Mathematical Modelling and Analysis. Ed. A.Buikis
Volume 12 Number 1, 2007, pages 101–106
 ISSN 1392-6292 print, ISSN 1648-3510 online

STABILITY OF SLOWLY DIVERGING FLOWS IN SHALLOW WATER [1]

A. KOLYSHKIN and S. NAZAROVS

Riga Technical University

1 Meza street Riga, Latvia

E-mail: akoliskins@rbs.lv; nazarow@yahoo.com

Received September 9, 2006; revised November 20, 2006; published online February 10, 2007

Abstract. Methods of weakly nonlinear theory are used in the present paper in order to study the development of instability in shallow water for the case where the flow is assumed to be slightly non-parallel. An asymptotic scheme where slow divergence of the base flow is taken into account is applied to shallow water equations with averaging coefficients. An amplitude evolution equation for the most unstable mode is derived.

Key words: momentum correction coefficients, shallow wake flows, asymptotic analysis

1. Introduction

Shallow turbulent mixing layers, jets and wakes are widespread in nature and engineering. The understanding of mass, momentum and energy exchange is important for flows in compound and composite channels, rivers, estuaries, and in the atmosphere. Such flows are characterized by the presence of turbulent eddies whose transverse horizontal length scale is considerably larger than the water depth. As a result, the limited water depth prevents the development of three-dimensional instabilities.

Different methods of analysis of two-dimensional structures in shallow water flows are considered in [9]. Several authors [3, 10] analyzed different aspects of the linear stability of flows in shallow water. The base flows used in [2, 3, 10], for stability analyses are assumed to be parallel. Experimental data show that

[1] This work has been partly supported by the European Social Fund within the National Programme "Support for the carrying out doctoral study programs and post-doctoral researches" project "Support for the development of doctoral studies at Riga Technical University" and the Latvian Council of Science under the Project No. 04.1239.

the width of shallow wake [1] and the width of the mixing layer [12] are slowly changing with respect to the longitudinal coordinate. The slow divergence of the base flow in the downstream direction allows one to construct an asymptotic scheme which takes non-parallelism of shallow water flows into account. Such formulations have been applied in the past to the spatial stability analysis of slowly diverging shear flows in deep water [4, 5]. Recently such a scheme has also been applied to shallow wakes [7].

Stability of shallow water flows is usually analyzed under the assumption that flow characteristics are independent on the vertical coordinate. In some cases, however, this assumption may not be valid. Changes in roughness of the bottom boundary or flow regimes can lead to large deviations from the above-mentioned assumption [13, 14]. Momentum correction coefficients [13, 14] are sometimes used in order to take into account the non-uniformity of the velocity distribution with respect to the vertical coordinate. In particular, momentum correction coefficients are used in [6] for linear stability analysis of shallow mixing layers and in [11] for weakly nonlinear analysis of shallow wakes.

In the present paper weakly nonlinear spatial stability analysis of flows in shallow water is performed. An asymptotic scheme where slow divergence of the base flow is taken into account is applied to shallow water equations with averaging coefficients.

2. Weakly Nonlinear Spatial Stability Analysis

Consider slightly non-parallel two-dimensional shallow water flow. This means that the normal velocity component is small compared with the streamwise component and that the base flow quantities are weakly varying functions of the streamwise coordinate. The assumption of weak non-parallelism implies that the instability wavelength λ is much smaller than the length scale L associated with streamwise inhomogeneities of the base flow.

Two-dimensional shallow water equations with averaging coefficients are the following [11]:

$$(\Delta\psi)_t + \gamma_1(\psi_y\psi_{xy})_y - \beta_2(\psi_x\psi_{yy})_y + \gamma_2(\psi_x\psi_{xy})_x + \beta_2(\psi_{xx}\psi_y)_x - \gamma_3(\psi_x\psi_{xx})_y$$
$$+ S\psi_S\Delta\psi + \frac{S}{\psi_S}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) = 0, \quad (2.1)$$

where $\psi(x,y)$ is the stream function, x and y are the streamwise and transverse coordinates, respectively, c_f is the bottom friction coefficient, h is water depth, β_1, β_2 and β_3 are the averaging coefficients defined in [11]:

$$\gamma_1 = 2\beta_1 - \beta_2, \quad \gamma_2 = \beta_2 - 1, \quad \gamma_3 = 2\beta_3 - 1, \quad S = \frac{c_f}{2h}, \quad \psi_S = \sqrt{\psi_x^2 + \psi_y^2}.$$

