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FINITE SUPERELEMENTS METHOD FOR BIHARMONIC EQUATION

M. GALANIN[1], D. MILYUTIN[2] and E. SAVENKOV[1]

[1]*Keldysh Institute for Applied Mathematics of RAS*
Russia, 125047, Moscow, Miusskaya sq., 4
[2]*Bauman Moscow State Technical University*
Russia, Moscow, 2nd Baumanskaya st., 5
E-mail: galan@keldysh.ru;daniel_m@list.ru;savenkov@keldysh.ru

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Abstract. In this work finite superelements method (FSEM) for solution of biharmonic equation in bounded domains is proposed and developed. The method is based on decomposition of domain into subdomains with the solution of a number of intermediary problems, every of which is a boundary value problem for biharmonic equation with boundary condition being basis for interpolation of solution at superelements boundaries. The initial problem solution is found as an expansion on the constructed function system. It is shown that the solution of general problem can be recovered using functions and traces found above. Error estimates for one case of FSEM are obtained.
Key words: Biharmonic equation, finite superelements method, Poincaré-Steklov operators

1. Introduction

There exists a wide class of problems, solutions of which contain sharp inhomogeneities that become apparent on small space-scales against size of domain of interest. The numerical solution of such problems requires special meshes for resolution of these singularities to be used. Either adaptive to solution meshes concentrating in singularity neighbourhood or rather fine meshes with uniform step h and large number of points should be used for these purposes. The first alternative requires special algorithms and the second one consumes a very large amount of the computer memory. For solution of such problems the finite superelements method (FSEM) was proposed by L.G. Strakhovskaya and R.P. Fedorenko [2, 16, 17, 18].

FSEM is based on the representation of the solution as an expansion in "basis" functions system with finite support. But, in contrast to the finite elements method (FEM), FSEM doesn't require the measure of such supports (mesh H) to be small. Actually, the size of this mesh can be so large that it certainly doesn't allow to resolve singularities of the solution using FEM for this mesh. Another distinction is in construction of the "basis" functions. "Basis" functions are purposely made containing the significant part of the solution of a given problem. Exactly this choice of the "basis" functions let us to get a good numerical solution with a very coarse decomposition of the domain of reference.

One approach for the analysis of the method was suggested in works [14, 15]. Another approach for the analysis of FSEM approximations using Poincaré-Steklov operators (see [11] and review in [12]) and variational equations for such problems as "well" problems for the Laplace operator equation, elasticity problem, velocity skin-layer problem is proposed in [4, 5, 6, 7, 8, 9, 10]. In these works differential equations of the second order were considered.

In this paper FSEM approximations for solution of the biharmonic equation with one type of boundary conditions is developed. A variational formulation for the traces of the problem and its approximations are presented. The presented approach was previously used by the authors for the analysis of FSEM approximations for a scalar second-order Laplace equation. In order to show a generality of the suggested approach we tried to follow it as much as possible. The only significant difference is the procedure of obtaining the errors of the approximate solution. To perform it in a most easy way we use here some particular properties of the problem under consideration, i.e. the case of biharmonic equation.

In its theoretical part the paper generally follows the theoretical approach presented by the authors earlier (see [4, 5, 6, 7, 8, 9, 10]). In spite of the fact that this paper covers a particular case of the biharmonic equation, we use the following main assumptions. During the procedure of obtaining the variational equation for traces we essentially use the facts of existence of the appropriate Green formula for operator under consideration and existence and uniqueness of solutions of all intermediate problems. For the error analysis it is additionally assumed that the problem is linear and originally it has a form of $A^2u = f$, where A is some linear continuous positively defined operator. Generalization to that case is quite straightforward comparing to the particular case of the biharmonical equation.

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2. Basic Notations

In the paper we use bold fonts for vector valued functions and operators as well as for elements of direct product type functional spaces and normal fonts for scalar functions and variables.

Let $\Omega \in \mathbb{R}^2$ be some bounded domain with piecewise smooth boundary $\partial\Omega$ and $\boldsymbol{x} = (x_1, x_2)^T \in \Omega$ be an arbitrary point from Ω. We use $(\ldots)$ brackets to define components of vector-valued variables and $\{\ldots\}$ to define elements of other direct product type spaces.

Further we will widely use conventional Sobolev spaces $W_2^1(\Omega)$, $\overset{\circ}{W}{}_2^1(\Omega)$, space $L_2(\Omega)$ and corresponding trace spaces $W_2^{1/2}(\partial\Omega)$ and $L_2(\partial\Omega)$. Hereafter for any space S we denote its dual space as S'. For the spaces mentioned above we set

$$\left(W_2^1(\Omega)\right)' = W_2^{-1}(\Omega), \quad (L_2(\Omega))' = L_2(\Omega), \quad \left(W_2^{1/2}(\partial\Omega)\right)' = W_2^{-1/2}(\partial\Omega).$$

Dot products in these spaces are defined as:

$$(u,v)_{L_2(\Omega)} := (u,v)_\Omega = \int\limits_\Omega uv\,d\Omega, \quad (u,v)_{L_2(\partial\Omega)} := \langle u,v\rangle_{\partial\Omega} = \int\limits_{\partial\Omega} uv\,d\gamma,$$

$$(u,v)_{W_2^1(\Omega)} = \int\limits_\Omega \left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} + uv\right) d\Omega.$$

We also use notation $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ for dualities between $W_2^1(\Omega)$ and $W_2^{-1}(\Omega)$ and $W_2^{1/2}(\partial\Omega)$ and $W_2^{-1/2}(\partial\Omega)$ correspondingly. For any space S mentioned above we set

$$\|u\|_S^2 = (u,u)_S,$$

and for any $W = S^2 \equiv S \times S$ we set

$$(\boldsymbol{u},\boldsymbol{v})_W = (u_1,v_1)_S + (u_2,v_2)_S, \quad \|\boldsymbol{u}\|_W^2 = (\boldsymbol{u},\boldsymbol{u})_W, \quad \boldsymbol{u},\boldsymbol{v} \in W,$$

where $\boldsymbol{u} = \{u_1,u_2\}$, $\boldsymbol{v} = \{v_1,v_2\}$, $u_i, v_i \in S$, $i = 1,2$. Let

$$V(\Omega) = W_2^1(\Omega) \times W_2^1(\Omega), \quad V_0(\Omega) = \overset{\circ}{W}{}_2^1(\Omega) \times \overset{\circ}{W}{}_2^1(\Omega),$$

$$H(\Omega) = L_2(\Omega) \times L_2(\Omega), \quad V_\Gamma(\partial\Omega) = W_2^{1/2}(\partial\Omega) \times W_2^{1/2}(\partial\Omega),$$

$$H_\Gamma(\partial\Omega) = L_2(\partial\Omega) \times L_2(\partial\Omega)$$

and the corresponding dual spaces are given by

$$V'(\Omega) = V_0'(\Omega) = W_2^{-1}(\Omega) \times W_2^{-1}(\Omega), \quad H'(\Omega) = H(\Omega),$$

$$V_\Gamma'(\partial\Omega) = W_2^{-1/2}(\partial\Omega) \times W_2^{-1/2}(\partial\Omega), \quad H_\Gamma'(\partial\Omega) = H_\Gamma(\partial\Omega).$$

