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The Reproducing Kernel Method for Some Variational Problems Depending on Indefinite Integrals

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Abstract. In this paper we introduce the reproducing kernel method to solve a class of variational problems (VPs) depending on indefinite integrals. We discuss an analysis of convergence and error for the proposed method. Some test examples are presented to demonstrate the validity and applicability of method. The results of numerical examples indicate that the proposed method is computationally very simple and attractive.

Keywords: variational problems, reproducing kernel, Hilbert space, convergence analysis, error analysis.

AMS Subject Classification: 49J20; 34K28.

1 Introduction

In the large number of problems arising in analysis, mechanics, geometry, etc., it is necessary to determine the maximum and minimum of a certain functional. Problems in which it is required to investigate a function for a maximum or minimum are called variational problems [26]. Finding the brachistochrone, or path of quickest descent, is a historically interesting problem that is discussed in all textbooks dealing with the calculus of variations.The solution of the brachistochrone problem is often cited as the origin of the calculus of variations as suggested in [26]. In the strict sense of the word, isoperimetric problems are problems in which one has to find a geometric figure of maximum area for a given perimeter. These extremum problems, which were even studied in ancient Greece, are also variational problems like, and their aim is to find a closed curve, without self-intersection, of a given length bounding a maximum area. Though the solution of this problem was known in ancient Greece, its peculiar variational nature was understood only at the end of the seventeenth century [23]. The brachistochrone, geodesics and isoperimetric problems have played an important role in the development of calculus of variations [12, 13].

The theory of reproducing kernels [2] was used for the first time at the beginning of the 20th century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions. This theory has been successfully applied to fractal interpolation [6], solving ordinary differential equations [3, 4, 8, 14, 15, 18, 19, 20, 22, 30, 31] and partial differential equations [7, 24]. The books [5, 9, 11] provide an excellent overview of the existing reproducing kernel methods for solving various model problems such as integral and integro-differential equations. In this study, a general technique is proposed for solving some systems in the reproducing kernel space. The main idea is to construct the reproducing kernel space satisfying the conditions for determining solution of the system. The analytical solution is represented in the form of series through the function value at the right side of the equation. The advantages of the approach lie in the following facts. The approximate solution $x_n(t)$ converges uniformly to the analytical solution $x(t)$. The method is mesh-free, easily implemented and capable in treating various boundary conditions. Also we can evaluate the approximate solution $x_n(t)$ for fixed n once and use it over and over. The rest of paper is organized as follows. In section 2, we give some necessary optimality conditions for a class of VPs depending on indefinite integrals. In section 3, we characterize different reproducing kernel Hilbert spaces and derive some theorems. In Section 4 we establish conditions under which exact solution of the Euler-Lagrange, a nonlinear differential equation, exists. We also construct and develop an algorithm for solving nonlinear differential equation in this section. The proposed methods are applied to several examples in Section 5. We conclude the paper in Section 6.

2 A necessary condition for an extremum

The Euler-Lagrange equation forms the centerpiece of the necessary condition for a functional to have an extremum. Now, we give the necessary optimality conditions for a class of variational problems depending on indefinite integrals which are used in the proceeding sections.

Consider functional of

$$\mathrm{J}[x] = \int_0^1 F(t, x(t), x^{'}(t), z(t))dt, \quad z(t) = \int_0^t l(s, x(s), x^{'}(s))ds, \tag{2.1}$$

defined on the set of continuous curves $x : [0, 1] \to \Re$, where F has continuous derivatives with respect to the second, third and fourth variables and l has continuous derivatives with respect to the second and third variables. Among all functions $x(t)$ which have continuous derivatives and satisfy the boundary conditions $x^{(k)}(0) = 0$ $(k = 0 \ or \ k = 1)$ and $x^{(l)}(1) = 0$ $(l = 0 \ or \ l = 1)$, we find the function $x(t)$ for which the functional (2.1) has an extremum.

Let us denote this problem by P. A necessary condition for problem P is given by the next theorem.

Theorem 1. *([1]) A necessary condition for $J[x]$ to have a extremum for a given function $x(t)$ is that $x(t)$ satisfies the generalized Euler-Lagrange equation*

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial\, x^{'}} + \int_t^1 \frac{\partial F}{\partial z} ds \cdot \frac{\partial l}{\partial x} - \frac{d}{dt}(\int_t^1 \frac{\partial F}{\partial z} ds \cdot \frac{\partial l}{\partial x^{'}}) = 0. \tag{2.2}$$

Remark 1. If $x(t)$ extremizes

$$\mathrm{J}[x] = \int_0^1 F(t, x(t), x^{'}(t))dt,$$

under the boundary conditions $x^{(k)}(0) = 0$ ($k = 0$ *or* $k = 1$) and $x^{(l)}(1) = 0$ ($l = 0$ *or* $l = 1$), then $x(t)$ satisfies the following Euler-Lagrange equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial\, x^{'}} = 0.$$

3 Reproducing kernel space

In order to solve VPs in reproducing kernel space, we introduce several reproducing kernel spaces. For more details see [16, 17]. Throughout this paper, we discuss problem on the domain $[0, 1]$.