In the spirit of the WKB approximation [8] we introduce a small parameter $\varepsilon = \dfrac{\lambda}{L} \ll 1$, a slow streamwise coordinate $X = \varepsilon x$ and decompose the total stream function of the flow, $\psi(x,y,t)$, into basic (ψ_0) and perturbed (ψ_f) components, respectively:

$$\psi(x,y,t) = \psi_0(y,X) + \psi_f(x,y,t). \tag{2.2}$$

Note that $\psi_f(x,y,t)$ in (2.2), in general, is not small (that is, $\psi_f(x,y,t)$ is not proportional to a positive power of ε). Linearizing (2.1) in the neighborhood of the base flow, dropping the subscript "f" and retaining only the terms of order ε we obtain

$$\begin{aligned}
&\psi_{xxt} + \psi_{yyt} + (2\beta_1 - \beta_2)(U_y\psi_{xy} + U\psi_{xyy}) - \beta_2(U_y\psi_{xy} + U_{yy}\psi_x) + \beta_2 U\psi_{xxx} \\
&+ \frac{c_f}{2h}\Big[U(\psi_{xx} + 2\psi_{yy}) + 2U_y\psi_y\Big] + \varepsilon\Big\{(2\beta_1 - \beta_2)(U_X\psi_{yy} + U_{Xy}\psi_y) \\
&- \beta_2(U_X\psi_{yy} - V\psi_{yyy}) + (\beta_2 - 1)(U_X\psi_{xx} - V\psi_{xxy}) + \beta_2 U_X\psi_{xx} \\
&- (2\beta_3 - 1)(U_X\psi_{xx} - V\psi_{yy}) + \frac{c_f}{2h}\Big[2U_X\psi_x - 2V\psi_{xy} + V\psi_x\frac{U_y}{U}\Big]\Big\} = 0,
\end{aligned} \tag{2.3}$$

where $U = \psi_{0y}$ and $V = -\psi_{0X}$.

The method of normal modes is a classical method of stability analysis of parallel steady flows (see, for example, [8]). In such cases the stream function is represented in the form

$$\psi(x,y,t) = \varphi(y)\exp\left[i(kx - \omega t)\right], \tag{2.4}$$

where k is the wavenumber of a perturbation and ω is the frequency of oscillation. An arbitrary perturbation consists of a superposition of perturbed components of the form (2.4) over the range of all wavenumbers. However, in order to find a necessary condition for instability, it is enough to consider only one component of the form (2.4) (see, for example, [8]). If the base flow is slightly non-parallel, then the perturbation stream function $\psi(x,y,t)$ is decomposed into a slowly varying amplitude function $\varphi(y,X,\omega)$ and a fast varying phase function $\theta(X,\omega)/\varepsilon$:

$$\psi(x,y,\omega,t) = \varphi(y,X,\omega)\exp\left[i\Big(\frac{\theta(X,\omega)}{\varepsilon} - \omega t\Big)\right]. \tag{2.5}$$

We also assume that $\varphi(y,X,\omega)$ can be represented by a power series in ε in the form

$$\varphi(y,X,\omega) = \varphi_1(y,X,\omega) + \varepsilon\varphi_2(y,X,\omega) + \dots \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.3) and collecting the terms that do not contain ε we obtain

$$\mathcal{L}[\varphi_1] = 0, \tag{2.7}$$

where

$$\begin{aligned}
\mathcal{L}[\varphi_1] = \varphi_1''\Big[(2\beta_1 - \beta_2)U - \frac{\omega}{k} - \frac{ic_f U}{kh}\Big] + \varphi_1'\Big[2(\beta_1 - \beta_2)U_y - \frac{ic_f U_y}{kh}\Big] \\
+ \varphi_1\Big[\omega k - \beta_2 U_{yy} - \beta_2 k^2 U + \frac{ic_f kU}{2h}\Big].
\end{aligned} \tag{2.8}$$

The primes in (2.8) represent the derivatives with respect to y and $k = k(X,\omega) = \theta_X$. Thus, equation (2.7) is the modified Rayleigh equation, which is obtained in [11] under parallel flow approximation. Equation (2.7) together with zero boundary conditions forms an eigenvalue problem (where the eigenvalues are $k = k(X,\omega)$). The values of $k = k(X,\omega) = \theta_X$ can be obtained as a result of the numerical solution of the eigenvalue problem. In addition, a normalized eigenfunction of the linear stability problem, $\Phi(y,X,\omega)$, can be calculated. Note that the coordinate X appears in (2.7) as a parameter.

