d^0) may be represented by $(x(k), T(k), d(k))$, $k = 1, \ldots, N+1$, where

$$x(1) = x, \ \ d(1) = d, \ \ d(N+1) = d_T, \ \ x(N+1) = x_T,$$

$$x(k+1) = x(k) + y(k)), \ T(k+1) = T(k) + t(k), \ d(k+1) = D_+(x(k), \ I_T(k)),$$

$(y(k), t(k))$ satisfy (1.2) and

$$I_T(k) = \{j \in I_1(d(k)) \mid x_j(k+1) = x_{Tj}\} \neq \emptyset,$$
$$x_j(k+1) < x_{Tj}, \ \ j \in I_1(d(k)) \setminus I_T(k).$$

The optimum succession $S^0 = \{(x^0(k), \ T^0(k), \ d^0(k)), \ i = 1, \ldots, N^0+1\}$ minimizes

$$t(1) + \ldots + t(N),$$

and $W(x^0, d^0) = t^0(1) + \ldots + t^0(N)$.

Let $x^1 \in X(d^0)$, $\Delta x(1) = x^1 - x^0$ and let $\|\Delta x(1)\|$ be small. We assess $|W(x^1, d^0) - W(x^0, d^0)|$ by constructing a possible succession

$$S^1 = \{(x^1(k), \ T^1(k), \ d^1(k)), \ i = 1, \ldots, N^1+1\},$$

that is close to S^0 in a certain sense.

Let $u^0(k) = y^0(k)/t^0(k)$. Setting $y^1(1) = u^0(1)t^1(1)$, we satisfy (1.2); the value of $t^1(1)$ is determined via the condition $x_i^1(1) + u_i^0(1)t^1(1) = x_{Ti}$, $i \in I_T^1(1)$, $x_i^1(1) + u_i^0(1)t^1(1) < x_{Ti}$, $i \notin I_T^1(1)$. Thus we conclude that

$$t^1(1) = t^0(1) + \min\{-\Delta x_i(1)/u_i^0(1) \mid i \in I_T^0(1)\},$$
$$|t^1(1) - t^0(1)| \leq \|\Delta x(1)\|_\infty / u_{\min},$$

where $u_{min} = \min\{u_{\min i}, i = 1, \ldots, n\}$, $\|a\|_\infty = \min\{|a_i|, i = 1, \ldots, m\}$ for $a \in R^m$, $I_T^1(1) = \operatorname{Arg\,min}\{-\Delta x_i(1)/u_i^0(1), i \in I_T^0(1)\} \subseteq I_T^0(1)$,

$$\|\Delta x(2)\|_\infty \leq \|\Delta x(1)\|_\infty + u_{\max}|t^1(1) - t^0(1)| \leq \|\Delta x(1)\|_\infty(1 + u_{\max}/u_{\min}),$$
$$u_{\max} = \min\{u_{\max i}, i = 1, \ldots, n\}.$$

Taking into account that $x_i^0(2) = x_{Ti}$, $i \in I_T^0(1) \setminus I_T^1(1)$, we conclude that all these jobs will terminate not later than $t^1(1) + \|\Delta x(2)\|_\infty / u_{\min}$. Therefore, on S^1 the state $d^0(2)$ is achieved as $d^1(k_2)$ at the time $T^1(k_2)$, where

$$|T^1(k_2) - T^0(2)| \leq q_T \|\Delta x(1)\|_\infty, \quad \|x^1(k_2) - x^0(2)\| \leq q_X \|\Delta x(1)\|_\infty,$$

and $q_T = (2u_{\min} + u_{\max})/(u_{\min})^2$, $q_X = (u_{\min} + u_{\max})u_{\max}/(u_{\min})^2$. Setting $y^1(k_2) = u^0(2)t^1(k_2)$ and repeating analogous computations $N-1$ times we conclude that

$$W(x^1, d^0) - W(x^0, d^0) = T^1(k_{N+1}) - T^0(N+1) \leq q_T(q_X^N - 1)/(q_X - 1)\|\Delta x(1)\|_\infty.$$

Interchanging x^1 and x^0, we conclude that for any sufficiently small Δx

$$|W(x^0 + \Delta x, d^0) - W(x^0, d^0)| \leq q_T(q_X^N - 1)/(q_X - 1)\|\Delta x\|_\infty.$$

So the continuity of $W(x, d)$ is proved.