Definition 1. Let H be a real Hilbert space of functions $x : \Omega \to \Re$, H is called a reproducing kernel space if for each $t \in \Omega$, there exists a positive constant c_t such that $|x(t)| \leq c_t \|x\|_H$ for all x in H.

3.1 The reproducing kernel space $W_2^1[0, 1]$

Definition 2. (see [28]) The reproducing kernel space $W_2^1[0, 1]$ for function x is defined as the set of functions such that x is absolutely continuous on $[0, 1]$, and $x^{'} \in L^2[0, 1]$, for $t \in [0, 1]$.

The inner product in $W_2^1[0, 1]$ is of the form

$$\langle x(t), y(t)\rangle_{W_2^1} = \int_0^1 (x(t)y(t) + x^{'}(t)y^{'}(t))dt$$

and the norm $\|x\|_{W_2^1}$ is defined by

$$\|x\|_{W_2^1} = \sqrt{\langle x(t), x(t)\rangle_{W_2^1}},$$

where $x(t), y(t) \in W_2^1[0, 1]$.

3.2 The reproducing kernel space $W_2^r[0, 1]$

Definition 3. (see [29]) The reproducing kernel space $W_2^r[0, 1]$ for function x is defined as the set of functions such that $x^{(r-1)}$ is absolutely continuous on $[0, 1]$, and $x^{(r)} \in L^2[0, 1]$, for $t \in [0, 1]$, where r is a positive integer.

The inner product in $W_2^r[0,1]$ is of the form

$$\langle x(t), y(t)\rangle_{W_2^r} = \sum_{i=0}^{r-1} x^{(i)}(0)y^{(i)}(0) + \int_0^1 x^{(r)}(t)y^{(r)}(t)dt$$

and the norm $\|x\|_{W_2^r}$ is defined by

$$\|x\|_{W_2^r} = \sqrt{\langle x(t), x(t)\rangle_{W_2^r}},$$

where $x(t), y(t) \in W_2^r[0,1]$. We construct the subspace ${}_oW_2^r[0,1]$ of the function space $W_2^r[0,1]$ by imposing a homogeneous boundary condition on $W_2^r[0,1]$ and it is defined as follows

$$\begin{aligned}{}_oW_2^r[0,1] = \{x(t) | x(t) \in W_2^r[0,1],\ & x^{(k)}(0) = 0\ (k = 0\ or\ k = 1),\\ & x^{(l)}(1) = 0\ (l = 0\ or\ l = 1)\}.\end{aligned}$$

3.2.1 The reproducing kernel

Lemma 1. *(see [28, 29]) Let H be a real reproducing kernel space of functions $x : \Omega \to \Re$. For each $t \in \Omega$, there exists a unique element $R_t \in H$ such that $\langle x, R_t\rangle_H = x(t)$ for all $x \in H$.*

DEFINITION 4. (see [28, 29]) Let H be a real reproducing kernel space of functions $x : X \to \Re$ and t be a point in Ω. The mapping $R : \Omega \times \Omega \to \Re$ defined by $R(t, s) = R_t(s)$ is called the reproducing kernel of H.

Theorem 2. *(see [28]) For each $t \in [0,1]$, there exists a unique element $R_t^{\{1\}}(.) \in W_2^1[0,1]$ such that $\langle x(.), R_t^{\{1\}}(.)\rangle_{W_2^1} = x(t)$ for all $x \in {}_oW_2^r$ and the reproducing kernel $R_t^{\{1\}}(.)$ is given by*

$$R_t^{\{1\}}(s) = \frac{1}{2\sinh(1)}[\cosh(t + s - 1) + \cosh(|t - s| - 1)].$$

Theorem 3. *(see [29]) For each $t \in [0,1]$, there exists a unique element $R_t^{\{r\}}(.) \in {}_oW_2^r[0,1]$ such that $\langle x(.), R_t^{\{r\}}(.)\rangle_{{}_oW_2^r} = x(t)$ for all $x \in {}_oW_2^r$ and the reproducing kernel $R_t^{\{r\}}(.)$ can be denoted by*

$$R_t^{\{r\}}(s) = \begin{cases} C_t^{\{r\}}(s) = \sum_{i=1}^{2r} c_i(t)s^{i-1}, & s \le t,\\ D_t^{\{r\}}(s) = \sum_{i=1}^{2r} d_i(t)s^{i-1}, & s > t,\end{cases}$$

where, the coefficients $c_i(t),\ d_i(t),\ i = 1, ..., 2r$, are determined as follows

