Mathematical Modelling and Analysis. Ed. A.Buikis
Volume 12 Number 1, 2007, pages 51–60
 ISSN 1392-6292 print, ISSN 1648-3510 online

EXPRESSIONS FOR FUČIK SPECTRA FOR STURM-LIOUVILLE BVP

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Received October 27, 2006; revised November 13, 2006; published online February 10, 2007

Abstract. We provide explicit formulas for the Fučik spectra of boundary value problems with the Sturm-Liouville boundary conditions.
Key words: Fučik spectrum; Sturm-Liouville boundary conditions; boundary value problem

1. Introduction

We consider equation with the piece-wise linear right side

$$x'' = -\mu^2 x^+ + \lambda^2 x^-, \quad \mu,\ \lambda \in \mathbb{R}, \tag{1.1}$$

where $x^{\pm}(t) = \max\{\pm x, 0\}$, with the Sturm-Liouville boundary conditions

$$\begin{cases} x(0)\cos\alpha - x'(0)\sin\alpha = 0, \\ x(\pi)\cos\beta - x'(\pi)\sin\beta = 0. \end{cases} \tag{1.2}$$

We are looking for those values of (λ, μ) for which the problem has a nontrivial solution.

Definition 1. The set of all values (λ, μ) such that a nontrivial solution for problem (1.1), (1.2) exists, is called the *Fučik spectrum* for boundary value problem (1.1), (1.2).

In this paper we show that the spectrum of problem (1.1), (1.2) is a collection of curves and we obtain the formulas for the spectrum. The branches of the spectrum are denoted by F_n^+ and F_n^-, where the lower index indicates how many zeros in the interval $(0, \pi)$ has the respective solution $x(t)$ of (1.1), (1.2) and the upper index $(+)$ shows that $x'(0) > 0$, respectively $(-)$ shows that $x'(0) < 0$.

The paper is organized as follows. Section 2 provides auxiliary results and describes the technique we used in the sequel. Section 2 is devoted to main results, i.e. to the case $0 \le \alpha \le \frac{\pi}{2} \le \beta \le \pi$. In Section 4 we consider other cases of ordering of α and β.

Properties of the Fučik spectrum for the Sturm-Liouville problem and for specific cases of boundary conditions, namely, the Neumann boundary conditions, the mixed boundary conditions, are considered in [1, 2, 4].

2. Auxiliary Results

Our technique is based on a regular usage of polar coordinates. Let us introduce them by the formulas

$$x = \rho \sin \varphi, \quad x' = \rho \cos \varphi.$$

The piece-wise linear function $f(x) = -\mu^2 x^+ + \lambda^2 x^-$, in polar coordinates looks as

$$f(\rho, \varphi) = \begin{cases} -\mu^2 \rho \sin \varphi, & \sin \varphi \ge 0, \\ -\lambda^2 \rho \sin \varphi, & \sin \varphi < 0. \end{cases}$$

Theorem 1. *Let $\varphi(t)$ be the angle function for solutions of the Cauchy problem* (1.1)

$$\varphi(0) = \varphi_0, \quad \rho(0) = \rho_0.$$

The difference $\varphi(T) - \varphi(0)$ is independent of the choice of $\rho_0 > 0$, that is, any trajectory starting at the time moment $t = 0$ from the first of the straight lines (1.2) *on a phase plane, ends at some other straight line*

$$x(T) \cos \varphi(T) - x'(T) \sin \varphi(T) = 0.$$

Proof. By using the polar coordinates we convert equation (1.1) to the form

$$\begin{cases} \rho' = \rho \sin \varphi \cos \varphi + f(\rho, \varphi) \cos \varphi, \\ \varphi' = -\dfrac{f(\rho, \varphi)}{\rho} \sin \varphi + \cos^2 \varphi. \end{cases}$$

Let us rewrite the second equation in the form

$$\varphi' = F(\varphi) = \begin{cases} \mu^2 \sin^2 \varphi + \cos^2 \varphi, & \sin \varphi \ge 0, \\ \lambda^2 \sin^2 \varphi + \cos^2 \varphi, & \sin \varphi < 0. \end{cases} \tag{2.1}$$