It is possible to establish an ordering on A_D. Let $A_D(0)=\{d_T\}$. The other classes are determined recursively: for known $A_D(i)$, $i = 0, \ldots, j$, we have

$$A_D(j+1) = \{d \in A_D \setminus (A_D(0) \cup \ldots \cup A_D(j)) \mid D_+(d) \subseteq A_D(0) \cup \ldots \cup A_D(j)\}.$$

For $d = d_T$ we have $W(x, d) = 0$, that is a particular case of a continuous piecewise-linear function of x. So the theorem is valid for $d \in A_D(0)$. Suppose that it is valid for $d \in A_D(i), i = 0, \ldots, j$. We shall prove that it is valid for $d \in A_D(j+1)$. According to Lemma 1 for any $d' \in A_D(i)$, $i = 0, \ldots, j$, for the function $W(x', d')$ we have

$$\begin{aligned} &W(x', d') = W_l(x', d') \equiv C_{X0l}(d')x' + C_{0l}(d'), \\ &x' \in X_l(d') = \{x' \mid C_{Xrl}(d')x' + C_{rl}(d') \leq 0, \ r = 1, \ldots, n_l(d')\}. \end{aligned}$$

Let $d \in A_D(j+1)$. According to the supposition for $W(x, d, d')$ we have

$$\begin{aligned} \min\{t + W_l(x+y, d') \,|\, (y,t) \in P(x,d,d'),\ (x+y) \in X_l(d'),\ l = 1, \ldots, L(d')\} \\ = \min\{t + C_{Xr0}(d')(x+y) + C_{r0}(d') \mid (y,t) \in P_l(x,d,d'), \\ (x+y) \in X_l(d'), l = 1, \ldots, L(d')\}. \end{aligned}$$

The set $P_l(x, d, d') = \{(y, t) \mid (y, t) \in P(x, d, d'),\ (x + y) \in X_l(d')\}$ is defined by conditions (1.2), (2.1) and the following inequality

$$C_{Xrl}(d')(x+y) + C_{rl}(d') \leq 0, \ \ r = 1, \ldots, n_l(d'). \tag{2.4}$$

In fact the minimum of $t + C_{X0l}(d)(x+y) + C_{0l}$ on the polyhedron $P_l(x, d, d')$ is reached on its vertex. Any possible vertex corresponds to some set I of inequalities (1.2), (2.1), (2.4) and all equalities (1.2), (2.1) for which $\dim I = n+1$. In more abstract way, forming a vector of coordinates of the vertex $v = (y, t)$ we represent this condition as

$$\begin{aligned} &b_i^T(d,d')v + a_i^T(d,d')x \leq c_i(d,d'), \ \ i \in I_{1l}, \\ &b_i^T(d,d')v + a_i^T(d,d')x = c_i(d,d'), \ \ i \in I_{2l}. \end{aligned}$$

Let us denote $B(I,d,d')$, $A(I,d,d')$, $c(I,d,d')$ the matrices which rows are $b_i(d,d')$, $a_i(d,d')$, $i \in I$, and the vector which components are $c_i(d,d')$, $i \in I$, respectively. Then we have

$$v(I,d,d',x) = B^{-1}(I,d,d')\big(A(I,d,d')x - c(I,d,d')\big).$$

This vector $v(I,d,d')$ is a vertex of $P_l(x,d,d')$ iff it satisfies the rest of (1.2), (2.1), (2.4) constraints, i.e., for $i \in I_1 \setminus I$ we have

$$b_i^T(d,d')\left(B^{-1}(I,d,d')(A(I,d,d')x - c(I,d,d')\right) + a_i^T(d,d')x \le c_i(d,d'). \quad (2.5)$$

The set of inequalities (2.5) defines the domain $X(I,d,d')$ in R^n for which the vector $v(I,d,d')$ is a vertex of $P_l(x,d,d')$. So all possible particular formulas for $W(x,d)$ are

$$W(x,d,I) = t(I,d,d',x) + C_{X0l}(d')\,(x + y(I,d,d',x)) + C_{0l}(d').$$

All functions on the right-hand side are linear, so $W(x,d)$ is a continuous piecewise-linear function of x. $\square$

Remark. From the proof of Lemma 1 and Theorem 1 a quasi-constructive way to define parameters of $W(x,d)$ may be derived. Given values of $L(d')$, $n_l(d')$, C_{lr} and values of C_{Xlr} components, we may generate the set $A_I(d,d')$ of all possible sets I of the above type and calculate coefficients of any linear vector function $v(I,d,d',x)$. To know whether $X(I,d,d') \neq \emptyset$ means to test compatibility of linear restrictions (2.5). This problem may be reduced to the solution of some linear programming problem which yields coordinates of an internal point for a non-empty polyhedron (2.5) as well. So we define the set $A_{I0}(d,d') = \{I \in A_I(d,d') \,|\, X(I,d,d') \neq \emptyset\}$. The domain on which $W(x,d) = W(x,d,I)$ may be represented with the formula

$$X(I,d) = \bigcap_{J \in A_I(d,d')} (X(I,d,d') \setminus (X(J,d,d') \cap \{x \,|\, W(x,d,I) \ge W(x,d,J)\})).$$

Thus the domain on which the Bellman function is expressed with any of its linear formulas is the result of operations of conjunction and difference which primary operands are convex polyhedra. So the result is a polyhedron as well, but probably non-convex. If it is non-convex, then it must be represented as a union of convex polyhedra.

