NUMERICAL SIMULATION OF THE HEAT CONDUCTION IN ELECTRICAL CABLES

R. ČIEGIS[1], A. ILGEVIČIUS[1], H. LIESS[2], M. MEILŪNAS[1], O. SUBOČ[1]

[1] *Vilnius Gediminas Technical University*

Saulėtekio al. 11, LT-10223, Vilnius, Lithuania

E-mail: {rc,mm,os}@fm.vgtu.lt; Audrius.Ilgevicius@varikonta.lt

[2] *University of the Federal Army Forces in Munich*

Werner-Heisenberg Weg 39, Munich, Germany

E-mail: hdliess@gmx.net

Abstract. The modelling of the heat conduction in electrical cables is a complex mathematical problem. To get a quantitative description of the thermo-electrical characteristics in the electrical cables, one requires a mathematical model for it. It must involve the different physical phenomena occurring in the electrical cables, i.e. heat conduction, convection and radiation effects, description of heat sources due to current transitions. Since the space in mobile systems is limited and weight is always reduced, wire conductor sizes must be kept as small as possible. Thus the main aim is to determine optimal conductor cross-sections for long standing loads. In this paper we develop and validate a set of mathematical models and numerical algorithms for the heat transfer simulation in cable bundles. The numerical algorithms are targeted to the two-dimensional transient heat transfer mathematical models. Finally, a validation procedure for the coefficient validation of the differential equations is carried out. Results of numerical experiments are presented.

Key words: heat conduction, finite volume method, optimization, mathematical modelling, electrical cables

1. Introduction

In modern cars electrical and electronic equipment is of great importance. With increasing number of devices using the electrical power, the amount of wires and the wire sizes rises also. Since the space in mobile systems is limited and weight is always reduced, wire conductor sizes must be kept as small as possible. Thus the main aim is to determine optimal conductor cross-sections for long standing loads.

The number and weight of the electrical conductors in cars increased very much during the last decades. Nowadays, an upper class car has already more than 3 km of cables with a mass of more than 40 kg. This is, compared to the whole mass of a car, more than a tolerable increase. One expects even, that this trend will dramatically continue in the future. At present the wires for the cable harness for mobile application are still selected according to design rules and specifications, which were elaborated for stationary use and not necessarily suited for mobile application. For this kind of application completely different rules would be valid. In a passenger car, for instance, the cable length does not exceed 8 m and in most cases it is even considerably shorter. Power and voltage losses due to higher use of its capacities are not very important. On the other hand, the way of placing the various cables or cable bundles in car body structure, sometimes in foam, in special tubes or cable channels and the temperature of the environment requires much more care of the thermal situations. To be on the save side in most cases the wire cross sections are therefore oversized.

The modelling of these processes is a complex mathematical problem. To get a quantitative description of the thermo-electrical characteristics in the electrical cables, one requires a mathematical model for it. It must involve the different physical phenomena occurring in the electrical cables, i.e. heat conduction, convection and radiation effects, description of heat sources due to current transitions, etc. At the next stage the model is discretized and numerical algorithms must be developed. Using the discrete model extensive simulation experiments are done, and the results of the simulations have to be verified by experiments. The final goal is optimization of the commercial products with the main goal to minimize the subsequent weight and costs of electro cables used in car industry.

For this reason, it is very important to develop accurate mathematical models and fast numerical algorithms for heat transfer simulation in cable bundles. Fundamentals of the theory heat distribution in cables are given [19]. For further readings we refer to [14, 15, 26].

Numerical algorithms for solution of parabolic and elliptic problems with discontinuous coefficients have been widely investigated in many papers. The application of standard finite element (FE) method to solution of interface problems is equivalent to arithmetic averaging of discontinuous coefficients, the mixed FEM leads to the harmonic averaging if special quadrature formulae are used (see, e.g. recent works [3, 13]).

Conservative finite-difference schemes for approximation of such problems were derived by Tikhonov and Samarskii [24, 27]. They are robust and use only general assumptions on the position of the interface. Recently new finite difference schemes were proposed, which approximate with the second order of accuracy not only the solution, but also the normal flux through the interface (see [18, 22]).

Very promising seems the new approach when the finite difference method is combined with techniques of the finite-volume. Such discretizations are capable to produce accurate approximations. They use a minimal discretization stencil and preserve a local discrete conservation. Applications of such finite-

volume schemes for solution of computational fluid dynamics (CFD) problems and simulation of multiphase flows in porous media are given in [4, 12, 20]. New finite volume schemes, which has $\mathcal{O}(h^2)$ local truncation error for the normal component of the flux in all grid points (including near-interface points) are derived by Iliev in [16, 17]. They can be applied in situations, when the interfaces are aligned with the finite volume surfaces or when arbitrary located interfaces are orthogonal to a coordinate axis. It is important to note that grid stencils for these schemes are minimal.