- $\partial^k R_t^{\{r\}}(0) = 0,\ \partial^l R_t^{\{r\}}(1) = 0,$
- $\frac{\partial^i R_t^{\{r\}}(0)}{\partial s^i} - (-1)^{r-i-1}\frac{\partial^{2r-i-1} R_t^{\{r\}}(0)}{\partial s^{2r-i-1}} = 0,\ i = 0, 1, 2, ..., r-1,\ i \neq k,$
- $\frac{\partial^{2r-i-1} R_t^{\{r\}}(1)}{\partial s^{2r-i-1}} = 0,\ i = 0, 1, 2, ..., r-1,\ i \neq l,$

- $\frac{\partial^i C_t^{\{r\}}(s)}{\partial s^i}|_{s=t} = \frac{\partial^i D_t^{\{r\}}(s)}{\partial s^i}|_{s=t}, \quad i = 0, 1, 2, ..., 2r-2,$
- $(-1)^r (\frac{\partial^{2r-1} C_t^{\{r\}}(s)}{\partial s^{2r-1}}|_{s=t} - \frac{\partial^{2r-1} D_t^{\{r\}}(s)}{\partial s^{2r-1}}|_{s=t}) = 1.$

Then the solution of above equations yields the expression of the reproducing kernel $R_t^{\{r\}}(s)$.

4 Solution guidelines for Euler-Lagrange equation

In this section, we present an efficient technique for solving Euler-Lagrange equation. The Euler-Lagrange equation (2.2), can generally be divided into linear part Ł, and nonlinear part N. Therefore, the Euler-Lagrange equation (2.2) can be converted into the equivalent form as follows

$$\begin{cases} Ł[x(t)] + N[x(t)] - g(t) = 0, \\ x^{(k)}(0) = 0 \ (k = 0 \text{ or } k = 1), \ x^{(l)}(1) = 0 \ (l = 0 \text{ or } l = 1), \end{cases} \tag{4.1}$$

where $x(t) \in {}_oW_2^r[0,1], (r > 2)$ is an unknown function which should can be determined.

In this paper, we assume that under adequate conditions Eq. (4.1) has a unique solution. The existence and uniqueness of solutions to Eq. (4.1) have been proved in Refs. [21, 27]. In order to represent the analytical solution of Eq. (4.1), we can assume that $Ł : {}_oW_2^r[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Choose a countable dense subset $T = \{t_1, t_2, ...\}$ in the domain $[0,1]$, and put $\xi_i(t) = R_{t_i}^{\{1\}}(t)$, where $R_t^{\{1\}}(s)$ is reproducing kernel of $W_2^1[0,1]$. Since Ł is a bounded linear operator, then adjoint operator of Ł defined by $Ł^* : W_2^1[0,1] \to {}_oW_2^r[0,1]$ is uniquely determined.

Now let $\chi_i(t) = (Ł^*\xi_i)(t)$. The orthonormal system $\{\overline{\chi}_i(t)\}_{i=1}^{\infty}$ can be derived from the Gram-Schmidt orthogonalization process of $\{\chi_i(t)\}_{i=1}^{\infty}$

$$\overline{\chi}_i(t) = \sum_{k=1}^{i} \beta_{ik}\chi_k(t), \quad (\beta_{ii} > 0, \ i = 1, 2, ...).$$

So $\{\overline{\chi}_i(t)\}_{i=1}^{\infty}$ is the orthonormalized sequence and β_{ik} are orthogonal coefficients.

Lemma 2. *(see [7]) If* $T = \{t_1, t_2, ...\}$ *is a dense subset in the domain* $[0,1]$ *and the solution of Eq. (4.1) be unique. If for any* $i \in N$ *and for each fixed* $x(t) \in {}_oW_2^r[0,1]$, $\langle x(t), \chi_i(t)\rangle_{{}_oW_2^r} = 0$, *then* $x(t) = 0$.

Theorem 4. *(see [7]) Suppose that* $T = \{t_1, t_2, ...\}$ *is a dense subset in the domain* $[0,1]$ *and the solution of Eq. (4.1) be unique. Then* $\{\overline{\chi}_i(t)\}_{i=1}^{\infty}$ *is the complete system of* ${}_oW_2^r[0,1]$ *and*

$$\overline{\chi}_i(t) = \sum_{k=1}^{i} \beta_{ik} L_s R_t^{\{r\}}(s)|_{s=t_k},$$

where the subscript s *of the operator Ł indicates that the operator Ł is applied to the function of* s.