As can be seen from (2.1), the derivative of $\varphi(t)$ is independent of $\rho(t)$. Notice that the function $\varphi(t)$ is increasing, since $\varphi' > 0$. ■

Theorem 2. *A solution of the problem*

$$\begin{cases} \varphi' = k^2 \sin^2 \varphi + \cos^2 \varphi, \ k > 0, \\ \varphi(t_0) = \varphi_0 \end{cases}$$

is given by

$$\frac{1}{k}\arctan(k\tan\varphi) - \frac{1}{k}\arctan(k\tan\varphi_0) = t - t_0,$$

for $0 \le \varphi_0 \le \varphi \le \frac{\pi}{2}$ *or* $\frac{\pi}{2} \le \varphi_0 \le \varphi \le \pi$*, and it is given by*

$$\frac{1}{k}\arctan(k\tan\varphi) + \frac{\pi}{k} - \frac{1}{k}\arctan(k\tan\varphi_0) = t - t_0,$$

for $0 \le \varphi_0 < \frac{\pi}{2} < \varphi \le \pi$.

Proof. One gets by integrating the given differential equation that

$$t - t_0 = \int\limits_{\varphi_0}^{\varphi} \frac{d\varphi}{k^2\sin^2\varphi + \cos^2\varphi} = \int\limits_{\varphi_0}^{\varphi} \frac{\frac{d\varphi}{\cos^2\varphi}}{k^2\tan^2\varphi + 1} = \frac{1}{k}\int\limits_{\varphi_0}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2 + 1}$$
$$= \frac{1}{k}\arctan(k\tan\varphi)\Big|_{\varphi_0}^{\varphi} = \frac{1}{k}\arctan(k\tan\varphi) - \frac{1}{k}\arctan(k\tan\varphi_0).$$

Consider the case when there is a value $\varphi_* = \frac{\pi}{2}$ in the interval $(\varphi_0;\varphi)$. Then we have

$$t - t_0 = \frac{1}{k}\int\limits_{\varphi_0}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2 + 1} = \frac{1}{k}\Bigg(\int\limits_{\varphi_0}^{\frac{\pi}{2}} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2 + 1} + \int\limits_{\frac{\pi}{2}}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2 + 1}\Bigg)$$
$$= \frac{1}{k}\Bigg(\lim_{\varepsilon_1\to 0}\int\limits_{\varphi_0}^{\frac{\pi}{2}-\varepsilon_1} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2 + 1} + \lim_{\varepsilon_2\to 0}\int\limits_{\frac{\pi}{2}+\varepsilon_2}^{\varphi} \frac{d(k\tan\varphi)}{(k\tan\varphi)^2 + 1}\Bigg)$$
$$= \frac{1}{k}\Bigg(\lim_{\varepsilon_1\to 0}\Big[\arctan(k\tan(\frac{\pi}{2} - \varepsilon_1)) - \arctan(k\tan\varphi_0)\Big]$$
$$+ \lim_{\varepsilon_2\to 0}\Big[\arctan(k\tan\varphi) - \arctan(k\tan(\frac{\pi}{2} + \varepsilon_2))\Big]\Bigg)$$
$$= \frac{1}{k}\Big(\frac{\pi}{2} - \arctan(k\tan\varphi_0) + \arctan(k\tan\varphi) + \frac{\pi}{2}\Big)$$
$$= \frac{1}{k}\arctan(k\tan\varphi) + \frac{\pi}{k} - \frac{1}{k}\arctan(k\tan\varphi_0).$$

■

Let us interpret the Sturm-Liouville boundary conditions (1.2) on a phase plane. We have from (1.2)

$$\begin{cases} x(0)/x'(0) = \tan\alpha, \quad \varphi_0 = \alpha, \\ x(\pi)/x'(\pi) = \tan\beta, \quad \varphi_1 = \beta + \pi n, \quad \text{for some } n = 0, 1, 2, \ldots, \end{cases}$$

where $\varphi_0 = \varphi(0)$ and $\varphi_1 = \varphi(\pi)$.