The proof of Lemma 1 using formula (2.3) gives a more simple description of convex domains on which $W(x,d)$ is expressed with definite linear formulas. For each of such polyhedra it is necessary to establish what linear formula expresses the Bellman function on it. To know this, it is sufficient to compute an arbitrary internal point and to determine whether it belongs to $X(I,d)$ by testing its belongingness to polyhedra $X(I,d,d')$ and $X(J,d,d') \cap \{x \,|\, W(x,d,I) \ge W(x,d,J)\}$.

3 Resource Planning as a Problem of a Transforming Process Optimization

In paper [8] the given problem was represented as an optimization problem for a hybrid system [3]. This form of the model enables us to perform non-local optimization with the use of special optimality conditions and iteration method of branch-and-bound type. In this paper additional opportunities resulting from linear form of relationships are studied. It was noticed in [8] that the process may have different *scenarios*, i.e., sequences $D = (d(1), \ldots, d(N))$ of qualitative states of the project. According to this approach the search of the optimum solution is based on three types of calculations: optimization within a given scenario, testing the optimality of this scenario and shifting to a better adjacent scenario.

For a given scenario the optimum schedule is found by solving the following dynamic linear programming problem (DLP):

$$T(N) \to \min; \tag{3.1}$$

$$T(0) = 0; \quad T(k) = T(k-1) + t(k), \ k = 1, \ldots, N; \tag{3.2}$$

$$x(0) = 0; \quad x_i(k) = x_i(k-1) + y_i(k); \tag{3.3}$$

$$u_{\min i} t(k) \le y_i(k) \le u_{\max i} t(k), \ \ i \in I_1(d(k)); \ y_i(k) = 0, i \notin I_1(d(k)); \tag{3.4}$$

$$\sum_{i \in I_{Rj}} y_i(k) \le u_{Rj} t(k), \quad j = 1, \ldots, m, \quad t(k) \ge 0; \tag{3.5}$$

$$x_i(k) + y_i(k) = x_{Ti}, \quad i \in I_2(d(k+1)). \tag{3.6}$$

First of all, DLP problem is a particular case of a linear programming problem, so its exact solution may be found with a finite method. Besides, there are decomposition methods that enhance the efficiency of optimum search, e.g. [4]. All these methods guarantee reaching the optimum (within a given scenario). If more than one work terminates at the end of some stage, then other scenario representations of the project schedule exist and it is necessary to test whether the same schedule is optimal within these adjacent scenarios.

For a class of hybrid models, including (3.1)–(3.6), the necessary optimality conditions were established in [2]. Here we strengthen these results taking into account linearity of the model relationships. To formulate them we represent the model in other forms that are more general. Let us denote the generalized vectors of state $z(k) = (x_1(k), \ldots, x_n(k), T(k))$ and control $v(k) = (y_1(k), \ldots, y_n(k), t(k))$. For a given scenario D the problem (3.1)–(3.6) (and some similar problems) may be represented as

$$z_{n+1}(N+1) \to \min; \tag{3.7}$$

$$z(k+1) = z(k) + v(k), \qquad k = 1, \ldots, N; \tag{3.8}$$

$$b^T_{Vi}(k,D)v(k) \le 0, \ i \in I_{V1}(k,D), \ b^T_{Vi}(k,D)v(k) = 0, \quad i \in I_{V2}(k,D); \tag{3.9}$$

$$b^T_{Zi}(k,D)z(k) + b^T_{Vi}(k,D)v(k) + c_i(k,D) = 0, \qquad i \in I_T(k,D). \tag{3.10}$$

We introduce formally $b_{Zi}(k,D) = 0$ for $i \in I_{V1}(k,D) \cup I_V(k,D)$ to represent (3.9) and (3.10) in the similar form. In this representation (3.7) corresponds to (3.1), (3.8) to (3.2) and (3.3), (3.9) to (3.4) and (3.5), (3.10) to (3.6).