Let $S = \overline{\{\overline{\chi}_i(t)\}_{i=1}^{\infty}}$ and $S^{\perp}$ be the orthogonal complement of S in ${}_oW_2^r[0,1]$, thus ${}_oW_2^r[0,1] = S \bigoplus S^{\perp}$.

Theorem 5. *(see [10]) Suppose that the following conditions are satisfied: (i)* $T = \{t_1, t_2, ...\}$ *is a countable dense subset in the domain* $[0,1]$*, (ii) the solution of Eq. (4.1) is unique. Then the exact solution of Eq. (4.1) in* ${}_oW_2^r$ *is given by*

$$x(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}[-\mathrm{N}[x(t_k)] + g(t_k)]\overline{\chi}_i(t). \tag{4.2}$$

4.1 Representation of approximate solution

Case (i): If $N[x(t)] = 0$, then the approximate solution can be obtained directly from Eq. (4.2) and

$$x_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}[-N[x(t_k)] + g(t_k)]\overline{\chi}_i(t).$$

Case (ii): If $N[x(t)]$ is nonlinear, then the exact solution can be obtained using the following method.

We note that, $N[x(t)]$ in Eq. (4.2) is unknown. First we construct the iterative sequence $x_n(t)$. For numerical computation, we give initial function $x_0 \in {}_oW_2^r$ and the by using Eq. (4.2), an iterative sequence is constructed as

$$\begin{cases} \mathrm{L}[z_n(t)] = -N[x_{n-1}(t)] + g(t), \\ x_n(t) = P_n z_n(t), \end{cases} \tag{4.3}$$

where $z_n \in {}_oW_2^r$ is the solution of Eq. (4.3) and

$$P_n : {}_oW_2^r \to \{\overline{\chi}_1(t), \overline{\chi}_2(t), ..., \overline{\chi}_n(t)\}$$

is an orthogonal projection operator.

Theorem 6. *Suppose that the following conditions are satisfied:*
(i) $T = \{t_1, t_2, ...\}$ *is a countable dense subset in the domain* $[0,1]$*, (ii) the solution of Eq. (4.3) is unique. Then the solution of Eq. (4.3) is given by*

$$z_n(t) = \sum_{i=1}^{\infty} \boldsymbol{H}_i \overline{\chi}_i(t), \quad n = 1, 2, ...,$$

where $\boldsymbol{H}_i = \sum_{k=1}^{i} \beta_{ik}[-\mathrm{N}[x_{n-1}(t_k)] + g(t_k)]$.

Proof. The proof is similar to that of Theorem 5. □

Therefore considering the numerical computation, we define the n-term approximation $x_n(t)$ to $x(t)$ by

$$x_n(t) = P_n z_n(t) = \sum_{i=1}^{n} \mathbf{H}_i \overline{\chi}_i(t), \quad n = 1, 2, ..., \tag{4.4}$$

where

$$\begin{cases} \mathbf{H}_1 = \beta_{11}[-N[x_0(t_1)] + g(t_1)]], \\ \mathbf{H}_2 = \sum_{k=1}^{2} \beta_{2k}[-N[x_{k-1}(t_k)] + g(t_k)]], \\ \mathbf{H}_3 = \sum_{k=1}^{3} \beta_{3k}[-N[x_{k-1}(t_k)] + g(t_k)]], \\ \cdots \end{cases}$$

4.1.1 The existence of solution in ${}_oW_2^r$ and convergence analysis

Now, we will prove that the solution of Eq. (4.3) exists, and $\{x_n(t)\}_{n=1}^{\infty}$ is convergent.

Lemma 3. *(see [25]) For any $x \in {}_oW_2^r$ we have the following statement*

$$\|x^{(j)}\|_\infty \leq \|\partial_t^j R_t^{\{r\}}(s)\|_{{}_oW_2^r} \ \|x\|_{{}_oW_2^r} \leq D_j \ \|x\|_{{}_oW_2^r}, \ j = 0, 1, 2, ..., r-1,$$

for some D_j independent of x.