3. Main Results

Czech mathematician S. Fučik in 70-th of 20-th century formulated and solved a number of problems which relate to the theory of nonlinear differential equations depending on two parameters. The second order Dirichlet boundary value problem was considered in the book [3].

We consider now a more general case. Notice that the Dirichlet boundary conditions are particular cases of the Sturm–Liouville boundary conditions.

Theorem 3. *For the case* $0 \le \alpha \le \frac{\pi}{2} \le \beta \le \pi$ *the spectrum of problem* (1.1), (1.2) *consists of separate branches* (*for* $k = 0, 1, 2, \dots \quad \lambda > 0,\ \mu > 0$) :

$$F^+_{2k} : \left[\frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha)\right] + \frac{(k-1)\pi}{\mu} + \frac{k\pi}{\lambda} + \left[\frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta)\right] = \pi,$$

$$F^-_{2k} : \left[\frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha)\right] + \frac{k\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi,$$

$$F^+_{2k+1} : \left[\frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha)\right] + \frac{k\pi}{\mu} + \frac{k\pi}{\lambda} + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi,$$

$$F^-_{2k+1} : \left[\frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha)\right] + \frac{k\pi}{\mu} + \frac{k\pi}{\lambda} + \left[\frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta)\right] = \pi.$$

Proof. One can find spectrum of problem (1.1), (1.2) by solving the equation

$$\varphi' = \begin{cases} \mu^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi \ge 0, \\ \lambda^2\sin^2\varphi + \cos^2\varphi, & \text{if} \quad \sin\varphi < 0 \end{cases}$$

together with the boundary conditions $\varphi(0) = \alpha$, $\varphi(\pi) = \beta$.

Consider a solution of the problem (1.1), (1.2), which has no zeros in the interval $(0;\pi)$. Then

$$\varphi_0 = \varphi(0) = \alpha \in [0; \frac{\pi}{2}], \quad \varphi_1 = \varphi(\pi) = \beta \in [\frac{\pi}{2}; \pi].$$

This means that for any $t \in (0;\pi)$ a solution $x(t) > 0$. Thus $x(t)$ is a solution of $x'' = -\mu^2 x$ and in polar coordinates satisfies

$$\begin{cases} \varphi'(t) = \mu^2\sin^2\varphi + \cos^2\varphi, \\ \varphi(0) = \alpha, \quad \varphi(\pi) = \beta. \end{cases}$$

By Theorem 2 we get

$$t - t_0 = \pi = \frac{1}{\mu}\arctan(\mu\tan\beta) + \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha). \tag{3.1}$$

Since λ is arbitrary positive number, we get

$$F^+_0 = \{\ (\mu_0, \lambda) :\ \mu_0 \ \text{ is a solution of (3.1)}\}\,.$$

When treating the case of $x(t) < 0$ for any $t \in (0; \pi)$, we use also the result of Theorem 2. Similarly, $F_0^- = \{(\lambda_0, \mu),\ \mu \in \mathbb{R}^+\}$, and λ_0 can be obtained from

$$\pi = \frac{1}{\lambda} \arctan(\lambda \tan \beta) + \frac{\pi}{\lambda} - \frac{1}{\lambda} \arctan(\lambda \tan \alpha).$$

Next we consider the case of $x(t)$ having exactly one zero, say, at $t = t_1$ and $x'(0) > 0$. Then for $0 \le t \le t_1$ we use $x'' = -\mu^2 x$ and the length of the interval $[0; t_1]$ is, by Theorem 2

$$t_1 - 0 = \frac{1}{\mu} \arctan(\mu \tan \pi) + \frac{\pi}{\mu} - \frac{1}{\mu} \arctan(\mu \tan \alpha) = \frac{\pi}{\mu} - \frac{1}{\mu} \arctan(\mu \tan \alpha).$$

For $t_1 \le t \le \pi$, $x(t)$ is non positive, therefore we consider a solution of $x'' = -\lambda^2 x$ and by Theorem 2 the length of the interval $[t_1, \pi]$ is equal to

$$\pi - t_1 = \frac{1}{\lambda} \arctan(\lambda \tan(\beta + \pi)) + \frac{\pi}{\lambda} - \frac{1}{\lambda} \arctan(\lambda \tan \pi) = \frac{1}{\lambda} \arctan(\lambda \tan \beta) + \frac{\pi}{\lambda}.$$