3. Problem Statement

Let's consider the following boundary value problem for some scalar valued function u defined in Ω:

$$\Delta\Delta u = f,$$

with the following BCs defined on $\partial\Omega$:

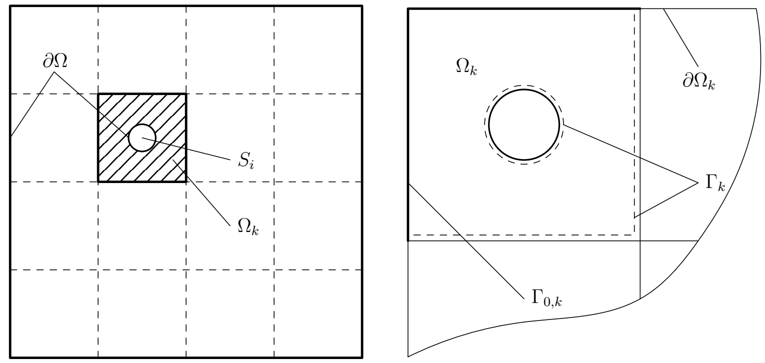

Figure 1. Computational domain and superelements.

$$u = g_1, \quad -\Delta u = g_2.$$

It can be rewritten as a system of equations of the form

$$-\Delta u_1 = u_2,\ -\Delta u_2 = f \text{ in } \Omega, \quad u_1 = g_1,\ u_2 = g_2 \text{ at } \partial\Omega,$$

or in the vector form

$$-\Delta \boldsymbol{u} + \mathbf{A}\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ at } \partial\Omega, \tag{3.1}$$

where column-vectors $\boldsymbol{u} = (u_1, u_2)^T$, $\boldsymbol{g} = (g_1, g_2)^T$ and $\boldsymbol{f} = (0, f)^T$ are some sufficiently smooth functions and

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Setting $\boldsymbol{L} = -\Delta + \mathbf{A}$ we can write (3.1) as

$$\boldsymbol{L}\boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \text{ at } \partial\Omega.$$

We suppose that problem (3.1) has a unique solution.

We solve the problem in a multiply connected domain Ω which is generated from a simply connected domain Ω_0 by elimination of a number of small disjoint disks (or "wells") S_i, i.e. $\Omega = \Omega_0 \setminus \cup S_i$. The boundary conditions for $\boldsymbol{u}$ define the trace of the solution at the whole disconnected boundary of Ω (see Fig. 1).

4. Weak Formulation

Multiplying equation (3.1) in $H(\Omega)$ by an arbitrary two-component function $\boldsymbol{v} = (v_1, v_2) \in V_0(\Omega)$ which components vanish at $\partial\Omega$ one can formally obtain the following variational equation for $\boldsymbol{u} \in V$:

$$L_{\Omega}\left(\boldsymbol{u},\boldsymbol{v}\right)=\left(\boldsymbol{f},\boldsymbol{v}\right)_{\Omega}, \quad \forall \boldsymbol{v}\in V_0, \tag{4.1}$$

$$\boldsymbol{u}|_{\partial\Omega}=\boldsymbol{g}, \tag{4.2}$$

where $\boldsymbol{g}\in V_{\Gamma}(\partial\Omega)$ and bilinear form $L_{\Omega}(\cdot,\cdot)$ is defined as follows:

$$L_{\Omega}(\boldsymbol{u},\boldsymbol{v})=a_{\Omega}(\boldsymbol{u},\boldsymbol{v})+\int\limits_{\Omega}\mathbf{A}\boldsymbol{u}\cdot\boldsymbol{v}\,d\Omega,$$

$$a_{\Omega}(\boldsymbol{u},\boldsymbol{v})=\int\limits_{\Omega}\frac{\partial\boldsymbol{u}}{\partial x_1}\cdot\frac{\partial\boldsymbol{v}}{\partial x_1}+\frac{\partial\boldsymbol{u}}{\partial x_2}\cdot\frac{\partial\boldsymbol{v}}{\partial x_2}\,d\Omega.$$

Here we used the following result from [13]:

Theorem 1. *There exists unique operator* $\boldsymbol{\delta}:V(\Omega)\to V'_{\Gamma}(\Omega)$ *such that Green's formula*

$$L_{\Omega}\left(\boldsymbol{u},\boldsymbol{v}\right)=\left(\boldsymbol{L}\boldsymbol{u},\boldsymbol{v}\right)_{\Omega}+\left\langle\boldsymbol{\delta}\boldsymbol{u},\boldsymbol{\gamma}\boldsymbol{v}\right\rangle_{\partial\Omega}, \quad \forall\boldsymbol{u}\in D\left(\boldsymbol{L}\right),\forall\boldsymbol{v}\in V$$

holds. Here $\boldsymbol{\gamma}$ *is a trace operator:*

$$\boldsymbol{\gamma}:V(\Omega)\to V_{\Gamma}(\Omega), \quad \boldsymbol{u}\mapsto\boldsymbol{\gamma}\boldsymbol{u}=\boldsymbol{u}|_{\partial\Omega},$$

and

$$D\left(\boldsymbol{L}\right)=\left\{\boldsymbol{u}\in V(\Omega): \quad \boldsymbol{L}\boldsymbol{u}\in H(\Omega)\right\}.$$

This theorem holds for domains with Lipschitz boundary. For domain with piecewise smooth boundary and sufficiently smooth function $\boldsymbol{u}=(u_1,u_2)^T$:

$$\boldsymbol{\delta}\boldsymbol{u}=\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}=n_1\frac{\partial\boldsymbol{u}}{\partial x_1}+n_2\frac{\partial\boldsymbol{u}}{\partial x_2}=\begin{pmatrix}\partial u_1/\partial\boldsymbol{n}\\ \partial u_2/\partial\boldsymbol{n}\end{pmatrix},$$

where $\boldsymbol{n}=(n_1,n_2)^T$ is an outer unit normal to the domain boundary.