Let us substitute $z(k)$ by $v(1) + \ldots + v(k-1)$ according to (3.7) and denote $I_1(D) = I_{V1}(1,D) \cup \ldots \cup I_{V1}(N,D)$, $I_2(k,D) = I_{V2}(k,D) \cup I_T(k,D)$, $I_2(D) = I_2(1,D) \cup \ldots \cup I_2(N,D)$, $v = (v(1), \ldots, v(N))$. Then we may express residuals in (3.9), (3.10) as $F_i(v, D)$, the target functional as $F_0(v, D)$ and formulate the problem (3.1)–(3.6) in the most general form

$$F_0(v, D) \equiv a_{V0}^T(D)v \to \min, \tag{3.11}$$
$$F_i(v, D) \equiv a_{Vi}^T(D)v \le 0, \qquad i \in I_1(D), \tag{3.12}$$
$$F_i(v, D) \equiv a_{Vi}^T(D)v + a_{0i}(D) = 0, \qquad i \in I_2(D). \tag{3.13}$$

We say that the vector δv of the same dimension as v determines a *feasible direction* [6] in $v \in V(D)$, if such $\alpha_1(\delta v) > 0$ exists that for any $0 \le \alpha \le \alpha_1(\delta v)$ we have that $v + \alpha\delta v \in V(D)$. Let $v_1 \in V(D)$. From the convexity of $V(D)$ we find that $\delta v = v_1 - v$ is a feasible direction in v with $\alpha_1(\delta v) = 1$.

For a given $v \in V(D)$ we treat a restriction (3.12) as an *active* one, if $F_i(v, D) = 0$, and denote the set of all active restrictions as $I_{10}(v, D)$. All restrictions (3.13) for $v \in V(D)$ are active too. We use further the following notations for sets and subsets of active restrictions:

$$I_0(v, D) = I_{10}(v, D) \cup I_2(v, D), \quad I_{V10}(v, k, D) = I_{V1}(k, D) \cap I_{10}(v, D),$$
$$I_0(v, k, D) = I_{V10}(v, k, D) \cup I_{V2}(k, D) \cup I_T(k, D).$$

In general, the problem (3.11)–(3.13) may be treated as regular if for any $v \in V(D)$ gradients of active restrictions $a_{Vi}(D)$, $i \in I_0(v, D)$, are linearly independent. For our problem the desired linear independence may take place for any $v \in V(D)$, except such $v \in V(D)$ for which $t(k) = 0$ for some k (i.e. $K_0(v, D) = \{k \mid v_{n+1}(k) = 0\} = \emptyset$). We suppose further that the following regularity condition is satisfied.

Condition 1. For any $k = 1, \ldots, N$ and $v(k)$ satisfying (3.9) and such that $v_{n+1}(k) > 0$ we have that

1. $\dim I_{V10}(v, k, D) \cup I_{V2}(k, D)) \le n + 1$;
2. for any $J \subseteq I_T(k, D)$ for which

$$\dim(I_{V10}(v, k, D) \cup I_{V2}(k, D) \cup J) \le n + 1,$$

vectors $b_{Vi}(k, D)$, $i \in I_{V10}(v, k, D) \cup I_{V2}(k, D) \cup J$, are linearly independent.

Let $I_{Rj} \cap I_{Ri} = \emptyset$, $i \ne j, i, j = 1, \ldots, m$. From the form of restrictions (3.4)–(3.6) we come to the conclusion that Condition 1 takes place if for any $j = 1, \ldots, m$ and any $I_{\min} \subseteq I_{Rj}$, $I_{max} \subseteq I_{Rj}$, for which $I_{min} \cap I_{max} = \emptyset$,

$$\sum_{i \in I_{\min} \cap I_{Rj}} u_{\min i} + \sum_{i \in I_{\max} \cap I_{Rj}} u_{\max i} \ne u_{Rj}.$$

Under the regularity condition the use of decomposition approach formulated in [8] is possible. We propose here a concrete decomposition scheme successfully used earlier by the author for the solution of numerous dynamical optimization problems.

4 Usage of a Decomposition Scheme

The method proposed here originates from author's generalization [7] of the computational construction proposed by Boltyanski [2] to simplify optimality conditions for discrete-time processes.

Let $v \in V(D)$, $K_0(v,D) = \emptyset$. Let us determine for any $k = 1,\dots,N$ $P(k) \subseteq I_T(k,D)$ and $N_M(k)$ according to the conditions:

$$\dim(I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k)) \le n+1,\ L(k) = I_T(k,D) \setminus P_T(k),$$

$$N_L = \dim(L(1)) + \ldots + \dim(L(N)),$$

$$N_M(k) \le n+1-\dim(I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k)),$$

$$N_{MS}(k) = N_M(1) + \ldots + N_M(k) \ge N_{LS}(k) = \dim(L(1)) + \ldots + \dim(L(N)),$$

$$N_{MS}(N) = N_{LS}(N),\ M(k) = \{N_{MS}(k-1)+1,\dots,N_{MS}(k)\}.$$

We can determine $(n+1)\times(n+1)$ matrices $C(k)$ and a set of linearly independent vectors $g^m(k) \in R^{n+1}$, $m \in M(k)$ by solving the following systems of linear equations:

$$\begin{gathered} b_{Zi}^T(k,D) + b_{Vi}^T(k,D)C(k) = 0,\ b_{Vi}^T(k,D)g^l(k) = 0,\ l \in M(k), \\ i \in I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k). \end{gathered} \tag{4.1}$$