The sum of two intervals is π, and we get the expression for $F_1^+ = \{(\lambda, \mu)\}$, where λ and μ can be obtained from

$$F_1^+ : \left[\frac{\pi}{\mu} - \frac{1}{\mu} \arctan(\mu \tan \alpha)\right] + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda} \arctan(\lambda \tan \beta)\right] = \pi.$$

Similarly F_1^- is given by

$$F_1^- : \left[\frac{\pi}{\lambda} - \frac{1}{\lambda} \arctan(\lambda \tan \alpha)\right] + \left[\frac{\pi}{\mu} + \frac{1}{\mu} \arctan(\mu \tan \beta)\right] = \pi.$$

For any solution of problem (1.1), (1.2), which has exactly $n > 0$ zeros in the interval $(0; \pi)$, the interval $[0; \pi]$ can be decomposed in $n + 1$ subintervals $J_{T_1} := [0; T_1]$, $J_{T_2} := [T_1; T_1 + T_2]$, $J_{T_3} := [T_1 + T_2; T_1 + T_2 + T_3]$, …, $J_{T_{n+1}} := [\sum_{n=1}^{n} T_n; \pi]$ (the lower index refers to the length of subinterval) so, that in any of those subintervals the sign of a solution does not change (see Fig. 1). If $x(t) \ge 0$ then we use the equation $x'' = -\mu^2 x$, and if $x(t) < 0$ then the equation $x'' = -\lambda^2 x$ is used. We can compute the length of each subinterval by using results of Theorem 2. The total length of all $n + 1$ subintervals is π. This is a basis for proving relations between μ and λ.

We decompose the main interval in subintervals

$$T_1 = \frac{\pi}{\mu} - \frac{1}{\mu} \arctan(\mu \tan \alpha), \quad T_2 = T_4 = \ldots = \frac{\pi}{\lambda}, \quad T_3 = T_5 = \ldots = \frac{\pi}{\mu},$$

$$T_{n+1} = \begin{cases} \dfrac{\pi}{\mu} + \dfrac{1}{\mu} \arctan(\mu \tan \beta), & n \text{ is even}, \\ \dfrac{\pi}{\lambda} + \dfrac{1}{\lambda} \arctan(\lambda \tan \beta), & n \text{ is odd}, \end{cases}$$

when finding analytical descriptions of the branches F_n^+ $(\forall n \in \mathbb{N})$. In case of the branches F_n^- $(\forall n \in \mathbb{N})$ this decomposition is

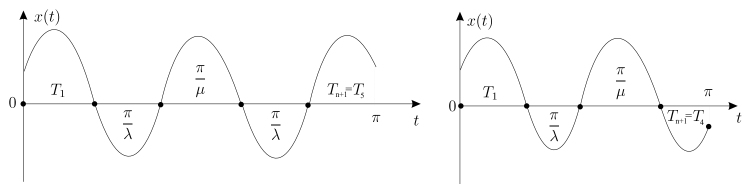

Figure 1. The example of solutions of problem (1.1), (1.2) with four and three zeros in the interval $(0;\pi)$.

$$T_1 = \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha), \quad T_2 = T_4 = \ldots = \frac{\pi}{\mu}, \quad T_3 = T_5 = \ldots = \frac{\pi}{\lambda},$$

$$T_{n+1} = \begin{cases} \dfrac{\pi}{\lambda} + \dfrac{1}{\lambda}\arctan(\lambda\tan\beta), & n \text{ is even}, \\ \dfrac{\pi}{\mu} + \dfrac{1}{\mu}\arctan(\mu\tan\beta), & n \text{ is odd}. \end{cases}$$

We use the fact, that the sum of the lengths of all intervals J_{T_1}, J_{T_2}, J_{T_3}, ..., $J_{T_{n+1}}$ is π and obtain the Fučik spectrum for problem (1.1), (1.2). ■

4. Other Cases

In previous sections analytical expressions for branches of the Fučik spectrum were obtained in the case of $0 \le \alpha \le \dfrac{\pi}{2} \le \beta \le \pi$. Let us consider the three remaining cases.