In order to obtain the equation for the amplitude of a perturbation we assume that

$$\varphi_1(y,X,\omega) = A(X,\omega)\Phi(y,X,\omega), \tag{2.9}$$

where $A(X,\omega)$ is an unknown complex amplitude and $\Phi(y,X,\omega)$ is a normalized eigenfunction of the linear stability problem. Substituting (2.5), (2.6) and (2.9) into (2.3) and collecting the terms containing ε we obtain

$$\mathcal{L}[\varphi_2] = g, \tag{2.10}$$

where

$$\begin{aligned} g = \frac{i}{k}\frac{dA}{dX}\Big\{&2\omega k\Phi + \big(2\beta_1 - \beta_2\big)\big(U_y\Phi' + U\Phi''\big) \\ &- \beta_2\big[U_y\Phi' + U_{yy}\Phi + 3U\Phi k^2\big] + \frac{ic_fUk\Phi}{h}\Big\} \\ - \frac{i}{k}A\Big\{&2\omega k\Phi_X + \omega\frac{dk}{dx}\Phi + \big(2\beta_1 - \beta_2\big)\big[U_y\Phi'_X + U\Phi''_X + U_X\Phi'' + U_{Xy}\Phi'\big] \\ - \beta_2\big[&U_y\Phi'_X + U_{yy}\Phi_X + 3Uk^2\Phi_X + 3U\Phi k\frac{dk}{dX} + U_X\Phi'' - V\Phi''' + U_Xk^2\Phi\big] \\ + \big(&\beta_2 - 1\big)\big(Vk^2\Phi' - k^2U_X\Phi\big) - \big(2\beta_3 - 1\big)\big[k^2V\Phi' - k^2U_X\Phi\big] \\ + \frac{c_f}{2h}&\Big[2iUk\Phi_X + iU\Phi\frac{dk}{dX} + 2ikU_X\Phi - 2ikV\Phi' + i\frac{V}{U}U_yk\Phi\Big]\Big\}. \end{aligned}$$

An amplitude evolution equation for $A(X,\omega)$ is obtained from Fredholm's alternative, namely, equation (2.10) has a solution if and only if the function g is orthogonal to all eigenfunctions $\tilde{\Phi}$ of the corresponding adjoint problem. Using the solvability condition

$$\int_{-\infty}^{\infty} g\tilde{\Phi}\,dy = 0$$

we obtain the equation for the function $A(X,\omega)$ in the form

$$M(X,\omega)\frac{dA}{dX} + N(X,\omega)A = 0,$$

where

$$\begin{aligned} M(X,\omega) = \frac{i}{k}\int_{-\infty}^{\infty}&\Big\{2\omega k\Phi + (2\beta_1 - \beta_2)(U_y\Phi' + U\Phi'') \\ &- \beta_2[U_y\Phi' + U_{yy}\Phi + 3U\Phi k^2] + \frac{ic_fUk}{h}\Phi\Big\}\tilde{\Phi}\,dy, \end{aligned}$$

$$N(X,\omega) = \frac{i}{k}\int_{-\infty}^{\infty}\Big\{2\omega k\Phi_X + \omega\frac{dk}{dX}\Phi$$
$$+ (2\beta_1 - \beta_2)\big[U_y\Phi_X' + U\Phi_X'' + U_X\Phi'' + U_{Xy}\Phi'\big]$$
$$- \beta_2\big[U_y\Phi_X' + U_{yy}\Phi_X + 3Uk^2\Phi_X + 3U\Phi k\frac{dk}{dX} + U_X\Phi'' - V\Phi''' + U_Xk^2\Phi\big]$$
$$+ (\beta_2 - 1)\big[Vk^2\Phi' - k^2U_X\Phi\big] - (2\beta_3 - 1)\big[k^2V\Phi' - k^2\Phi U_X\big]$$
$$\times\frac{c_f}{2h}\Big[2ikU\Phi_X + iU\frac{dk}{dX}\Phi + 2ikU_X\Phi - 2ikV\Phi' + i\frac{V}{U}U_yk\Phi\Big]\Big\}\tilde{\Phi}\,dy.$$