It is shown in [7] for a more general model than (3.7)–(3.10) that any feasible direction δv may be defined stage-wise in such a way:

$$\delta v(k) = \delta_0 v(k) + C(k)\delta z(k) + \sum_{l \in M(k)} \mu_l g^l(k), \tag{4.2}$$

where the following conditions are valid for the every $\delta_0 v(k)$

$$b_{Vi}^T(k,D)\delta_0 v(k) \le 0,\ i \in I_{V10}(k,D), \tag{4.3}$$

$$b_{Vi}^T(k,D)\delta_0 v(k) = 0,\ i \in I_{V2}(k,D) \cup P_T(k)), \tag{4.4}$$

and the given below conditions on variables μ_l, $l \in M(k)$, are valid. Note that for any $\delta z \in R^{n+1}$ and any μ_l, $l \in M(k)$, $\delta F_i = F_i(v,D) - F_i(v+\delta v, D)$, $i \in I_{V10}(v,k,D) \cup I_{V2}(k,D) \cup P_T(k)$, we have the formula

$$b_{Zi}^T(k,D)\delta z(k) + b_{Vi}^T(k,D)\delta v(k) = b_{Vi}^T(k,D)\delta_0 v(k). \tag{4.5}$$

Using the following conjugate equations for the target functional and restrictions from $L(k)$:

$$p^0(N+1) = (0,\dots,0,-1),\ \ p^0(k) = (E + C(k))p^0(k+1),\ k = N,\dots,1; \tag{4.6}$$

$$p^i(k'+1) = 0,\ \ k' = k+2,\dots,N,\ \ p^i(k+1) = b_{Zi}^T(k,D), \tag{4.7}$$

$$p^i(k') = (E + C(k'))p^i(k'+1),\ \ k' = k,\dots,1,$$

and letting $L_S = L(1) \ldots \cup L(N)$ we get the following formulas for their variations:

$$\delta F_i(v) = \sum_{k=1}^{N} (p^i(k+1), \delta_0 v(k) + \sum_{l \in M(k)} \mu_l g^l(k)), \;\; i \in \{0\} \cup L_S. \tag{4.8}$$

The relationships to determine all values of μ_l, $l \in M(k)$, $k = 1, \ldots, N$, are given as

$$\delta F_i(v) = 0, \quad i \in L_S. \tag{4.9}$$

With the formulas (4.8) they are reduced to a system of linear equations. Let $G_{il} = (p^i(k+1), g^l(k))$, $l \in L_S$, $l \in M(k)$, $Q = G^{-1}$, then

$$\mu_l = -\sum_{i \in L_S} Q_{il} \sum_{k=1}^{N} (p^i(k+1), \delta_0 v(k)), \quad l \in M(k), \quad k = 1, ..., N. \tag{4.10}$$

With the substitution of (4.10) to (4.8) we have the final expression for $\delta F_0(v)$

$$\delta F_0(v, D) = \sum_{k=1}^{N} (q(k+1), \delta_0 v(k)), \quad q(k+1) = p^0(k+1)$$
$$+ \sum_{i \in L_S} \Big(\sum_{k'=1}^{N} \sum_{l \in M(k')} Q_{il} \left(p^0(k'+1), g^l(k') \right) \Big) p^i(k+1). \tag{4.11}$$

The efficiency of the decomposition scheme depends mainly on its dimension, i.e., $N_{LS}(N)$; in practice, as a rule, $N_{LS}(N)$ is much less than the dimension of v.

With the use of formulas (4.3)–(4.4), (4.11) we formulate the criterion of optimality.

Theorem 2. *The control $v \in V(D)$ for which $K_0(v, D) = \emptyset$ is the solution of problem* (3.7)–(3.10) *if and only if for any $\delta_0 v(k)$, $k = 1, \ldots, N$ satisfying* (4.3)–(4.4) *the following condition is valid*

$$(q(k+1), \delta_0 v(k)) \geq 0. \tag{4.12}$$

Proof. The necessary condition follows from the theorem proved in [7]. Thus we consider the sufficiency case.

Let the conditions of Theorem 2 are valid but there is $v^* \in V(D)$, $F_0(v^*) < F_0(v)$. Then $\delta v^* = v^* - v$ is a feasible direction, $\delta_0 v^*$ corresponds to δv^*, thus $\delta_0 v^*(k)$ satisfies (4.3)–(4.4) and therefore (4.12) is valid for it. We get that

$$\delta F_0(v) = a^T_{V0}(D) \delta v = F_0(v^*) - F_0(v) < 0.$$

Then due to the statement of the theorem and formula (4.11) $\delta F_0(v)$ is a sum of non-negative terms so $\delta F_0(v) \geq 0$. The obtained contradiction ends the proof. □

If $K_0(v, D) \neq \emptyset$, then the control v' received from v by cancelling stages of zero duration (and hence $v(k) = 0$) and joining $I_T(k)$ to $I_T(k-1)$ corresponds to another scenario D'. For $v' \in V'(D')$ the optimality criterion of Theorem 2 may be tested. It is possible, however, to test the optimality of v within the original scenario and the other adjacent scenarios with the below optimality conditions.