Lemma 4. *Suppose that, $\|x_n\|_{{}_oW_2^r}$ is bounded in Eq. (4.4), then there exists a constant K_j, such that $\|x_n^{(j)}\|_\infty \leq K_j, \ j = 0, 1, 2, ..., r-1$.*

Proof. Since $\|x_n\|_{{}_oW_2^r}$ is bounded and by Lemma 4, $\|x_n^{(j)}\|_{{}_oW_2^r}$ is also bounded and $\|x_n^{(j)}\|_\infty \leq K_j, \ j = 0, 1, 2, ..., r-1$, which completes the proof. □

Lemma 5. *$\Pi = \{x_n | \ \|x_n\|_{{}_oW_2^r} \leq K\} \subset C[0, 1]$ is a bounded set, where K is a constant.*

Proof. From Lemma 4, there exists a positive constant $K < \infty$ such that $|x_n(t)| \leq K$ for each $t \in [0, 1]$ and each $x_n \in \Pi$. □

Lemma 6. *$\Pi = \{x_n | \ \|x_n\|_{{}_oW_2^r} \leq K\} \subset C[0, 1]$ is equicontinuous, where K is a constant.*

Proof. For an arbitrary $x_n \in \Pi$, we deduce

$$\begin{aligned} |x_n(t') - x_n(t)| &= |\langle x_n(s), R_{t'}^{\{r\}}(s) - R_t^{\{r\}}(s)\rangle| \\ &\leq \|x_n(s)\|_{{}_oW_2^r} \ \|R_{t'}^{\{r\}}(s) - R_t^{\{r\}}(s)\|_{{}_oW_2^r} \\ &\leq \|x_n(s)\|_{{}_oW_2^r} \ \|\frac{d}{dt} R_\zeta^{\{r\}}(s)|_{\zeta \in [t', t]}\|_{{}_oW_2^r} \ |t' - t| \leq C|t' - t|, \end{aligned}$$

where C is a positive constant. Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $t, t' \in [0, 1]$:

$$|t' - t| < \delta \Rightarrow |x_n(t') - x_n(t)| < \epsilon, \quad \forall x_n \in \Pi.$$

This completes the proof. □

Lemma 7. *Suppose that the following conditions hold (i)* $T = \{t_1, t_2, ...\}$ *be a countable dense subset in the domain* $[0,1]$, *(ii)* $\Pi = \{x_n | \ \|x_n\|_{{}_oW_2^r} \leq K\} \subset C[0,1]$, *(iii)* $\mathrm{N}[x(t)]$ *is continuous as* $t \in [0,1]$ *and* $x = x(t) \in \Re$. *Then there exists a subsequence* $\{x_{n_q}\}_{q\geq 1} \subseteq \Pi$ *which* $\{x_{n_q}\}_{q\geq 1}$ *converges uniformly to* $\widehat{x}$, *as* $q \to \infty$, *where*

$$\widehat{x}(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}[-\mathrm{N}[\widehat{x}(t_k)] + g(t_k)]\overline{\chi}_i(t).$$

Proof. It follows from Lemmas 5 and 6 that Π is a precompact set. Then any sequence in Π has a uniformly convergent subsequence whose limit belongs to Π. Applying this principle we find that there exists a sequence $\{n_q\}_{q\geq 1}$ with $n_1 < n_2 < ....$ such that subsequence $\{x_{n_q}\}_{q\geq 1}$ is uniformly convergent and

$$\widehat{x} = \lim_{q\to\infty} x_{n_q} \in \Pi.$$

Since $N[x(t)]$ is continuous as $t \in [0,1]$, and $x = x(t) \in \Re$, then

$$\widehat{x}(t) = \lim_{q\to\infty} \sum_{i=1}^{n_q} \mathbf{H}_i \overline{\chi}_i(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}[-N[\widehat{x}(t_k)] + g(t_k)]\overline{\chi}_i(t).$$

This completes the proof. □

Theorem 7. *Suppose that the conditions of Lemma 7 hold, then there exists a subsequence* $\{x_{n_q}\}_{q\geq 1} \subseteq \Pi$ *in which,* $\{x_{n_q}^{(j)}\}_{q\geq 1}$, *for each* $j \in \{0,1,2,...,r-1\}$ *converges uniformly to* $\widehat{x}^{(j)}$, *as* $q \to \infty$, *where*

$$\widehat{x}(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}[-\mathrm{N}[\widehat{x}(t_k)] + g(t_k)]\overline{\chi}_i(t).$$

Proof. By Lemma 7, there exists a subsequence $\{x_{n_q}\}_{q\geq 1} \subseteq \Pi$ in which $\{x_{n_q}\}_{q\geq 1}$ converges uniformly to $\widehat{x}$, *as* $q \to \infty$, where

$$\widehat{x}(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}[-N[\widehat{x}(t_k)] + g(t_k)]\overline{\chi}_i(t).$$