There are specialized commercial software tools, such as CableCad, AN-SYS, which can be used for solution of heat conduction problems. For example, ANSYS software is a coupled physics tool combining structural, thermal, CFD, acoustic and electromagnetic simulation capabilities in a single engineering software solution [25].

Our goal is to optimize the final product, thus we need to have a set of differential and discrete models, simulating the heat conduction processes with the different level of accuracy. The essential step of the whole solution procedure is a systematic reduction of the full (and therefore, very complex and computationally expensive models) to simplified models, which can be used efficiently in solution of optimization step. During this step the direct problem is solved for very large number of different sets of parameters.

The paper is organized as follows. Section 2 gives a brief description of the model, describing the main processes of heat conduction in electrical cables. A discrete scheme is proposed in Section 3. It is based on the finite volume method and approximates the differential problem on the non-rectangular region with a smooth boundary. A special attention is given to the approximation of the third-type boundary conditions. The stability and convergence of the difference scheme is presented in Section 4. The fitting of the proposed mathematical model to the experimental data is done in Section 5. The heat conduction coefficient in the PVC is used as a free parameter and a formula, based on the geometrical averaging is proposed for its estimation. Some final conclusions are given in Section 6.

2. Problem Formulation

The main aim of the research is to develop and validate a set of mathematical models and numerical algorithms for the heat transfer simulation in cable bundles. The numerical algorithms should be targeted to the two-dimensional transient heat transfer mathematical models. Because of non-linear temperature dependence of the material coefficients, fast iterative methods for solving non-linear system of equations must be used. Finally, a validation procedure for the coefficient validation of the differential equations must be carried out, this fitting procedure is based on experimental data.

Figure 1 shows the cross sectional view of a cable bundle and mechanisms of heat transport. In domain $D \times (0, t_F]$, where

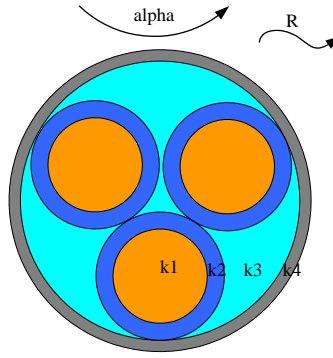$$D = \{X = (x_1, x_2): \quad x_1^2 + x_2^2 \le R^2\},$$

Figure 1. Cross sectional view of a cable:

we solve the nonlinear non-stationary problem, which describes a distribution of the temperature $T(X, t)$ in electrical cable. All wires which makes up the cable bundle need to fulfill electrical and thermal requirements. Thus, temperature of a wire isolation, which is usually PVC, must not exceed a maximal permissible value. The temperature rise is essentially caused by ohmic heating of current-carrying parts. The main mechanisms of heat transfer are:

1. Conduction in solid bodies (conductors, PVC isolation);
2. Conduction in the air and PVC mixture;
3. Convection and radiation from the outer side of the bundle isolation to the environment.

Because the heat transfer mechanism in the air between the wires of a bundle is complicated and non-relevant, the model can be simplified. This non-relevance can be explained by the fact that in a close vicinity to a wire, the main heat transfer mechanism is heat conduction, while in a distance, the dominating mechanism is the motion of molecules. Since all the wires in a bundle are tightly pressed together, only heat conduction is relevant. This simplification helps to increase the efficiency of the numerical algorithm.

The mathematical model consists of the parabolic differential equation [19]:

$$\rho(X)c(X,T)\frac{\partial T}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k(X)\frac{\partial T}{\partial x_i}\right) + f(X,T), \quad (X,t) \in D \times (0, t_F], \quad (2.1)$$

subject to the initial condition

$$T(X, 0) = T_a, \quad X \in \bar{D} = D \cup \partial D, \qquad (2.2)$$

and the nonlinear boundary conditions of the third kind

$$k(X,T)\frac{\partial T}{\partial \eta} + \alpha_k(T)\big(T(X,t) - T_a\big) + \varepsilon\sigma\big(T^4 - T_a^4\big) = 0, \quad X \in \partial D. \quad (2.3)$$