Equation (4.1) is strictly elliptical in domain Ω, i.e. there exist real positive constants c_1 and c_2 such that the following condition is fulfilled:

$$c_1\|\boldsymbol{u}\|^2_{V(\Omega)}\le a_{\Omega}\left(\boldsymbol{u},\boldsymbol{u}\right)\le c_2\left\|\boldsymbol{u}\right\|^2_{V(\Omega)}, \quad \forall\boldsymbol{u}\in V(\Omega).$$

Using Green's formula it can be shown that operator $\boldsymbol{L}:V(\Omega)\to V'(\Omega)$ is formally generated by bilinear form $L_{\Omega}(\cdot,\cdot):V(\Omega)\times V_0(\Omega)\to\mathbb{R}$, i.e.

$$L_{\Omega}\left(\boldsymbol{u},\boldsymbol{v}\right)=\left(L\boldsymbol{u},\boldsymbol{v}\right)_{\Omega}, \quad \forall\boldsymbol{u}\in V(\Omega),\boldsymbol{v}\in V_0(\Omega).$$

5. Special Weak Formulation

Results presented in this section are quite general in that sense that we don't use here a particular form of the operator $\boldsymbol{L}$ but only the form of appropriate Green's formula.

Let us suppose that domain $\bar{\Omega}_0\supseteq\bar{\Omega}$ is a union of K subdomains $\bar{\Omega}_{0,k}$,

$$\bar{\Omega}_0 = \bigcup_{k=1}^{K} \bar{\Omega}_{0,k}, \quad \bar{\Omega}_{0,k} = \Omega_{0,k} \cup \partial\bar{\Omega}_{0,k}.$$

We consider domains $\bar{\Omega}_{0,k}$ which are non-overlapping polygons, i.e. any couple of these polygons either are not overlapped or have either common edge or common vertex. Additionally we suppose that every $\Omega_{0,k}$ contains at most one "well" S_i and every "well" S_i is situated inside some subdomain $\Omega_{0,k}$. Let $\Omega_k = \Omega \cap \Omega_{0,k} \neq \emptyset$, then $\bar{\Omega} = \bigcup\limits_{k=1}^{K} \bar{\Omega}_k$.

Let $\partial\Omega_k$ be a boundary of subdomain Ω_k, then $\partial\Omega_k = \Gamma_{0,k} \cup \Gamma_k$, where $\Gamma_{0,k} = \partial\Omega_k \cap \partial\Omega$, $\Gamma_k = \partial\Omega_k \backslash \Gamma_{0,k}$ (Fig. 1, right hand plot). We will call domain Ω_k a superelement. Since all domains which introduce decomposition are polygons, an outer normal to their boundaries exists almost everywhere.

For any functional space $S = S(\Omega)$ over Ω we set $S_k = S(\Omega_k)$. We define spaces $\tilde{V}_k = \tilde{V}(\Omega_k)$ in the following way:

$$\tilde{V}_k = \tilde{V}(\Omega_k) = \begin{cases} V(\Omega_k), & \Gamma_{0,k} = \emptyset, \\ \left\{ \boldsymbol{v} \in V(\Omega_k) : \quad \boldsymbol{v}|_{\Gamma_{0,k}} = \boldsymbol{g}|_{\Gamma_{0,k}} \right\}, & \Gamma_{0,k} \neq \emptyset \end{cases}.$$

Let be $W = \prod\limits_{k=1}^{K} V_k$ and $\tilde{W} = \prod\limits_{k=1}^{K} \tilde{V}_k$. Element $\boldsymbol{v} \in W$ is a set of K functions $\boldsymbol{v}_k$, $\boldsymbol{v} = \{\boldsymbol{v}_k\}_{k=1}^{K}$, $\boldsymbol{v}_k \in V_k$, $k = \overline{1,K}$.

Space V can be canonically embedded into space W:

$$\boldsymbol{v} \in V \mapsto \{\boldsymbol{v}_k\}_{k=1}^{K} \in W, \quad \boldsymbol{v}_k = \boldsymbol{v}|_{\Omega_k} \in V_k, \quad k = \overline{1,K},$$

and space W can be canonically embedded into space H in the same way.

The following result can be proved ([4]):

Lemma 1. *Let $\boldsymbol{v} = \{\boldsymbol{v}_k\}_{k=1}^{K} \in W$ and*

$$\sum_{k=1}^{K} \langle \gamma_k \boldsymbol{v}_k, \boldsymbol{\mu}_k \rangle_{\partial\Omega_k} = 0, \quad \forall \boldsymbol{\mu} = \{\boldsymbol{\mu}_k\}_{k=1}^{K} \in M^2 = M \times M,$$

where

$$M = \Big\{ \boldsymbol{\eta} = \{\eta_k\}_{k=1}^{K} \in \prod_{k=1}^{K} W_2^{-1/2}(\partial\Omega_k) :$$
$$\exists \psi_1, \psi_2 \in \overset{\circ}{W}{}_2^1(\Omega), \text{ such that } \eta_k = \psi_1 n_{k,1} + \psi_2 n_{k,1}, \quad k = \overline{1,K} \Big\},$$

and $n_{k,i}$ is the i-th component of an outer unit normal $\boldsymbol{n}_k = (n_{k,1}, n_{k,2})^T$ to the boundary $\partial\Omega_k$ of domain Ω_k. Then $\boldsymbol{v} \in V$.

Roughly speaking this lemma gives conditions of "weak continuity" under which piecewise differentiable function from W is globally differentiable function from V.