Different scenario representations exist for the process with $v \in V(D)$ for which $K_0(v, D) = \emptyset$ and $K_1(D) = \{k \,|\, \dim(I_T(k)) > 1\} \neq \emptyset$. We treat the scenario D' as adjacent to D if there exists a set of stages $K' \subseteq K_1(D)$ for which every $k \in K'$ (for D) corresponds in D' to the succession of stages that we numerate with compound indices $(k, 1)$ or $k, (k, 2), \dots, (k, n(k))$, these stages terminating with sets of finished jobs $I_T(k, 1), \dots,$ $I_T(k, n(k))$, where $I_T(k) = I_T(k, 1) \cup \dots \cup I_T(k, n(k))$, the rest stages $k \notin K'$ having the same $I_T(k)$. Obviously $v' \in V'(D')$, if $v'(k) = v(k)$, $k = 1, \dots, N$, $v'(k, i) = 0$, $k \in K'$, $i = 1, \dots, n(k)$, here we denote $v_{TR}(v, D, D')$ by v'.

Let $v \in V(D)$ be the solution of problem (3.7)–(3.10). To establish whether the adjacent scenario D' is not better than D, it is necessary to compare $F_0(v_{TR}(v, D, D'),\, D') = F_0(v, D)$ with $F_0(v', D')$ for near controls $v' \in V(D')$ with at least one $v'(k, i) \neq 0$. To construct such a v' we use the following variant of the formula (4.2). Let $v'(k, i)$ satisfy (3.9) for all $k \in K'$, $i = 1, \dots, n(k)$. To determine $v' \in V(D')$ with given values of $v'(k, i)$ we take into consideration that:

1) restrictions (3.9) on values $v'(k)$, $k = 1, \dots, N$, stay the same as for D and for additional steps they must be satisfied;

2) the total amount of restrictions (3.10), or (3.6), stay the same as on D;

3) for an additional step formulas for (3.6) include additional terms and may be expressed as

$$z_i'(k) + v_i'(k) + v_i'(k, 2) + \dots + v_i'(k, n(k)) - x_{Ti} = 0, \; k \in K', \; i \in I_T(k, r), \quad (4.13)$$

here we take into account that $v_i'(k, j) = 0$ for $j > r$. Let us define $\delta v(k)$, $k \in K'$, as

$$\delta v(k) = C(k)\Big(\delta z(k) + \sum_{r=1}^{n(k)} v(k, r)\Big) + \sum_{l \in M(k)} \mu_l g^l. \quad (4.14)$$

According to the definition of $C(k)$, $g^l(k)$ (see also formula (4.5)) we conclude that for any $\delta z(k)$, $k \in K'$

$$b_{Zi}^T(k, D)\delta z(k) + b_{Vi}^T(k, D)\delta v(k) = 0, \; i \in I_{V10}(v, k, D') \cup I_{V2}(k, D') \cup P_T(k).$$

Due to (4.13)–(4.14) we have for $\delta F_i(v', D')$, $i \in L_T(k)$, $k = 1, \dots, N$, and for $\delta z(k+1)$, $k \in K'$, the same expression as for the scenario D with

$$\delta_0 v(k) = (E + C(k)) \sum_{r=1}^{n(k)} v(k, r). \quad (4.15)$$

Let us determine μ_l, $l \in M(k)$, $k = 1, \dots, N$, from (4.10) as for the scenario D with $\delta_0 v(k) = 0$ for $k \notin K'$, using the formula (4.15) for $k \in K'$. Then we have from (4.11)

$$\delta F_0(v') = \sum_{k\in K'} \Big(q(k+1), (E+C(k)) \sum_{r=1}^{n(k)} v(k,r)\Big). \tag{4.16}$$