The following continuity conditions are specified at the points of discontinuity of coefficients

$$[T(x,t)] = 0, \quad \left[k \frac{\partial T}{\partial x_i} \right] = 0.$$

Here $c(X, T)$ is the specific heat capacity:

$$c = \begin{cases} 381 + 0.17\,T, & 0 \leq T \leq 200^0\,C, \text{ for copper,} \\ 920 - 1.3\,T + 0.074\,T^2, & 0 \leq T \leq 100^0\,C, \text{ for PVC,} \end{cases}$$

$\rho(X)$ is the density, $k(X)$ is the heat conductivity coefficient:

$$\rho = \begin{cases} 8960, & \text{for copper,} \\ 1350, & \text{for PVC,} \end{cases} \qquad k = \begin{cases} 401, & \text{for copper,} \\ 0.17, & \text{for PVC.} \end{cases}$$

The density of the energy source $f(X, T)$ is defined as

$$f = \left(\frac{I}{A} \right)^2 \rho_0 \big(1 + \alpha_\rho (T - 20) \big),$$

here I is the density of the current, A is a area of the cross-section of the cable, ρ_0 is the specific resistivity of the conductor, T_a is the temperature of the environment.

By using this mathematical model, we can investigate the dynamics of the non-stationary solution. It can be noted that in many cases only a stationary solution is required.

3. Finite Volume Algorithm

Robustness of numerical algorithms for approximation of the heat conduction equation with discontinuous diffusion coefficients is very important for development of methods to be used in simulation of various properties of electrical cables. The differential problem is approximated by the discrete problem by using the finite volume method which is applied on the vertex centered grids. In [17] the discretization is done on cell centered grids. The vertex centered grids are very convenient when the boundary conditions are of the first or third kind. Finite difference schemes for linear elliptic boundary value problems of the third kind are derived in [11]. They have investigated a supraconvergence of such schemes in fractional order Sobolev spaces.

The main challenges of our paper arise due to the fact that the simulation grid is general and it not aligned with the interfaces where the diffusion coefficient is discontinuous and not coincides with the boundary of the computational domain.

3.1. Discretization of the domain

In this section, we introduce a general grid $\bar{D}_h$. First, we define an auxiliary grid $\widetilde{D}_h = \Omega_h \cap \bar{D}$, which is defined as intersection of the equidistant rectangular grid Ω_h with the computational domain $\bar{D}$:

$$\Omega_h = \{X_{ij} = (x_{1i}, x_{2j}): \quad x_{1i} = L_1 + ih_1, \quad i = 0, \ldots, I, \quad x_{1I} = R_1,$$
$$x_{2j} = L_2 + jh_2, \quad j = 0, \ldots, J, \quad x_{2J} = R_2\}.$$

For each node $X_{ij} \in \widetilde{D}_h$ X_{ij} we define a set of neighbours

$$N(X_{ij}) = \{X_{kl}: \quad X_{i\pm1,j} \in \widetilde{D}_h, \quad X_{i,j\pm1} \in \widetilde{D}_h\}.$$

If some neighbour point do not exist, i.e. $X_{i\pm1,j} \notin \widetilde{D}_h$ or $X_{i,j\pm1} \notin \widetilde{D}_h$, then such a neighbour is denoted by NULL.

The computational grid $\bar{D}_h = D_h \cup \partial D_h$ is obtained after deletion from $\widetilde{D}_h$ those nodes X_{ij}, for which both neighbours in some direction do not belong to $\widetilde{D}_h$, i.e. $X_{i\pm1,j} \notin \widetilde{D}_h$ or $X_{i,j\pm1} \notin \widetilde{D}_h$ (see Figure 2). The set of neighbours $N(X_{ij})$ is also modified in a similar way.
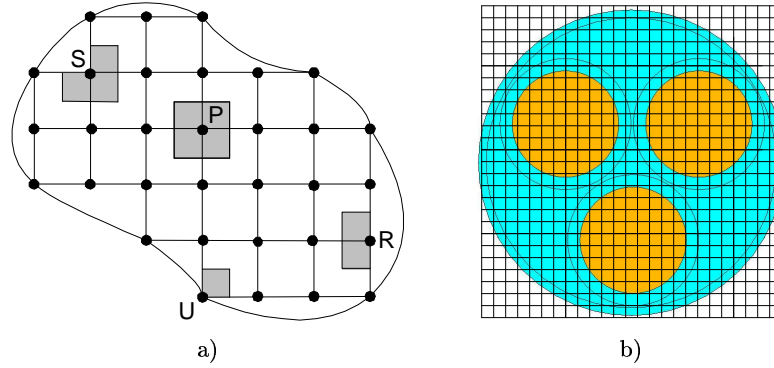

a)   b)