4.1. The case of $0 \le \beta \le \pi/2 \le \alpha \le \pi$

Consider the problem (1.1), (1.2) under the additional assumption that

$$0 \le \beta \le \frac{\pi}{2} \le \alpha \le \pi. \tag{4.1}$$

In the polar coordinates one gets

$$\begin{cases} \varphi' = \begin{cases} \mu^2\sin^2\varphi + \cos^2\varphi, & \text{when} \quad \sin\varphi \ge 0, \\ \lambda^2\sin^2\varphi + \cos^2\varphi, & \text{when} \quad \sin\varphi < 0, \end{cases} \\ \varphi(0) = \alpha, \quad \varphi(\pi) = \beta, \quad 0 \le \beta \le \frac{\pi}{2} \le \alpha \le \pi. \end{cases}$$

We are looking for solutions such that the respective trajectories on a phase plane starting at the straight line with the angle $\dfrac{\pi}{2} \le \alpha \le \pi$, rotate to the angle $\pi n \le \beta \le \dfrac{\pi}{2} + \pi n \ \forall n \in \mathbb{N}$ for the time period $t = \pi$.

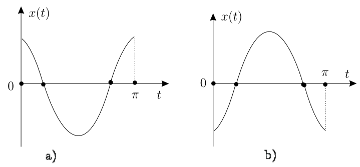

Figure 2. Visualization of solutions of problem (1.1), (1.2) with the additional condition (4.1): a) x^+_{2k}, b) x^-_{2k} .

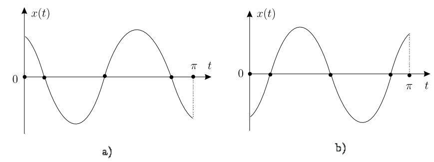

Figure 3. Visualization of solutions of problem (1.1), (1.2) under the additional condition (4.1): a) x^+_{2k-1}, b) x^-_{2k-1}.

Remark 1. The spectrum of problem (1.1), (1.2) with the restrictions (4.1) relates to solutions which are depicted in Fig. 2 and Fig. 3.

Lemma 1. *If $\beta < \alpha$, then spectrum of problem* (1.1), (1.2) *has no branches* $F^\pm_0$.

Proof. It follows from $\varphi'(t) > 0$ that the function $\varphi(t)$ is monotonically increasing. There are no branches $F^\pm_0$ for problem (1.1), (1.2) in the case of $\beta < \alpha$, therefore solutions of problem (1.1), (1.2) do not exist, which satisfy the boundary conditions and have no zeros in the interval $(0;\pi)$. ■

Corollary 1. There are no branches $F^\pm_0$ for problem (1.1), (1.2) with the condition (4.1).

We got analytical expressions for branches of the spectrum for the problem under consideration, making use of results of Theorem 2:

$$
\begin{aligned}
F^+_{2k-1}:&\ -\frac{1}{\mu}\arctan(\mu\tan\alpha)+\frac{(k-1)\pi}{\mu}+\frac{(k-1)\pi}{\lambda}+\frac{1}{\lambda}\arctan(\lambda\tan\beta)=\pi,\\
F^-_{2k-1}:&\ -\frac{1}{\lambda}\arctan(\lambda\tan\alpha)+\frac{(k-1)\pi}{\mu}+\frac{(k-1)\pi}{\lambda}+\frac{1}{\mu}\arctan(\mu\tan\beta)=\pi,\\
F^+_{2k}:&\ -\frac{1}{\mu}\arctan(\mu\tan\alpha)+\frac{k\pi}{\lambda}+\frac{(k-1)\pi}{\mu}+\frac{1}{\mu}\arctan(\mu\tan\beta)=\pi,\\
F^-_{2k}:&\ -\frac{1}{\lambda}\arctan(\lambda\tan\alpha)+\frac{k\pi}{\mu}+\frac{(k-1)\pi}{\lambda}+\frac{1}{\lambda}\arctan(\lambda\tan\beta)=\pi,\quad \forall k\in\mathbb{N}.
\end{aligned}
$$