Theorem 3. *Let the pair $(D, v \in V(D))$ be such that $K_1(D) \neq \emptyset$ and let $K_0(v,D) = \emptyset$ be the solution of problem (3.7)–(3.10). The optimum values of problem (3.7)–(3.10) for all adjacent scenarios satisfy $F^*(D') \geq F^*(D) = F_0(v,D)$ if and only if for any $k \in K_1(D), I_{T1}(k) \subset I_T(k)$, $I_{T1}(k) \neq \emptyset$ and $\delta v' = (\delta y_1', \dots, \delta y_n', 1)$ satisfying*

$$u_{\min i} \leq \delta y_i' \leq u_{\max i},\ i \in I_1(d(k,2)),\ \ \delta y_i' = 0,\ i \notin I_1(d(k,2)); \tag{4.17}$$

$$\sum_{i\in I_{Rj}} \delta y_i' \leq u_{Rj},\ j = 1,\dots,m, \quad d(k,2) = D_+(d(k),\ I_{T1}(k)), \tag{4.18}$$

the following inequality is valid

$$\delta F_0(v) = (q(k+1), (E+C(k))\delta v') \geq 0. \tag{4.19}$$

Proof. *Necessity.* Let $k \in K_1(D)$, $I_{T1}(k) \subset I_T(k)$, and $\delta v' = (\delta y_1', \dots, \delta y_n', 1)$ satisfy (4.17)–(4.18). Let us define the scenario D' differing from D in two aspects: 1) $I_T'(k) = I_{T1}(k)$; 2) after the k-th stage a stage $(k,2)$ with $I_T'(k,2) = I_T(k) \setminus I_{T1}(k)$ is inserted. Let us determine a family of $v'(\alpha)$, $\alpha > 0$ as $v'(\alpha,k') = v(k') + \alpha\delta v(k')$, where $\delta v(k')$ is expressed by (4.2) with $\delta_0 v(k') = 0$, $k' \neq k$, $v'(\alpha,(k,2)) = \alpha\delta v'$, $v'(\alpha,k) = v(k) + \alpha\delta v'(k)$ and $\delta v(k)$ is expressed by (4.14). Due to the definition of $C(k)$, $g^l(k)$, and formulas (4.17)–(4.18) we have

$$F_i(v'(\alpha,k), D') = F_i(v,D),\ i \in I_{10}(v,D) \cup I_2(D), \tag{4.20}$$

$$F_i(v'(\alpha,k), D') = F_i(v,D) + \alpha(\nabla F_i(v,D), \delta v'),\ i \in I_1(D) \setminus I_{10}(v,D), \tag{4.21}$$

$$F_i(v'(\alpha,k), D') = \alpha(b_{Vi}^T(k,D)\delta v') \leq 0,\ i \in I_1(D') \setminus I_1(D) = I_{V1}((k,2),D'), \tag{4.22}$$

$$F_i(v'(\alpha,k), D') = 0,\ i \in I_2(D') \setminus I_2(D) = I_{V2}((k,2),D'). \tag{4.23}$$

From formulas (4.20)–(4.23), we come to the conclusion that for sufficiently small α we get that $v'(\alpha,k) \in D'$, so $F_0(v'(\alpha),D') \geq F_0(v,D)$ and hence $\delta F_0(v) = (F_0(v'(\alpha),D') - F_0(v,D))/\alpha \geq 0$. With the use of (4.16) we conclude that (4.19) is valid.

Sufficiency. Let $D_1 \in S_D(D)$, $v^{(1)} \in V(D_1)$. Let us determine $v^{(0)} \in V(D_1)$ so $v^{(0)}(k) = v(k)$, $k = 1,\dots,N$, $v^{(0)}(k,r) = 0$, $r = 2,\dots,n(k)$, $k \in K'(D_1)$. From (3.1)–(3.6) we conclude that $v^{(0)} \in V(D_1)$. Let

$$v^{(1,\alpha)} = \alpha v^{(1)} + (1-\alpha)v^{(0)}, \quad \alpha \in [0,1/2].$$

The control $v^{(2,\alpha)} \in V(D_1)$, $\alpha \in$[0,1/2], is determined with $v^{(2,\alpha)}(k) = v^{(0)}(k) + \alpha\delta v(k)$, $k = 1,\dots,N$, $v^{(2,\alpha)}(k,r) = \alpha v^{(1)}(k,r) = v^{(1,\alpha)}(k,r)$, $r = 2,\dots,n(k)$, $k \in K'(D)$. Here $\delta v(k)$ is determined due to (4.2) with $\delta_0 v(k) = 0$ for $k \notin$

$K'(D_1)$ and (4.14) for $k \in K'(D_1)$, where μ_l, $l \in M(k)$, $k = 1, \ldots, N$, are computed as if vectors $\delta_0 v(k)$ for $k \in K'(D_1)$ would determined from (4.15). Then

$$F_i(v^{(1,\alpha)}, D_1) = \alpha F_i(v^{(1)}, D_1) + (1-\alpha)(F_i(v^{(0)}, D_1) \leq 0,\ i \in I_1(v^{(0)}, D_1),$$
$$F_i(v^{(1,\alpha)}, D_1) = \alpha F_i(v^{(0)}, D_1) + (1-\alpha)(F_i(v^{(1)}, D_1) = 0,\ i \in I_2(D_1).$$