It follows from Lemma 4 that for each $j \in \{0,1,2,...,r-1\}$, the sequence $\{x_{n_q}^{(j)}\}_{q\geq 1}$ is bounded. Then for each $j \in \{0,1,2,...,r-1\}$, there exists subsequence $\{x_{n_{q_l}}^{(j)}\}_{l\geq 1}$, such that $\{x_{n_{q_l}}^{(j)}\}_{l\geq 1}$, is uniformly convergent. Consequently

$$\|x_{n_{q_l}}^{(j)} - \widehat{x}^{(j)}\|_\infty \to 0, \ j = 0,1,2,...,r-1, \ as \ l \to \infty.$$

Now, without of generality, we replace $\{x_{n_q}\}_{q\geq 1}$ with $\{x_{n_{q_l}}\}_{l\geq 1}$. This completes the proof. □

Lemma 8. *Suppose that the conditions of Theorem 7 hold, then* $\widehat{x}^{(j)}(t)$, $j = 0,1,...,r-2$, *are absolutely continuous functions.*

Proof. For each $j \in \{0,1,2,...,r-2\}$ and for arbitrary $\epsilon > 0$, choose $\delta_j = \frac{\epsilon}{M_j}$. Let $\{(a_k, b_k)\}_{k=1}^{m}$ be a set of mutually disjoint open intervals $(a_k, b_k) \subset [0,1]$, satisfying $\sum_{k=1}^{m}(b_k - a_k) < \delta_j$. It follows by Theorem 7 that there exists a subsequence $\{x_{n_q}\}_{q\geq 1} \subseteq \Pi$ which $\{x_{n_q}^{(j)}\}_{q\geq 1}$ converges uniformly to $\widehat{x}^{(j)}$, as $q \to \infty$. Then for each $j \in \{0,1,2,...,r-2\}$,

$$
\begin{aligned}
|\widehat{x}^{(j)}(b_k) - \widehat{x}^{(j)}(a_k)| &= \lim_{q\to\infty} |x_{n_q}^{(j)}(b_k) - x_{n_q}^{(j)}(a_k)| \\
&= \lim_{q\to\infty} |\langle x_{n_q}(s), \partial_t^j R_t^{\{r\}}(s)|_{t=b_k} - \partial_t^j R_t^{\{r\}}(s)|_{t=a_k}\rangle| \\
&= \lim_{q\to\infty} \|x_{n_q}(s)\|_{{}_oW_2^r} \|\partial_t^j R_t^{\{r\}}(s)|_{t=b_k} - \partial_t^j R_t^{\{r\}}(s)|_{t=a_k}\|_{{}_oW_2^r} \\
&\leq K\|\partial_t^j R_t^{\{r\}}(s)|_{t=b_k} - \partial_t^j R_t^{\{r\}}(s)|_{t=a_k}\|_{{}_oW_2^r} \\
a &= K\|\partial_t^{j+1} R_t^{\{r\}}(s)|_{t=\zeta_j\in[a_k,b_k]}(b_k - a_k)\|_{{}_oW_2^r} \\
&\leq KD_j|b_k - a_k| = M_j|b_k - a_k|.
\end{aligned}
$$

Then, we have

$$
\sum_{k=1}^{m} |\widehat{x}^{(j)}(b_k) - \widehat{x}^{(j)}(a_k)| \leq \sum_{k=1}^{m} M_j|b_k - a_k| < \epsilon.
$$

This completes the proof. □

Lemma 9. *Suppose that the conditions of Theorem 7 hold, and the sequence $\{x_n^{(r)}(t)\}_{n\geq 1}$ is bounded, then $\widehat{x}^{(r-1)}(t)$ is absolutely continuous function.*

Proof. For arbitrary $\epsilon > 0$, choose $\delta = \frac{\epsilon}{C}$. Let $\{(a_k, b_k)\}_{k=1}^{m}$ be a set of mutually disjoint open intervals $(a_k, b_k) \subset [0,1]$, satisfying $\sum_{k=1}^{m}(b_k - a_k) < \delta$. By Theorem 7, there exists a subsequence $\{x_{n_q}\}_{q\geq 1} \subseteq \Pi$ in which, for each $j \in \{0,1,...,r-1\}$, $\{x_{n_q}^{(j)}\}$, converges uniformly to $\widehat{x}^{(j)}$, as $q \to \infty$, where $\widehat{x}(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}[-N[\widehat{x}(t_k)] + g(t_k)]\overline{\chi}_i(t)$. Since the sequence $\{x_{n_q}^{(r)}\}_{q\geq 1} \subseteq \{x_n^{(r)}(t)\}_{n\geq 1}$ is bounded, then there exists subsequence $\{x_{n_{q_l}}^{(r)}\}_{l\geq 1} \subseteq \{x_{n_q}^{(r)}(t)\}_{q\geq 1}$, such that $\{x_{n_{q_l}}^{(r)}\}_{l\geq 1}$, is uniformly convergent. Consequently