Our next step is the following generalization of weak statement of the problem (3.1): find $\boldsymbol{u} = \{\boldsymbol{u}_k\}_{k=1}^K \in \tilde{W}$:

$$\sum_{k=1}^{K} \langle \gamma_k \boldsymbol{u}_k, \boldsymbol{\mu}_k \rangle_{\partial\Omega_k} = 0, \quad \forall \boldsymbol{\mu} = \{\boldsymbol{\mu}_k\}_{k=1}^K \in M^2, \tag{5.1}$$

$$L_k(\boldsymbol{u}_k, \boldsymbol{v}) = (\boldsymbol{f}_k, \boldsymbol{v})_{\Omega_k}, \quad \forall \boldsymbol{v} \in V_0(\Omega_k), \quad k = \overline{1, K}, \tag{5.2}$$

$$\sum_{k=1}^{K} \langle \boldsymbol{\delta}_k \boldsymbol{u}_k, \boldsymbol{\gamma}_k \boldsymbol{v} \rangle_{\partial\Omega_k} = 0, \quad \forall \boldsymbol{v} \in V_0(\Omega), \tag{5.3}$$

where $L_k(\cdot,\cdot) \equiv L_{\Omega_k}(\cdot,\cdot)$ is defined as $L_\Omega(\cdot,\cdot)$ but for domain Ω_k:

$$L_k(\boldsymbol{u}, \boldsymbol{v}) = \int\limits_{\Omega_k} \left(\frac{\partial \boldsymbol{u}}{\partial x_1} \cdot \frac{\partial \boldsymbol{v}}{\partial x_1} + \frac{\partial \boldsymbol{u}}{\partial x_2} \cdot \frac{\partial \boldsymbol{v}}{\partial x_2} \right) d\Omega_k + \int\limits_{\Omega_k} \mathbf{A}\boldsymbol{u} \cdot \boldsymbol{v}\, d\Omega_k,$$

$$\boldsymbol{f}_k = \boldsymbol{f}|_{\Omega_k}, \quad (\boldsymbol{f}_k, \boldsymbol{v})_{\Omega_k} = \int\limits_{\Omega_k} \boldsymbol{f}_k \cdot \boldsymbol{v}\, d\Omega_k,$$

and $\boldsymbol{\delta}_k$ and $\boldsymbol{\gamma}_k$ are defined as the operators $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ for domain Ω_k. It can be shown (see [4]) that formulations (4.1)–(4.2) and (5.1)–(5.3) are equivalent, i.e. they lead to the same solution $\boldsymbol{u}$.

6. Variational Equation for Traces

6.1. General constructions

Let us introduce an operator $\boldsymbol{G} : V_\Gamma \to V$ which maps $\boldsymbol{\varphi} \in V_\Gamma$ to $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\varphi}$, where u is a solution of problem (4.1)–(4.2) with $\boldsymbol{g} = \boldsymbol{\varphi}$ and $\boldsymbol{f} = 0$.

We also define the Poincaré-Steklov (P.-S.) operator ([11]) as $\boldsymbol{P} = \boldsymbol{\delta}\boldsymbol{G}$. This operator is the main tool we use to reduce original problem (4.1), (4.2) to some problem for traces of solution at SE's boundaries. It is known ([11]) that operators $\boldsymbol{P}$ and $\boldsymbol{G}$ are linear and continuous. Let us return to the equations (5.1)–(5.3). We introduce $\boldsymbol{u} = \{\boldsymbol{u}_k\}_{k=1}^K$ as $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{U}$, $\boldsymbol{u}_k = \boldsymbol{w}_k + \boldsymbol{U}_k$, $k = \overline{1, K}$, where $\boldsymbol{w} = \{\boldsymbol{w}_k\}_{k=1}^K$ is such a function that

$$\boldsymbol{L}\boldsymbol{w}_k = \boldsymbol{f}_k, \quad \boldsymbol{w}_k|_{\partial\Omega_k} = \boldsymbol{\omega}|_{\partial\Omega_k}, \tag{6.1}$$

in every SE Ω_k, and $\boldsymbol{\omega} \in V$ is an arbitrary function fulfilling boundary conditions at $\partial\Omega$, $\boldsymbol{\omega}|_{\partial\Omega} = \boldsymbol{g}$. Actually there is no need to define function $\boldsymbol{\omega}$ explicitly inside superelements Ω_k, the only purpose of its definition is to provide correct (matching) BCs at SEs boundaries and at the entire domain boundary. Function $\boldsymbol{w}$ obviously satisfies conditions of Lemma 1.

It is proposed here that each boundary value problem (6.1) has unique solution. The purpose of such representation of $\boldsymbol{u}$ is to convert problem (4.1), (4.2) to the the one with homogeneous boundary conditions on $\partial\Omega$.

Substitution of $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{U}$ into equations (4.1), (4.2) gives the following system of equations to be solved for $\boldsymbol{U}$:

$$\boldsymbol{U} = \{\boldsymbol{U}_k\}_{k=1}^K \in \prod_{k=1}^K V_{00,k},$$

$$L_k(\boldsymbol{U}_k, \boldsymbol{v}) = 0, \qquad \forall \boldsymbol{v} \in V_0(\Omega_k),\ k = \overline{1, K}, \tag{6.2}$$

$$\sum_{k=1}^K \langle \boldsymbol{\gamma}_k \boldsymbol{U}_k, \boldsymbol{\mu}_k \rangle_{\partial\Omega_k} = 0, \qquad \forall \boldsymbol{\mu} = \{\boldsymbol{\mu}_k\}_{k=1}^K \in M^2, \tag{6.3}$$

$$\sum_{k=1}^K \langle \boldsymbol{\delta}_k \boldsymbol{U}_k, \boldsymbol{\gamma}_k \boldsymbol{v} \rangle_{\partial\Omega_k} = -\sum_{k=1}^K \langle \boldsymbol{\delta}_k \boldsymbol{w}_k, \boldsymbol{\gamma}_k \boldsymbol{v} \rangle_{\partial\Omega_k}, \quad \forall \boldsymbol{v} \in V_0(\Omega), \tag{6.4}$$

where

$$V_{00,k} = \begin{cases} V(\Omega_k), & \Gamma_{0,k} = \emptyset, \\ \left\{ \boldsymbol{v} \in V(\Omega_k) : \boldsymbol{v}|_{\Gamma_{0,k}} = \boldsymbol{0} \right\}, & \Gamma_{0,k} \neq \emptyset. \end{cases}$$