Figure 2. Discretization: a) discrete grid D_h and examples of control volumes, b) basic grid Ω_h and the obtained discretization of the computational domain

For each $X_{ij} \in \bar{D}_h$ a *control volume* is defined

$$e_{ij} = \sum_{k=0}^{3} e_k(X_{ij})\delta_{ijk},$$

$$e_0(X_{ij}) = \{(x_1, x_2): \quad x_{1,i-1/2} \le x_1 \le x_{1,i}, \quad x_{2,j} \le x_2 \le x_{2,j+1/2}\},$$

$$e_1(X_{ij}) = \{(x_1, x_2): \quad x_{1,i} \le x_1 \le x_{1,i+1/2}, \quad x_{2,j} \le x_2 \le x_{2,j+1/2}\},$$

$$e_2(X_{ij}) = \{(x_1, x_2): \quad x_{1,i} \le x_1 \le x_{1,i+1/2}, \quad x_{2,j-1/2} \le x_2 \le x_{2,j}\},$$

$$e_3(X_{ij}) = \{(x_1, x_2): \quad x_{1,i-1/2} \le x_1 \le x_{1,i}, \quad x_{2,j-1/2} \le x_2 \le x_{2,j}\},$$

$$\delta_{ijk} = \begin{cases} 1, & \text{if } e_k(X_{ij}) \in \bar{D}, \\ 0, & \text{if } e_k(X_{ij}) \notin \bar{D}. \end{cases}$$

For example, condition $e_1 \in \bar{D}$ is satisfied, if all three vertexes $X_{i+1,j}$, $X_{i,j+1}$, $X_{i+1,j+1}$ belong to $\bar{D}_h$. Let us denote the measure (or area) of e_{ij} by

$$S_{ij} := \frac{1}{4} h_1 h_2 \sum_{k=0}^{3} \delta_{ijk}.$$

3.2. Finite volume scheme

In $\bar{D}_h$ we define discrete functions

$$U_{ij}^n = U(x_{1i}, x_{2j}, t^n), \quad X_{ij} \in \bar{D}_h,$$

here $t^n = n\tau$ and τ is the discrete time step.

Integrating the differential equation over the control volume e_{ij} and approximating the obtained integrals with an individual quadrature for each term, the differential problem is discretized by the conservative scheme

$$S_{ij}\rho_{ij}c_{ij}(U_{ij}^n)\frac{U_{ij}^n - U_{ij}^{n-1}}{\tau} = \sum_{k=0}^{3} \delta_{ijk}J_{ijk}(U_{ij}^n)U_{ij}^{n\prime} + S_{ij}f_{ij}(U_{ij}^n), \quad X_{ij} \in \bar{D}_h, \tag{3.1}$$

where $J_{ijk}^n(U_{ij}^n)U_{ij}^n$ is the heat flux through a surface of the control volume $e_{ijk} = e_k(X_{ij})$ defined as

$$J_{ij0}(V_{ij})U_{ij} = \frac{h_2}{2}\left(-k_{i-1/2,j}\frac{U_{ij} - U_{i-1,j}}{h_1} + (1 - \delta_{ij1})\alpha_G(V_{ij})(U_{ij} - T_a)\right) \tag{3.2}$$

$$+ \frac{h_1}{2}\left(k_{i,j+1/2}\frac{U_{i,j+1} - U_{ij}}{h_2} + (1 - \delta_{ij3})\alpha_G(V_{ij})(U_{ij} - T_a)\right),$$

$$J_{ij1}(V_{ij})U_{ij} = \frac{h_2}{2}\left(k_{i+1/2,j}\frac{U_{i+1,j} - U_{ij}}{h_1} + (1 - \delta_{ij0})\alpha_G(V_{ij})(U_{ij} - T_a)\right)$$

$$+ \frac{h_1}{2}\left(k_{i,j+1/2}\frac{U_{i,j+1} - U_{ij}}{h_2} + (1 - \delta_{ij2})\alpha_G(V_{ij})(U_{ij} - T_a)\right),$$

$$J_{ij2}(V_{ij})U_{ij} = \frac{h_2}{2}\left(k_{i+1/2,j}\frac{U_{i+1,j} - U_{ij}}{h_1} + (1 - \delta_{ij3})\alpha_G(V_{ij})(U_{ij} - T_a)\right)$$