$$
\|x_{n_{q_l}}^{(r)} - \widehat{x}^{(r)}\|_\infty \to 0, s \to \infty.
$$

Note that

$$
\begin{aligned}
|\widehat{x}^{(r-1)}(b_k) - \widehat{x}^{(r-1)}(a_k)| &= \lim_{l\to\infty} |x_{n_{q_l}}^{(r-1)}(b_k) - x_{n_{q_l}}^{(r-1)}(a_k)| \\
&= \lim_{l\to\infty} |\langle x_{n_{q_l}}, \partial_t^{r-1} R_t^{\{r\}}(s)|_{t=b_k} - \partial_t^{r-1} R_t^{\{r\}}(s)|_{t=a_k}\rangle| \\
&\leq \lim_{l\to\infty} |\sum_{i=0}^{r-1} x_{n_{q_l}}^{(i)}(0)(\partial_{t^{r-1}s^i}^{r-1+i} R_t^{\{r\}}(0)|_{t=b_k} \\
&\quad - \partial_{t^{r-1}s^i}^{r-1+i} R_t^{\{r\}}(0)|_{t=a_k})| + \lim_{l\to\infty} |\int_0^1 x_{n_{q_l}}^{(r)}(s)(\partial_{t^{r-1}s^r}^{2r-1} R_t^{\{r\}}(s)|_{t=b_k}
\end{aligned}
$$

$$- \partial^{2r-1}_{t^{r-1}s^r} R^{\{r\}}_t(s)|_{t=a_k})ds| \leq \lim_{l\to\infty} |\sum_{i=0}^{r-1} x^{(i)}_{n_{q_l}}(0)(\partial^{r-1+i}_{t^{r-1}s^i} C^{\{r\}}_t(0)|_{t=b_k}$$
$$- \partial^{r-1+i}_{t^{r-1}s^i} C^{\{r\}}_t(0)|_{t=a_k})| + \lim_{l\to\infty} |\int_0^{a_k} x^{(r)}_{n_{q_l}}(s)(\partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=b_k}$$
$$- \partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=a_k})ds| + \lim_{l\to\infty} |\int_{a_k}^{b_k} x^{(r)}_{n_{q_l}}(s)(\partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=b_k}$$
$$- \partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=a_k})ds| + \lim_{l\to\infty} |\int_{b_k}^{1} x^{(r)}_{n_{q_l}}(s)(\partial^{2r-1}_{t^{r-1}s^r} D^{\{r\}}_t(s)|_{t=b_k}$$
$$- \partial^{2r-1}_{t^{r-1}s^r} D^{\{r\}}_t(s)|_{t=a_k})ds|.$$

Using Lemma 4 and the continuity of $\partial^{r-1+i}_{t^{r-1}s^i} C^{\{r\}}_t(0),\ i = 0, 1, ..., r-1$, with respect to t, and the differential mean value Theorem, we obtain

$$\lim_{l\to\infty} |\sum_{i=0}^{r-1} x^{(i)}_{n_{q_l}}(0)(\partial^{r-1+i}_{t^{r-1}s^i} C^{\{r\}}_t(0)|_{t=b_k} - \partial^{r-1+i}_{t^{r-1}s^i} C^{\{r\}}_t(0)|_{t=a_k})| \leq C_1|b_k - a_k|, \quad (4.5)$$

where C_1 is a positive constant. From the continuity of $\partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)$ with respect to t, the differential mean value Theorem and the Cauchy-Schwarz inequality, one obtains

$$\lim_{l\to\infty} |\int_0^{a_k} x^{(r)}_{n_{q_l}}(s)(\partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=b_k} - \partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=a_k})ds| \leq C_2|b_k - a_k|, \quad (4.6)$$

$$\lim_{l\to\infty} |\int_{a_k}^{b_k} x^{(r)}_{n_{q_l}}(s)(\partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=b_k} - \partial^{2r-1}_{t^{r-1}s^r} C^{\{r\}}_t(s)|_{t=a_k})ds| \leq C_3|b_k - a_k|, \quad (4.7)$$

$$\lim_{l\to\infty} |\int_{b_k}^{1} x^{(r)}_{n_{q_l}}(s)(\partial^{2r-1}_{t^{r-1}s^r} D^{\{r\}}_t(s)|_{t=b_k} r - \partial^{2r-1}_{t^{r-1}s^r} D^{\{r\}}_t(s)|_{t=a_k})ds| \leq C_4|b_k - a_k|, \quad (4.8)$$