Using P.-S. and Green's operators P and G we can rewrite equations (6.2), (6.4) as

$$\boldsymbol{U}_k = \boldsymbol{G}_k \boldsymbol{\varphi}_k, \quad k = \overline{1, K}, \tag{6.5}$$

$$\sum_{k=1}^K \langle \boldsymbol{P}_k \boldsymbol{\varphi}_k, \boldsymbol{\gamma}_k \boldsymbol{v} \rangle_{\partial\Omega_k} = -\sum_{k=1}^K \langle \boldsymbol{\delta}_k \boldsymbol{w}_k, \boldsymbol{\gamma}_k \boldsymbol{v} \rangle_{\partial\Omega_k}, \quad \forall \boldsymbol{v} \in V_0(\Omega). \tag{6.6}$$

Here $\boldsymbol{\varphi} = \{\boldsymbol{\varphi}_k\}_{k=1}^K = \{\boldsymbol{\gamma}_k \boldsymbol{U}_k\}_{k=1}^K$ is set of traces of given function $\boldsymbol{U} = \{\boldsymbol{U}_k\}_{k=1}^K$ on $\partial\Omega_k$ and $\boldsymbol{P}_k$,$\boldsymbol{G}_k$ are operators $\boldsymbol{P}$,$\boldsymbol{G}$ which correspond to subdomain Ω_k.

The last problem can be formally solved in two steps:

1. Using (6.6) one can formally find function $\boldsymbol{\varphi} = \{\boldsymbol{\varphi}_k\}_{k=1}^K$, which represents traces of unknown solution at SE's boundaries $\partial\Omega_k$;
2. To compute solutions $\boldsymbol{U}_k$ according to (6.5) (using $\boldsymbol{\varphi}_k$ found above).

To perform further analysis of the last problem we need some additional theoretical background. Let us introduce spaces

$$\tilde{H} = \prod_{k=1}^K L_2(\partial\Omega_k), \quad \|\mu\|_{\tilde{H}}^2 = \sum_{k=1}^K \|\mu_k\|_{L_2(\partial\Omega_k)}^2,$$

$$\tilde{X} = \prod_{k=1}^K W_2^{1/2}(\partial\Omega_k), \quad \|\mu\|_{\tilde{X}}^2 = \sum_{k=1}^K \|\mu_k\|_{W_2^{1/2}(\partial\Omega_k)}^2,$$

and subspaces

$$X = \left\{ \mu = \{\mu_k\}_{k=1}^K \in \tilde{X} : \quad \exists \nu \in W_2^1(\Omega),\ \mu_k = \nu|_{\partial\Omega_k} \right\},$$

$$X_0 = \left\{ \mu = \{\mu_k\}_{k=1}^K \in \tilde{X} : \quad \exists \nu \in \overset{\circ}{W}{}_2^1(\Omega),\ \mu_k = \nu|_{\partial\Omega_k} \right\},$$

$$H_0 = \left\{ \mu = \{\mu_k\}_{k=1}^K \in \tilde{H} : \quad \mu_k = 0|_{\partial\Omega\cap\partial\Omega_k} \text{ almost everywhere } \right\}.$$

We consider $\tilde{X}' = \prod\limits_{k=1}^{K} W_2^{-1/2}(\partial\Omega_k)$ as a dual space to $\tilde{X}$. The following inclusion is fulfilled

$$\tilde{X} \subset \tilde{H} \subset \tilde{X}'$$

where $\tilde{X}$ is everywhere dense in $\tilde{X}'$. We also define the direct products:

$$X^2 = X \times X, \quad X_0^2 = X_0 \times X_0, \quad \tilde{X}^2 = \tilde{X} \times \tilde{X}$$

and the corresponding dual space $X'^{,2} = X' \times X'$.

Using P.-S. operators $\boldsymbol{P}_k$ one can define bilinear form

$$B(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{k=1}^{K} \langle \boldsymbol{P}_k \boldsymbol{\mu}_k, \boldsymbol{\nu}_k \rangle_{\partial\Omega_k}, \quad \forall \boldsymbol{\mu}, \boldsymbol{\nu} \in \tilde{X}^2.$$

This form is continuous since P.-S. operators are continuous. Now one can rewrite equation (6.6) as

$$\boldsymbol{\varphi} \in X_0^2 : \quad B(\boldsymbol{\varphi}, \boldsymbol{\psi}) = F(\boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in X_0^2, \tag{6.7}$$

where

$$F(\boldsymbol{\psi}) = -\sum_{k=1}^{K} \langle \boldsymbol{\delta}_k \boldsymbol{w}_k, \boldsymbol{\psi}_k \rangle_{\partial\Omega_k}, \quad \forall \boldsymbol{\psi} \in X^2. \tag{6.8}$$

and $\boldsymbol{w} \in V$ is the one from (6.1). Equation (6.3) is fulfilled because of the choice of the space X_0. So we came to the following result:

Theorem 2. *Let* $F \in X'^{,2}$ *is defined as in* (6.8)*. Then the solution* $\boldsymbol{u} = \{\boldsymbol{u}_k\}_{k=1}^K \in V$ *of the problem* (4.1), (4.2) *has a form*

$$\boldsymbol{u}_k = \boldsymbol{G}_k \boldsymbol{\varphi}_k + \boldsymbol{w}_k, \quad k = \overline{1, K},$$

where $\boldsymbol{\varphi}$ *is the solution of problem* (6.7)–(6.8) *with given* F *and* $\boldsymbol{w} = \{\boldsymbol{w}_k\}_{k=1}^K \in V$ *is the one from* (6.1)*.*

It is possible also to rewrite the variational equation for traces presented above in matrix notation. Indeed, operator $\boldsymbol{G}$, as it was defined above, maps two-component function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ to another two-component function, say $\boldsymbol{u}$. Let $(\boldsymbol{\varphi})_i = \varphi_i$ be the i-th component of $\boldsymbol{\varphi}$, $i = 1, 2$. For any such $\boldsymbol{\varphi}$ we can write

$$\boldsymbol{\varphi} = \varphi_1 \boldsymbol{e}_1 + \varphi_2 \boldsymbol{e}_2, \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\boldsymbol{G}$ is linear we can write

$$\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\varphi} = \boldsymbol{G}(\varphi_1 \boldsymbol{e}_1) + \boldsymbol{G}(\varphi_2 \boldsymbol{e}_2). \tag{6.9}$$

Defining G_{ij} as

$$G_{ij}\varphi = \left(\boldsymbol{G}(\varphi \boldsymbol{e}_j)\right)_i, \quad i,j = 1,2$$

it is possible to rewrite equation (6.9) as

$$u_1 = G_{11}\varphi_1 + G_{12}\varphi_2,$$
$$u_2 = G_{21}\varphi_1 + G_{22}\varphi_2.$$

Operators G_{ij} are linear and act not on a two-component function like operator $\boldsymbol{G}$ but on scalar functions and return scalar values. They are linear and continuous since $\boldsymbol{G}$ is linear and continuous.