$$+ \frac{h_1}{2}\left(-k_{i,j-1/2}\frac{U_{ij} - U_{i,j-1}}{h_2} + (1 - \delta_{ij1})\alpha_G(V_{ij})(U_{ij} - T_a)\right),$$

$$J_{ij3}(V_{ij})U_{ij} = \frac{h_2}{2}\left(-k_{i-1/2,j}\frac{U_{ij} - U_{i-1,j}}{h_1} + (1 - \delta_{ij2})\alpha_G(V_{ij})(U_{ij} - T_a)\right)$$

$$+ \frac{h_1}{2}\left(-k_{i,j-1/2}\frac{U_{ij} - U_{i,j-1}}{h_2} + (1 - \delta_{ij0}).\alpha_G(V_{ij})(U_{ij} - T_a)\right),$$

here $\alpha_G(V)$ is the nonlinear generalized coefficient of convection and radiation:

$$\alpha_G(V) = \alpha_k(V) + \varepsilon\sigma(V^3 + V^2 T_a + V T_a^2 + T_a^3).$$

The diffusion coefficient is approximated by using the harmonic averaging formula

$$k_{i\pm1/2,j} = \frac{2k(X_{i\pm1,j})k(X_{ij})}{k(X_{i\pm1,j}) + k(X_{ij})}, \quad k_{i,j\pm1/2} = \frac{2k(X_{i,j\pm1})k(X_{ij})}{k(X_{i,j\pm1}) + k(X_{ij})}.$$

3.3. Predictor-corrector scheme

The derived finite difference scheme (3.1) defines a system of nonlinear equations. By using the predictor–corrector method we approximate it by the linear finite-difference scheme of the same order of accuracy

- **Predictor** ($\forall X_{ij} \in \bar{D}_h$):

$$S_{ij}\rho_{ij}c_{ij}(U_{ij}^{n-1})\frac{\widetilde{U}_{ij}^n - U_{ij}^{n-1}}{\tau} = \sum_{k=0}^{3}\delta_{ijk}J_{ijk}(U_{ij}^{n-1})\widetilde{U}_{ij}^n + S_{ij}f_{ij}(U_{ij}^{n-1}), \quad (3.3)$$

- **Corrector** ($\forall X_{ij} \in \bar{D}_h$):

$$S_{ij}\rho_{ij}c_{ij}(\widetilde{U}_{ij}^n)\frac{U_{ij}^n - U_{ij}^{n-1}}{\tau} = \sum_{k=0}^{3}\delta_{ijk}J_{ijk}(\widetilde{U}_{ij}^n)U_{ij}^n + S_{ij}f_{ij}(\widetilde{U}_{ij}^n), \quad (3.4)$$

The developed algorithm is presented in Figure 3.

```
Temperature (Vector U_New, Vector U_Old, Vector R, Matrix A)
begin
    (1)   ReadData();
    (2)   GenerateGrid();
    (3)   AllocateMemory();
    (4)   t=0;
    (5)   InitSolution(U_New);
    (3)   while  ( t < T ) do
    (4)           t = t + τ;
    (5)           U_Old = U_New;
                  // Predictor step (3.3)
    (6)           MakeSystem(U_Old, U_New, A, R);
                  // Solve system
    (7)           A U_New = R ;
                  // Corrector step (3.4)
    (8)           MakeSystem(U_Old, U_New, A, R);
                  // Solve system
    (9)           A U_New = R ;
          end do
end Temperature
```

Figure 3. Algorithm for the computation of temperature distribution in electric cables

3.4. Solution of systems of linear equations

At steps (7) and (9) of the algorithm we solve systems of linear equations

$$\mathcal{A}U = F,$$

where $\mathcal{A}$ is a sparse non-symmetric matrix of size $N \times N$ with at most five non-zero coefficients. Here N is equal to the number of points in the grid $\bar{D}_h$.

The monotonicity of the discrete solution is very important property of the discrete algorithm. It can be observed that most spatial finite difference approximations, used in computational experiments, yield M-matrices, this guarantees the monotonicity of the solution (see, [21, 23] and references given in these papers).

It is easy to check, that the matrix A, arising after the linearization of the proposed finite volume scheme, satisfies the maximum principle [24]

$$a_{ij} \leq 0, \quad i \neq j, \quad a_{ii} + \sum_{j=1}^{N} a_{ij} > 0, \ \ 1 \leq i, j \leq N.$$

In numerical experiments we use the BiCGStab iterative method with the Gauss-Seidel type preconditioner [8]. It is well known that estimating the rate of convergence of iterations for a system with nonsymmetric matrix is a difficult task. There is no general theory how this rate depends on the spectrum of the matrix even for a well-clustered spectrum (away from zero). Since the non-symmetry of the matrix arises only due to the third-type boundary conditions, we can expect to obtain the fast convergence rate of the BiCGStab iterations.