4.2. The case of $0 \le \alpha \le \pi/2$, $0 \le \beta \le \pi/2$

Consider a solution of problem (1.1), (1.2) with the additional condition

$$0 \le \alpha \le \frac{\pi}{2}, \quad 0 \le \beta \le \frac{\pi}{2}. \tag{4.2}$$

Its image on a phase plane starts from the angle $0 \le \alpha \le \frac{\pi}{2}$, and ends at the angle $\pi n \le \beta \le \frac{\pi}{2} + \pi n$, $\forall n \in \mathbb{N}$ for the time period $t = \pi$ (see Fig. 4).

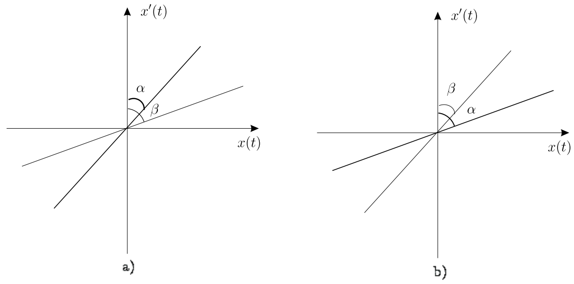

Figure 4. Interpretation of the boundary conditions for problem (1.1), (1.2) on a phase plane, with the additional condition (4.2): a) $\alpha < \beta$, b) $\alpha > \beta$.

Remark 2. Respective solutions of problem (1.1), (1.2) with the additional condition (4.2) are depicted in Fig. 5 and Fig. 6.

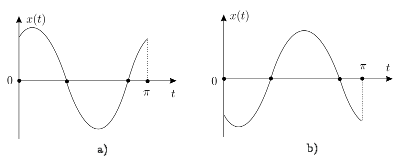

Figure 5. Visualization of solutions to problem (1.1), (1.2) with the additional condition (4.2): (a) x_{2k}^+, (b) x_{2k}^-.

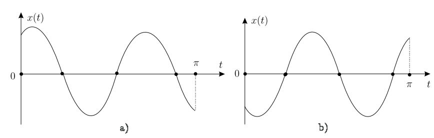

Figure 6. Visualization of solutions to problem (1.1), (1.2) with the additional condition (4.2): (a) x_{2k-1}^+, (b) x_{2k-1}^-.

Remark 3. If $\beta < \alpha$ (see Fig. 4), then there are no branches $F_0^{\pm}$ of the spectrum for problem (1.1), (1.2) with the condition (4.2).

Let us find formulas for branches of the spectrum for the problem under consideration. Branches $F_0^{\pm}$ of the Fučik spectrum relate to the case of $\alpha < \beta$

$$F_0^+ : \ -\frac{1}{\mu}\arctan(\mu\tan\alpha) + \frac{1}{\mu}\arctan(\mu\tan\beta) = \pi,$$

$$F_0^- : \ -\frac{1}{\lambda}\arctan(\lambda\tan\alpha) + \frac{1}{\lambda}\arctan(\lambda\tan\beta) = \pi.$$

Let us find analytical expressions for remaining branches:

$$F^+_{2k-1}: \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha) + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta) = \pi,$$

$$F^-_{2k-1}: \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha) + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \frac{1}{\mu}\arctan(\mu\tan\beta) = \pi,$$

$$F^+_{2k}: \frac{\pi}{\mu} - \frac{1}{\mu}\arctan(\mu\tan\alpha) + \frac{k\pi}{\lambda} + \frac{(k-1)\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta) = \pi,$$

$$F^-_{2k}: \frac{\pi}{\lambda} - \frac{1}{\lambda}\arctan(\lambda\tan\alpha) + \frac{k\pi}{\mu} + \frac{(k-1)\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta) = \pi, \ \forall k \in \mathbb{N}.$$

4.3. The case of $\pi/2 \leq \alpha \leq \pi$, $\pi/2 \leq \beta \leq \pi$

Consider solutions of problem (1.1), (1.2) with the additional condition

$$\frac{\pi}{2} \leq \alpha \leq \pi, \quad \frac{\pi}{2} \leq \beta \leq \pi. \tag{4.3}$$

Its image on a phase plane starts from the angle $\frac{\pi}{2} \leq \alpha \leq \pi$, and ends at the angle $\frac{\pi}{2} + \pi n \leq \beta \leq \pi + \pi n$, $\forall n \in \mathbb{N}$ for the time period $t = \pi$ (see Fig. 7).