Exactly the same analysis can be performed for P.-S. operators. We can define operators $P_{ij} = \left(\boldsymbol{\delta G}(\varphi \boldsymbol{e}_j)\right)_i$ and then expand $\boldsymbol{\psi} = \boldsymbol{P}\boldsymbol{\varphi}$ as

$$\psi_1 = P_{11}\varphi_1 + P_{12}\varphi_2,$$
$$\psi_2 = P_{21}\varphi_1 + P_{22}\varphi_2,$$

Now let's turn to the bilinear form B. It can be rewritten as:

$$B(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{i,j=1}^{2} b_{ij}(\mu_j, \nu_i),$$

where $\boldsymbol{\mu} = \mu_1 \boldsymbol{e}_1 + \mu_2 \boldsymbol{e}_2$, $\boldsymbol{\nu} = \nu_1 \boldsymbol{e}_1 + \nu_2 \boldsymbol{e}_2$ and bilinear forms $b_{ij}(\cdot,\cdot)$ are the ones of the form:

$$b_{ij}(\mu, \nu) = B(\mu \boldsymbol{e}_j, \nu \boldsymbol{e}_i),$$

We can write variational equation (6.7) as

$$b_{11}(\mu_1, \nu_1) + b_{12}(\mu_2, \nu_1) = F_1(\nu_1), \tag{6.10}$$
$$b_{21}(\mu_1, \nu_2) + b_{22}(\mu_2, \nu_2) = F_2(\nu_2), \tag{6.11}$$

where $\nu_1, \nu_2 \in X_0$ are arbitrary trial functions and

$$F_i(\nu) = F(\nu \boldsymbol{e}_i), \quad i = 1,2,$$

where F is a linear functional defined in (6.8).

6.2. Variational equation for biharmonic equation

Essential assumptions which were made during the previous considerations are the following:

1. Linearity of the problem under consideration;
2. Availability of Green's formula for operators under consideration;
3. Existence of solutions of all subsidiary problems.

It is useful also to point that availability of Green's formula of the desirable form is proven for a quite general case, i.e. so called "abstract" Green's formula exists, see [13]. So all constructions performed above are valid not only for the particular case of the biharmonic equation but for rather general class of problems.

In that sense it doesn't matter which particular form operator $\boldsymbol{L}$ has. For example it is valid for the case of an arbitrary matrix $\mathbf{A}$ under condition that all intermediate problems have unique solutions. Nevertheless to make further error analysis of the method more simple it is more convenient to utilize properties of the particular problem under consideration, which makes it possible to use some results obtained previously for the case of problems with conventional (scalar) Laplace operator.

Exactly, let us study the structure of operators G_{ij} and P_{ij}. It is easy to show that

$$\boldsymbol{G}(\varphi \boldsymbol{e}_1) = \begin{pmatrix} G_\Delta \varphi \\ 0 \end{pmatrix},$$

where G_Δ is Green's operator for the Laplace equation, it maps some φ defined at $\partial\Omega$ to u defined in Ω and being a solution of the problem:

$$\Delta u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \varphi.$$

See [5, 6, 7] for details. Hence $G_{11} = G_\Delta$ and $G_{21} = 0$.

In the same way it can be shown that

$$\boldsymbol{G}(\varphi \boldsymbol{e}_2) = \begin{pmatrix} G_{21} \varphi \\ G_\Delta \varphi \end{pmatrix},$$

where $u = G_{12}\varphi$ is a solution of the problem

$$-\Delta u + G_\Delta \varphi = 0, \quad u|_{\partial\Omega} = 0$$

with a given φ.

In the same way it can be shown that $P_{11} = P_{22} = P_\Delta$, $P_{12} = \delta_\Delta G_{12}$ and $P_{21} = 0$. Here δ_Δ is a conventional normal derivative operator for the Laplace equation, $\delta_\Delta u = \partial u / \partial \boldsymbol{n}$ for some u, P_Δ is P.-S. operator for the Laplace equation, P_Δ maps some φ defined at $\partial\Omega$ to $P_\Delta \varphi$ defined at $\partial\Omega$ by the rule:

$$P_\Delta \varphi = \frac{\partial}{\partial \boldsymbol{n}} G_\Delta \varphi.$$

See [5, 6, 7] for details. Due to the form of operators P_{ij} equations (6.10), (6.11) can be reduced to:

$$b_\Delta(\mu_1, \nu_1) + b_{12}(\mu_2, \nu_1) = F_1(\nu_1), \tag{6.12}$$

$$b_\Delta(\mu_2, \nu_2) = F_2(\nu_2), \tag{6.13}$$

where $b_\Delta(\cdot, \cdot)$ is a bilinear form which corresponds to the Laplace equation,

$$b_\Delta(\mu, \nu) = \sum_{k=1}^{K} \langle P_{\Delta,k} \mu, \nu \rangle_{\partial\Omega_k}.$$

It can be shown ([5, 6, 7]) that bilinear form $b_\Delta(\cdot, \cdot)$ is continuous and positively defined in X_0. Problem (6.12)–(6.13) can be solved sequentially.

7. FSEM Approximations

FSEM is a mesh-projection method for solution of equation (6.7). Only the Bubnov-Galerkin approximations are considered below, i.e. the case when spaces of basis and trial functions coincide. Nevertheless generalization to the more general the Galerkin-Petrov approach is quite straightforward.