4. Convergence Analysis

A general template for the convergence analysis of predictor-corrector type finite–difference schemes is developed in [6, 7], see also references given in these papers for more approaches. It was used to investigate nonlinear parabolic problems under assumption that coefficients of the problem are continuous functions.

Finite volume discretizations on 2D grids for linear elliptic problems with discontinuous diffusion coefficients are derived and investigated in many papers, see [1, 2, 10, 21]. The new interesting finite volume approximations are proposed in [17]. Not only the solution but also the fluxes are approximated with the second order of accuracy in the uniform norm. The finite volume schemes are derived under assumption that interfaces are aligned with the finite volume surfaces or when arbitrary located interfaces are orthogonal to a coordinate axis.

4.1. Analysis of a simplified 1D problem

In this section we consider a simplified elliptic problem with a discontinuous diffusion coefficient. Our main concern is to investigate the impact of the error introduced by the perturbed boundary of the computational domain.

We consider the boundary value problem

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = f(x), \quad 0 < x < 1, \tag{4.1}$$

$$-k_1 \frac{du}{dx}(0) + \alpha(u(0) - T) = 0, \quad x = 0,$$

$$k_1 \frac{du}{dx}(1) + \alpha(u(1) - T) = 0, \quad x = 1,$$

where

$$k(x) = \begin{cases} k_1, & 0 \le x < a, \\ k_2, & a \le x \le b, \\ k_1, & b < x \le 1, \end{cases} \qquad f(x) = \begin{cases} 0, & 0 \le x < a, \\ g(x) > 0, & a \le x \le b, \\ 0, & b < x \le 1. \end{cases}$$

We introduce a standard uniform grid

$$\omega_h = \{x_i : \ x_0 = \xi_0, \ x_i = x_0 + ih, \ i = 1, \dots, N, \ x_N = 1 - \xi_1\},$$

where $0 \le \xi_j \le h/2$, $j = 0, 1$, i.e. the grid boundary is perturbed and approximates the boundary of the domain with the accuracy $\mathcal{O}(h)$.

The finite volume method is used to approximate the differential problem. First the balance equation is written over the finite volume $(x_{i-1/2}, x_{i+1/2})$:

$$\frac{W_{i+1/2} - W_{i-1/2}}{h} = \varphi_i, \quad \varphi_i = \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)\, dx \approx f_i, \quad i = 1, \dots, N-1,$$

where $W(x) = -k(x)\dfrac{du}{dx}$ denotes the flux. Next, the flux is written in the form $\dfrac{du}{dx} = -\dfrac{W(x)}{k(x)}$ and integrated over the interval (x_i, x_{i+1}):

$$-(u_{i+1} - u_i) = \int_{x_i}^{x_{i+1}} \frac{W(x)}{k(x)}\, dx \approx W_{i+1/2} \int_{x_i}^{x_{i+1}} \frac{dx}{k(x)}.$$

We use a simple harmonic averaging of coefficient $k(x)$ over the volume (x_i, x_{i+1})

$$\left(\int_{x_i}^{x_{i+1}} \frac{dx}{k(x)} \right)^{-1} \approx a_{i+1/2} = 2\left(\frac{1}{k_i} + \frac{1}{k_{i+1}} \right)^{-1}.$$

This formula is independent on the position of the discontinuity of $k(x)$. In 1D case the exact averaging formula can be derived for a piecewise constant diffusion coefficient:

$$a_{i+1/2} = \left(\frac{\theta}{k_i} + \frac{1-\theta}{k_{i+1}} \right)^{-1},$$

here the interface point $\xi = x_i + \theta h$, $0 \le \theta \le 1$. We note that such an approximation can be generalized to 2D case only when the interfaces of the discontinuity of $k(x)$ are aligned with the grid.