Thus, using the WKB method, the leading order approximation of the stream function $\psi(x,y,\omega,t)$ has the form

$$\psi(x,y,\omega,t) \sim A(X,\omega)\Phi(y,X,\omega)\exp\Big[i\Big(\frac{1}{\varepsilon}\int_0^X k(X,\omega)\,dX - \omega t\Big)\Big]. \qquad (2.11)$$

3. Discussion

Formula (2.11) provides the connection between local parallel flow approximations and takes into account slow streamwise variation of the base flow. Following [4], a few important conclusions can be drawn from (2.11). First, all the three terms on the right-hand side of (2.11) contain information related to the amplitude and phase of the perturbation. Second, the growth rate and phase speed of the perturbation at any given downstream station depends on the choice of the perturbed quantities. Finally, the growth rate and phase speed depend even on the location where these quantities are calculated. In particular, it is shown in [4] that for any given flow variable Q one can define a local wavenumber k_l by the formula

$$k_l(x,y|Q) = -i\frac{\partial}{\partial x}\ln Q(x,y), \qquad (3.1)$$

where $k_l = k_{lr} + ik_{li}$ and the values of k_{lr} and k_{li} are interpreted as the local phase speed and local spatial growth rate. Thus, in order to make a meaningful comparison of the weakly nonlinear model (2.11) with experimental data one needs to choose a particular flow quantity Q (say, pressure or streamwise velocity), then measure it at a particular point and evaluate the right-hand side of (3.1) at the same point. In other words, in order to validate the weakly nonlinear model one needs to have either detailed experimental data or, alternatively, numerical solution of nonlinear two-dimensional shallow water equations.

References

[1] D. Chen and G. H. Jirka. Experimental study of plane turbulent wake in a shallow water layer. *Fluid Dynamic Research*, **16**, 1995.

[2] D. Chen and G. H. Jirka. Absolute and convective instabilities of plane turbulent wakes in a shallow water layer. *Journal of Fluid Mechanics*, **338**, 1997.

[3] V. H. Chu, J. H. Wu and R. E. Khayat. Stability of transverse shear flows in shallow open channel. *Journal of Hydraulic Engineering*, **117**, 1991.

[4] D. G. Crighton and M. Gaster. Stability of slowly diverging jet flow. *Journal of Fluid Mechanics*, **77**, 397–413, 1976.

[5] M. Gaster, E. Kit and J. Wygnanski. Large-scale structures in a forced turbulent mixing layer. *Journal of Fluid Mechanics*, **150**, 23–39, 1985.

[6] M. Ghidaoui, A. A. Kolyshkin and B. C. Yen. Influence of momentum correction coefficients on linear stability analysis of open-channel flows. In: *Proc. of XXVIII IAHR Congress, Theme E5, CD-ROM*, 1999.

[7] M. S. Ghidaoui and A. A. Kolyshkin. Weakly nonlinear spatial stability analysis of shallow water flows. In: *4th International Symposium on Environmental Hydraulics, Hong Kong*, 2004.

[8] P. Huerre and M. Rossi. Hydrodynamic instabilities in open flows. In: *Hydrodynamics and nonlinear instabilities*, 1998.

[9] G. H. Jirka. Large scale flow structures and mixing processes in shallow flows. *Journal of Hydraulic Engineering*, **39**, 2001.

[10] A. A. Kolyshkin and M. S. Ghidaoui. Stability analysis of shallow wake flows. *Journal of Fluid Mechanics*, **494**, 355–377, 2003.

[11] A. A. Kolyshkin and S. Nazarovs. Influence of averaging coefficients on weakly nonlinear stability of shallow flows. *IASME Transactions*, **2**, 86–91, 2005.

[12] B. C. van Prooijen and W. S. J. Uijttewaal. A linear approach for the evolution of coherent structures in shallow mixing layers. *Physics of Fluids*, **14**, 4105–4114, 2002.

[13] R. Xia and B. C. Yen. Significance of averaging coefficients in open-channel flow equations. *Journal of Hydraulic Engineering*, **120**, 169–189, 1994.

[14] B. C. Yen. Open-channel flow equations revisited. *Journal of the Engineering Mechanics Division*, **99**, 979–1009, 1973.