Following the Bubnov-Galerkin method we choose some finite-dimensional subspace of basis and trial functions $X_{0,h}^2 \subset X_0^2$ which is linear span of system of basis functions $\left\{\boldsymbol{\varphi}_h^{(s)}\right\}_{s=1}^N \subset X_0^2$. An approximate solution $\boldsymbol{\varphi}_h \in X_{0,h}^2$ of the problem (6.2) is of the form

$$\boldsymbol{\varphi}_h = \sum_{s=1}^N a_s \boldsymbol{\varphi}_h^{(s)} \in X_{0,h}^2$$

and has to satisfy the following equation:

$$B(\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) = F(\boldsymbol{\psi}_h), \quad \forall \boldsymbol{\psi}_h \in X_{0,h}^2. \tag{7.1}$$

One can obtain different versions of FSEM by choosing $X_{0,h}^2$ in different ways. Computational procedure can be applied formally in the following sequence:

1. Compute functions

$$\boldsymbol{u}_h^{(s)} = \boldsymbol{G}\boldsymbol{\varphi}_h^{(s)} = \left\{\boldsymbol{G}_k \boldsymbol{\varphi}_{h,k}^{(s)}\right\}_{k=1}^K, \quad \boldsymbol{\Pi}_h^{(s)} = \boldsymbol{P}\boldsymbol{\varphi}_h^{(s)} = \left\{\boldsymbol{P}_k \boldsymbol{\varphi}_{h,k}^{(s)}\right\}_{k=1}^K$$

   for each $\boldsymbol{\varphi}_h^{(s)}$. To compute $\boldsymbol{G}\boldsymbol{\varphi}_h^{(s)}$ it is needed to solve some auxiliary problem for equation under consideration in every SE independently with boundary conditions defined by $\boldsymbol{\varphi}_h^{(s)}$. Actually any convenient analytical or numerical approach can be used for this purpose. In the numerical example, presented below, conventional FEM in every SE separately is used. It allows us to use separate FE meshes in different SEs independently.
2. Assemble and solve finite-dimensional problem (7.1) for $\boldsymbol{\varphi}_h$. Approximate solution of the original problem (4.1), (4.2) is given as

$$\boldsymbol{u}_h = \sum_{s=1}^N a_s \boldsymbol{u}_h^{(s)} + \boldsymbol{w}_h.$$

   To define and compute additional function $\boldsymbol{w}_h$ we use conventional FEM in every SE. Boundary conditions for $\boldsymbol{w}_h$ can be defined in the simplest way, for example we can set $\boldsymbol{w}_h = \boldsymbol{g}_h$ at the SE edges which belong to the domain boundary, $\boldsymbol{w}_h = 0$ at the nodes of superelements which are situated strictly inside Ω and then use piecewise interpolation to define $\boldsymbol{w}_h$ at all SE edges which lies inside Ω. Here $\boldsymbol{g}_h$ denotes some approximation of $\boldsymbol{g}$. There is no necessity to define function $\boldsymbol{\omega}$ from (6.1) here, because it is used only for definition of the BCs for functions $\boldsymbol{w}_h$, which could be easily done explicitly while algorithm's implementation.

8. Error Estimates for the Bubnov-Galerkin Method

We consider system (6.12), (6.13) in a sequence, using terms of the previous section. So this section is essentially problem-specific.

Approximation of the equation (6.13) reads

$$b_\Delta\left(\varphi_{2,h},\nu_h\right)=F_2\left(\nu_h\right),\quad \forall\nu_h\in X_{0,h}. \tag{8.1}$$

Since form $b_\Delta(\cdot,\cdot)$ is continuous and positively defined, conditions of the Sea lemma are fulfilled ([1, 13]) and the following estimate can be immediately obtained:

$$\left\|\varphi_{2,h}-\varphi_2\right\|_{X_0}\le C\varepsilon\left(\varphi_2,X_{0,h}\right),$$

where

$$\varepsilon(y,Y_h)=\inf_{z\in Y_h}\|y-z\|$$

denotes the distance from some $y\in Y$ to subspace $Y_h\subset Y$ for some functional space Y and its subspace Y_h.

Approximation of the equation (6.12) reads

$$b_\Delta\left(\varphi_{1,h},\nu_h\right)+b_{12}\left(\varphi_{2,h},\nu_h\right)=F_1\left(\nu_h\right),\quad \forall\nu_h\in X_{0,h}. \tag{8.2}$$

Consider also the following subsidiary problem

$$b_\Delta\left(\tilde\varphi_{1,h},\nu_h\right)+b_{12}\left(\varphi_2,\nu_h\right)=F_1\left(\nu_h\right),\quad \forall\nu_h\in X_{0,h}. \tag{8.3}$$

The difference between (8.2) and (8.3) is that we use the exact value of φ_2 in (8.3) instead of its approximation in (8.2). Using triangle inequality we obtain

$$\left\|\varphi_{1,h}-\varphi_1\right\|_{X_0}\le\left\|\varphi_{1,h}-\tilde\varphi_{1,h}\right\|_{X_0}+\left\|\tilde\varphi_{1,h}-\varphi_1\right\|_{X_0}. \tag{8.4}$$

Equation (8.3) is the one for $\tilde\varphi_{1,h}$ since we assume that φ_2 is already defined, so we can write (8.3) as

$$b_\Delta\left(\tilde\varphi_{1,h},\nu_h\right)=\tilde F_1(\nu_h),\quad \tilde F_1(\nu_h)=F_1\left(\nu_h\right)-b_{12}\left(\varphi_2,\nu_h\right),\quad \forall\nu_h\in X_{0,h},$$

and the second term on the right hand side of (8.4) can be estimated again by using the Sea lemma:

$$\left\|\tilde\varphi_{1,h}-\varphi_1\right\|_{X_0}\le C\varepsilon\left(\varphi_1,X_{0,h}\right).$$

To estimate the first term on the right hand side of (8.4) we subtract (8.2) from (8.3) and substitute $\nu_h=\tilde\varphi_{1,h}-\varphi_{1,h}$ to obtain

$$b_\Delta\left(\tilde\varphi_{1,h}-\varphi_{1,h},\tilde\varphi_{1,h}-\varphi_{1,h}\right)=b_{12}\left(\varphi_{2,h}-\varphi_2,\tilde\varphi_{1,h}-\varphi_{1,h}\right).$$

Since $b_\Delta(\cdot,\cdot)$ is positively defined and $b_{12}(\cdot,\cdot)$ is continuous we obtain:

$$C_1\|\tilde\varphi_{1,h}-\varphi_{1,h}\|^2_{X_0}\le b_\Delta\left(\tilde\varphi_{1,h}-\varphi_{1,h},\tilde\varphi_{1,h}-\varphi_{1,h}\right)=$$
$$b_{12}\left(\varphi_{2,h}-\varphi_2,\tilde\varphi_{1,h}-\varphi_{1,h}\right)\le C_2\|\varphi_{2,h}-\varphi_2\|_{X_0}\|\tilde\varphi_{1,h}-\varphi_{1,h}\|_{X_0}$$

which leads to

$$\|\tilde{\varphi}_{1,h} - \varphi_{1,h}\|_{X_0} \le C\,\|\varphi_2 - \varphi_{2,h}\|_{X_0} \le C\varepsilon\,(\varphi_2, X_{0,h})\,.$$