Thus we get a system of discrete equations for interior grid points

$$-\left(a_{i-1/2} U_{\bar{x}} \right)_x = f_i, \quad i = 1, 2, \dots, N-1,$$

here we use notation

$$U_{\bar{x},i} = \frac{U_i - U_{i-1}}{h}, \quad U_{x,i} = \frac{U_{i+1} - U_i}{h}.$$

The boundary conditions are approximated by writing the balance equations over control volumes $(x_0, x_{1/2})$ and $(x_{N-1/2}, x_N)$ and using the boundary conditions of the differential problem:

$$-a_{1/2}U_{x,0} + \alpha(U_0 - T) = \frac{h}{2}f_0, \quad a_{N-1/2}U_{\bar{x},N} + \alpha(U_N - T) = \frac{h}{2}f_N.$$

Denote by $Z_i = U_i - u(x_i)$ the grid function for the global discrete errors. It satisfies the discrete problem

$$-\left(a_{i-1/2}Z_{\bar{x}}\right)_x = -\frac{\eta_{i+1/2} - \eta_{i-1/2}}{h} + \psi_i, \quad i = 1, 2, \ldots, N-1,$$

$$-a_{1/2}Z_{x,0} + \alpha Z_0 = -\eta_{1/2} + \widetilde{\psi}_0, \quad a_{N-1/2}Z_{\bar{x},N} + \alpha Z_N = \eta_{N-1/2} + \widetilde{\psi}_N,$$

where the local approximation errors are defined as

$$\eta_{i+1/2} = a_{i+1/2}u_{x,i} - W_{i+1/2} = \mathcal{O}(h), \quad \psi_i = \mathcal{O}(h),$$

$$\widetilde{\psi}_j = \psi_j + r_j, \quad \psi_j = \mathcal{O}(h^2), \quad j = 0, N.$$

Terms r_j estimate the error due to the perturbed position of the discrete boundary. For solutions sufficiently smooth in the neighbourhood of $x = 0$ and $x = 1$ we have that $r_j = \mathcal{O}(h)$.

Let us define the following scalar products of grid functions:

$$(U, V) = \sum_{i=1}^{N-1} h U_i V_i, \quad (U, V] = \sum_{i=1}^{N} h U_i V_i.$$

We compute a scalar product of the error equation at interior points with Z_i, use the boundary conditions to obtain the equality

$$(aZ_{\bar{x}}, Z_{\bar{x}}] + \alpha(Z_0^2 + Z_N^2) = (\eta, Z_{\bar{x}}] + (\psi, Z) + \widetilde{\psi}_0 Z_0 + \widetilde{\psi}_N Z_N.$$

Then applying a simple embedding theorem (see [24]) the estimate of the global error function in the uniform norm is obtained

$$\max_{0 \le i \le N} |Z_i| \le C(\|\eta\| + \|\psi\| + |\widetilde{\psi}_0| + |\widetilde{\psi}_N|) \le Ch.$$

Thus the discrete solution converges to the solution of the differential problem with the accuracy $\mathcal{O}(h)$.

4.2. Computational results

We have applied the numerical algorithm to simulate the heat conduction in industrial electrical cables, when the bundle is composed of 27 wires. The

radiuses of these wires varied between 0.34 and 1.421 *mm* and the current from 0.4 till 23 A was applied on 12 wires. We have carried out computations till the stationary state was reached. The number of grid points $N \times N$ was increased from $N = 100$ till $N = 400$. We have investigated the distribution of the maximal and minimal temperatures of the electrical cables. The results are given in Table 1, here T_{min} and T_{max} denote the minimal and maximal temperatures reached in some wire of the bundle, respectively, and CPU is a computational time.

Table 1. Computational results for different values of N

N	h_t (s)	T_{min}	T_{max}	CPU (s)
100	30	83.43	88.98	11.6
200	30	83.53	89.09	125
300	20	82.94	88.47	456
400	20	82.60	88.15	1892

The results are consistent with the theoretical complexity estimates of the algorithm (i.e. $\mathcal{O}(N^3)$). The second conclusion is that the algorithm is robust and the discrete model gives accurate results starting from quite rough grids.

5. Identification of the Model

Mixed heat conductivity of air and PVC k can be computed by the following simple relationship between the heat conductivity of air k_A , of PVC k_{Isol} and the filling factor of the bundle F:

$$k = k_A(1 - F) + k_{Isol}F, \tag{5.1}$$

where the filling factor F of the bundle is computed from the relationship between the total area of the wires $d = \sum_{i=1}^{n} d_i$ and the inside area of a bundle D

$$F = \sum_{i=1}^{n} d_i/(D - S),$$

here S is the area of the outer isolation of the bundle.

Such a formula for the filling factor F is heuristical. The improvement of this formula can be done in two directions: first, the mechanism of the heat transport in air includes not only the diffusion but also the convection and radiation, thus some generalized conductivity coefficient $k_A^g > k_A$ should be used; second, the filling coefficient averages the diffusion coefficients of PVC and air with the weight proportional to the areas of these phases.