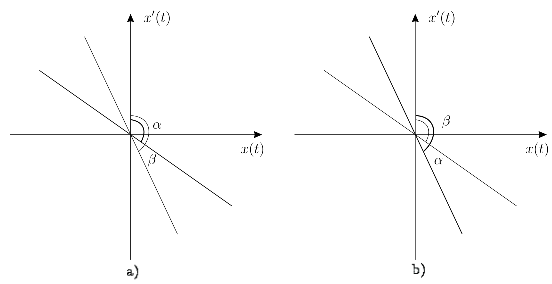

Figure 7. Interpretation of the boundary conditions for problem (1.1), (1.2) on a phase plane, with the additional condition (4.3): (a) $\alpha < \beta$, (b) $\alpha > \beta$.

Remark 4. Respective solutions of problem (1.1), (1.2) with the condition (4.3) are depicted in Fig. 8 and 9.

Remark 5. There are no branches $F_0^{\pm}$ for problem (1.1), (1.2) with the condition (4.3), if $\beta < \alpha$ (see Fig. 7).

Let us find formulas for branches of the spectrum for the problem under consideration. Branches $F_0^{\pm}$ relate to the case of $\alpha < \beta$:

$$F_0^+: -\frac{1}{\mu}\arctan(\mu\tan\alpha) + \frac{1}{\mu}\arctan(\mu\tan\beta) = \pi,$$

$$F_0^-: -\frac{1}{\lambda}\arctan(\lambda\tan\alpha) + \frac{1}{\lambda}\arctan(\lambda\tan\beta) = \pi.$$

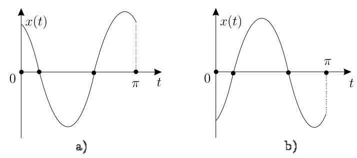

Figure 8. Visualization of solutions to problem (1.1), (1.2) with the additional condition (4.3): (a) x_{2k}^{+}, (b) x_{2k}^{-}.

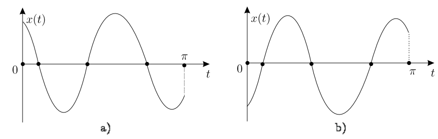

Figure 9. Visualization of solutions to problem (1.1), (1.2) with the additional condition (4.3): (a) x_{2k-1}^{+}, (b) x_{2k-1}^{-}.

Analytical expressions for remaining branches are given as:

$$
\begin{aligned}
F_{2k-1}^{+}: \ & -\frac{1}{\mu}\arctan(\mu\tan\alpha) + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} \\
& + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi, \\
F_{2k-1}^{-}: \ & -\frac{1}{\lambda}\arctan(\lambda\tan\alpha) + \frac{(k-1)\pi}{\mu} + \frac{(k-1)\pi}{\lambda} \\
& + \left[\frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta)\right] = \pi, \\
F_{2k}^{+}: \ & -\frac{1}{\mu}\arctan(\mu\tan\alpha) + \frac{k\pi}{\lambda} + \frac{(k-1)\pi}{\mu} \\
& + \left[\frac{\pi}{\mu} + \frac{1}{\mu}\arctan(\mu\tan\beta)\right] = \pi, \\
F_{2k}^{-}: \ & -\frac{1}{\lambda}\arctan(\lambda\tan\alpha) + \frac{k\pi}{\mu} + \frac{(k-1)\pi}{\lambda} \\
& + \left[\frac{\pi}{\lambda} + \frac{1}{\lambda}\arctan(\lambda\tan\beta)\right] = \pi, \quad \forall k \in \mathbb{N}.
\end{aligned}
$$

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