According to formulas (4.2), (4.14) we have

$$F_i(v^{(2,\alpha)}, D_1) = F_i(v^{(0)}, D_1), \quad i \in I_1(D) \cup I_2(D),$$
$$F_i(v^{(2,\alpha)}, D_1) = F_i(v^{(1,\alpha)}, D_1), \quad i \in (I_1(D_1) \cup I_2(D_1)) \setminus (I_1(D) \cup I_2(D)).$$

The rest constraints (3.9) for (k, r) are satisfied, since

$$b^T_{Vi}((k,r), D_1) v^{(2,\alpha)}(k,r) = \alpha b^T_{Vi}((k,r), D_1) v^{(1)}(k,r).$$

Let us consider vectors

$$\delta v^{(2)} = (v^{(1,\alpha)} - v^{(2,\alpha)})/\alpha, \quad \delta v^{(1)} = (\delta v^{(2)}(1), \ldots, \delta v^{(2)}(N)).$$

Taking into account the fact that $F_i(v^{(2,\alpha)}, D_1) = 0$, $i \in I_{10}(D)$, $\delta v^{(2)}(k,r) = 0$, $r = 2, \ldots, n(k)$, $k \in K'(D)$, as well as the form of restrictions (3.2)–(3.5), we have

$$a^T_{Vi}(D) \delta v^{(1)} = (F_i(v^{(1,\alpha)}, D_1) - F_i(v^{(2,\alpha)}, D_1))/\alpha \leq 0, i \in I_{10}(D),$$
$$a^T_{Vi}(D) \delta v^{(1)} = (F_i(v^{(1,\alpha)}, D_1) - F_i(v^{(2,\alpha)}, D_1))/\alpha = 0,\ i \in I_2(D_1).$$

Therefore, $\delta v^{(1)}$ is a feasible direction for v and for small α we get that $v + \alpha \delta v^{(1)} \in V(D)$ and due to optimality of v on D $F_0(v + \alpha \delta v^{(1)}, D) \geq F_0(v, D)$. Thus we have that

$$\begin{aligned} F_0(v^{(1,\alpha)}, D_1) - F_0(v^{(2,\alpha)}, D_1) &= \alpha a^T_{V0}(D_1) \delta v^{(2)} \\ &= \alpha a^T_{V0}(D) \delta v^{(1)} = F_0(v + \alpha \delta v^{(1)}, D) - F_0(v, D) \geq 0. \end{aligned}$$

Setting $\delta v' = v'(k,r)/v'_{n+1}(k,r)$, $I_{T1}(k) = I_T(k,1) \cup \ldots \cup I_T(k, r-1)$, we satisfy conditions (4.17)–(4.18), thus (4.19) is valid. Hence,

$$\begin{aligned} F_0(v^{(1,\alpha)}, D_1) - F_0(v^{(0)}, D_1) &\geq F_0(v^{(2,\alpha)}, D_1) - F_0(v^{(0)}, D_1) \\ &= \sum_{k \in K'} \sum_{r=1}^{n(k)} v'_{n+1}(k,r) \left(q(k+1), (E + C(k)) v'(k,r)\right) \geq 0. \end{aligned}$$

and $F_0(v^{(1)}, D_1) - F_0(v, D) = (F_0(v^{(1,\alpha)}, D_1) - F_0(v^{(0)}, D_1)/\alpha \geq 0$. Therefore, $F^*(D_1) \geq F^*(D)$ for all $D_1 \in S_D(D)$. $\square$

5 Principal Construction of the Computational Method

As it was formulated above, the numerical method based on the above decomposition constructions consists in the interchange of the solution of optimization problems (3.7)–(3.10) within a given D and the search of better adjacent scenarios by testing optimality conditions of Theorem 3. Most of the necessary calculations are reduced to direct computation of conjugate trajectories with (4.6)–(4.7), solution of algebraic linear equations (4.1) and (4.9), some linear transformations and testing optimality conditions by solution of linear programming problems, the latter being:

1. Minimization of $q^T(k+1)\delta_0 v(k)$ under constraints (4.3)–(4.4) and $|\delta_0 v_i(k)| \leq 1, \quad i = 1, \ldots, n$;
2. Minimization of $q^T(k+1)(E + C(k))\delta v'$ under constraints (4.17)–(4.18).

The dimension ($n{+}1$ and n variables, respectively) and the structure of both problems are very similar, no singularity being displayed.

The author's hypothesis is that in the set of D there are no local minima. It means that if $F(D)$ is less than $F(D')$ for all adjacent scenarios D' this value gives the global minimum. No contradictions with this hypothesis was found, some evidence is found in particular cases, but its formal substantiation is not obtained. If it is always true, it is not necessary to build a solution tree, because in that case every minimizing succession of scenarios lead to the globally optimum solution.

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