In numerical simulations two different heat conductivity coefficients of the media between the wires were used. In the first case , assuming that the gaps

between the wires are considerably small, the pure PVC isolation of the wire had been taken into account. In the second case, filling factor F and mixed heat conductivity k of air and PVC had been determined and put into the simulation program.
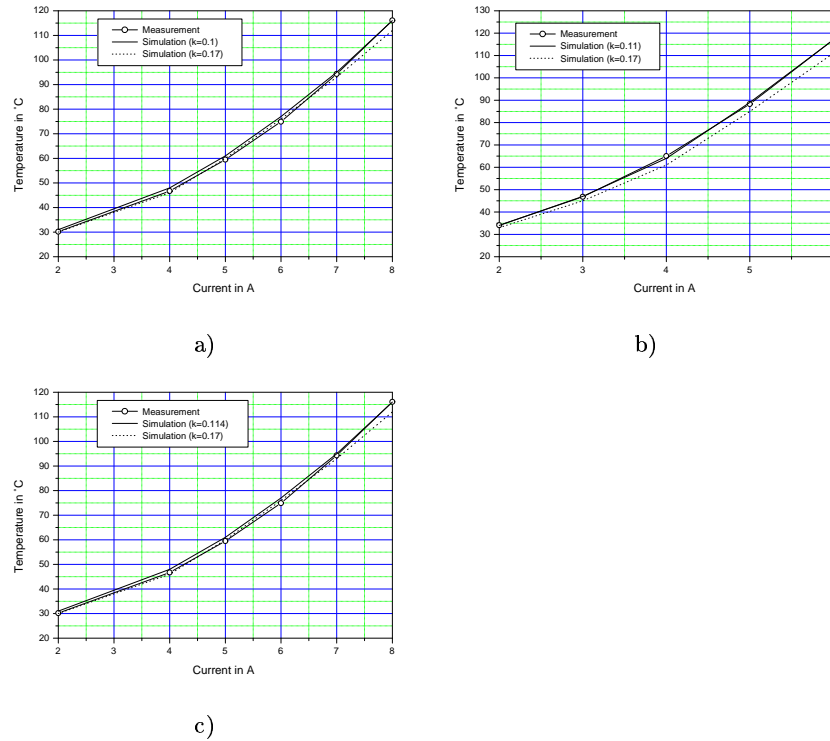

a)

b)

c)

Figure 4. Experimental and computer simulation results for electro-cables with different numbers of wires n: a) $n = 10$, bundle filling factor $F = 0.58$, heat conductivity of the material media between the wires in case F is not considered $k = 0.17$ and $k = 0.1$, when F is counted, b) $n = 20$, bundle filling factor $F = 0.6$, heat conductivity of the material media between the wires in case F is not considered $k = 0.17$ and $k = 0.11$, when F is counted, c) $n = 40$, bundle filling factor $F = 0.7$, heat conductivity of the material media between the wires in case F is not considered $k = 0.17$ and $k = 0.14$, when F is counted,

From Fig. 4 a) and b) parts, one can see that after the consideration of the filling factor F and the mixed conductivity coefficient k the simulation curve was pulled up towards the measurement curve. In Fig. 4 c), where the bundle with 40 wires was simulated, a smaller influence of the different heat conductivity coefficients is observed. This is due to the higher filling factor F of the bundle, which means that the influence of air in the bundle is less compared to the first two cases.

6. Conclusions

The paper discusses a robust finite volume approximation of the mathematical model describing the heat conduction in electrical cables. It defines a 2D nonlinear parabolic problem. The discretization gives discrete systems of nonlinear equations, the matrix of a linearized problem satisfies the maximum principle. It is proved for a simplified 1D problem that such a discretization introduces $\mathcal{O}(h)$ order errors in approximation of discontinuous diffusion coefficients and due to perturbed position of the boundary.

The algorithm can be parallelized efficiently by using the domain (or data) decomposition method and applying techniques developed in [5] for parallelization of a solver for simulation of flows in porous media. Another possibility is to use a classical domain decomposition discretization (see [9] and references given in this paper).

The simulation results give very good agreement with the measured ones if the adjusted heat conductivity coefficient k for PVC is considered. For large numbers of wires in the bundle (more than 40), one can avoid the consideration of the factor F (since $F \approx 1$) and use the coefficient k of PVC material. On the other hand, if the size of wires within the bundle have large deviations from each other, the quality of filling the bundle by the wires can be worse. In such a situation, the factor F and the mixed heat conductivity of air and PVC must be used.

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