Hence the following estimations hold for $\varphi_{1,h}$ and $\varphi_{2,h}$:

$$\|\varphi_{h,1} - \varphi_1\|_{X_0} \le C_1\varepsilon\,(\varphi_1, X_{0,h}) + C_2\varepsilon\,(\varphi_2, X_{0,h})\,,$$
$$\|\varphi_{2,h} - \varphi_2\|_{X_0} \le C_3\varepsilon\,(\varphi_2, X_{0,h})\,.$$

The final step is an estimation of $\varepsilon\,(\varphi, X_{0,h})$ for $\varphi = \varphi_1, \varphi_2$. In the simplest way it can be done as

$$\varepsilon\,(\varphi, X_{0,h}) \le \|\varphi - \tilde{\varphi}_h\|_{X_0},$$

where $\tilde{\varphi}_h$ stands for an interpolant of φ by the system of basis functions which forms $X_{0,h}$. So estimations of approximate solution are reduced to the estimations for interpolation procedure.

In the case of two-dimensional domain and one-dimensional boundaries (i.e. the case under consideration) it leads to ([4, 6, 7]):

$$\|\varphi_{1,h} - \varphi_1\|_{X_0} \le C_1 h, \quad \|\varphi_{2,h} - \varphi_2\|_{X_0} \le C_2 h$$

for the case of piecewise linear finite element approximation on boundaries. Here h is a step size of one-dimensional superelements mesh defined at SE boundaries. Constants C_1 and C_2 depend on domain and its decomposition into superelements but not on h.

9. Numerical Examples

As an example we introduce the following problem in a multiply-connected domain

$$\Omega = [0, 10]^2 \setminus (S_1 \cup S_2)\,, \tag{9.1}$$

where S_1 is a disk of radius $r_1 = 10^{-2}$ and center at $\boldsymbol{c}_1 = (0.5, 0.5)$ and S_2 is a disk of the same radius $r_2 = r_1$ and center at $\boldsymbol{c}_2 = (4.5, 4.5)$.

We choose

$$u(x) = \ln|\boldsymbol{x} - \boldsymbol{c}_1| + \ln|\boldsymbol{x} - \boldsymbol{c}_2| + 10\ln(5 + |\boldsymbol{x} - \boldsymbol{c}_3|)\,, \tag{9.2}$$

where

$$|\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2,$$

and $\boldsymbol{c}_3 = (-5, 15)$ as an exact solution and define boundary conditions considering u defined above at domain boundary.

Superelements are squares of size $H = 1$, "wells" S_1 and S_2 are situated in the center of the corresponding superelements. Contour plots of the approximate solution are presented on Fig. 2. A plot of an exact and approximate values of solution at the line $x_1 = x_2$ is presented on Fig. 3.

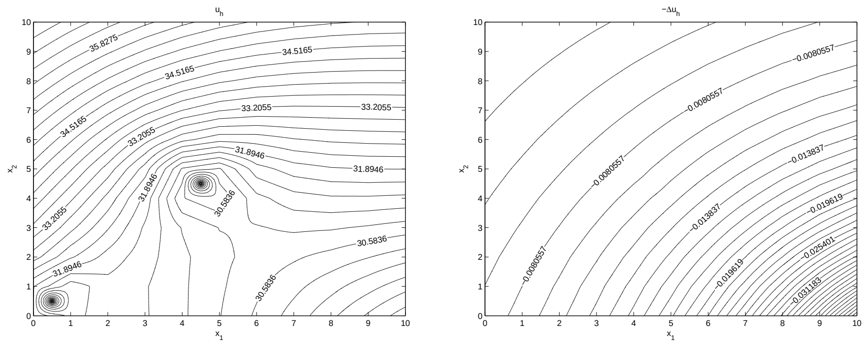

Figure 2. Approximate (u – left hand plot, $v = -\Delta u$ – right hand plot).

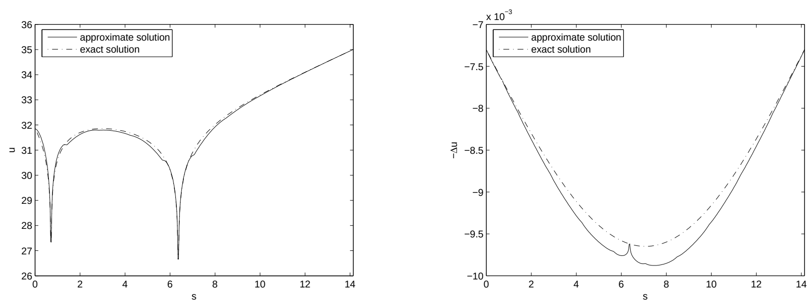

Figure 3. Exact and approximate values of u (left hand plot) and $-\Delta u$ (right hand plot) at the line $x_1 = x_2$, $s = \sqrt{2}x_1$.

Norms in $C(\Omega)$ of the error of the approximate solution for u and v are 0.3499 and 4.2151×10^{-4} correspondingly. So large difference in the values of error for different components of the solution is caused not by the properties of the method but by corresponding difference in characteristic values of the exact solution. Relative errors are of the same order and have values of 0.0111 and 0.0280 correspondingly. We also note that $\boldsymbol{u}$ errors achieve their maximum value in the "wells" neighbourhood.

Let us point here that the values of the relative errors are quite small for such a coarse decomposition (10×10 superelements) of computational domain. The error can be reduced by use of higher-order boundary basis functions and more accurate solution of subsidiary problems for SEs basis functions (see [3]).

10. Conclusion

FSEM for solution of biharmonic equation in bounded domain was analyzed in this paper. Special weak formulation for the traces of the solution of the original problem at superelements boundaries was suggested. Equivalence of formulations was shown. Computational model was constructed using developed formal approach. Abstract error estimates were obtained for one version of the method. The method have shown its efficiency while solving the given test problem. Good agreement between numerical and exact solutions was observed.

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