where C_2, C_3, and C_4 are constants. By (4.5)–(4.8), we have

$$\sum_{k=1}^{m} |\widehat{x}^{(r-1)}(b_k) - \widehat{x}^{(r-1)}(a_k)| \leq \sum_{k=1}^{m} C|b_k - a_k| < \epsilon,$$

where $C = C_1 + C_2 + C_3 + C_4$. This completes the proof. □

Lemma 10. *Suppose that the conditions of Lemmas 8 and 9 hold, and further the invertible operator* $Ł^{-1}$ *exists then* $\widehat{x} \in {}_oW^r_2[0,1]$*.*

Proof. By Lemmas 8 and 9, $N[\widehat{x}(t)]$ is absolutely continuous and furthermore $-\frac{\partial}{\partial t}N[\widehat{x}(t)] + g'(t) \in L^2[0,1]$. Then $-N[\widehat{x}(t)] + g(t) \in W^1_2[0,1]$. Consequently $Ł^{-1}(-N[\widehat{x}(t)] + g(t)) \in {}_oW^r_2[0,1]$, and we must have

$$Ł^{-1}(-N[\widehat{x}(t)] + g(t)) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}[-N[\widehat{x}(t_k)] + g(t_k)]\overline{\chi}_i(t) = \widehat{x}(t).$$

This completes the proof. □

By using Lemmas 8, 9 and 10 one obtain the following theorem.

Theorem 8. *Suppose that the conditions of Lemmas 8, 9 and 10 hold, then the solution of Eq. (4.1) is exists and is expressed as*

$$x(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[-\mathrm{N}[x(t_k)] + g(t_k)]\overline{\chi}_i(t) \in \ {}_oW_2^r[0,1].$$

Theorem 9. *Suppose that the conditions of Theorem 8 hold, and the solution of Eq. (4.1) is exists and is unique, then for each* $j \in \{0, 1, 2, , ..., r-1\}$,

$$\|x_n^{(j)} - x^{(j)}\|_\infty \to 0, \ as \ n \to \infty, \tag{4.9}$$

where $x(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}[-\mathrm{N}[x(t_k)] + g(t_k)]$.

Proof. If (4.9) is not true, then there exist a positive number ϵ_0 and a subsequence $\{x_{n_q}\}_{q\geq 1} \subset \Pi$ such that

$$\|x_{n_q}^{(j)} - x^{(j)}\|_\infty \geq \epsilon_0, \ (q = 1, 2, \ldots). \tag{4.10}$$

Since, for each $j \in \{0, 1, 2, ..., r-1\}$, $\{x_n^{(j)}\}_{n\geq 1}$ is precompact, there exists a subsequence of $\{x_{n_q}^{(j)}\}_{q\geq 1}$ which converges in $C[0,1]$ to some $x^{(j)} \in C[0,1]$. Without of generality, we may assume that $\{x_{n_q}^{(j)}\}_{q\geq 1}$ itself converges to $\widetilde{x}^{(j)}$:

$$\|x_{n_q}^{(j)} - \widetilde{x}^{(j)}\|_\infty \to 0, \ as \ q \to \infty. \tag{4.11}$$

Since the solution of Eq. (4.1) is unique, we have $x^{(j)} = \widetilde{x}^{(j)}$, and therefore, (4.11) contradicts (4.10). So the proof of Theorem 9 is completed. □

4.1.2 Error analysis

We now obtain the error estimate for the approximate solution of Eq. (4.1) in ${}_oW_2^r$. To achieve this aim, we establish and prove the next theorem.

Theorem 10. *Suppose that the conditions of Theorem 9 hold. Let* $P_n = \{0 = t_1 < t_2 < ... < t_n = 1\}$, *be a partition of interval* $[0,1]$ *and also* $x_n(t)$ *be the approximate solution of the Eq. (4.1) in the space* ${}_oW_2^r$. *The following relation holds,*

$$\|x(t) - x_n(t)\|_\infty \leq C \ h^{r-1}, \quad h = \max_{1\leq i\leq n-1}(t_{i+1} - t_i),$$

where C *is real constant.*

Proof. In each subinterval $[t_i, t_{i+1}]$, we can write

$$x^{(r-2)}(t) - x_n^{(r-2)}(t) = x^{(r-2)}(t)$$
$$-x^{(r-2)}(t_i) + x_n^{(r-2)}(t_i) - x_n^{(r-2)}(t) + x^{(r-2)}(t_i) - x_n^{(r-2)}(t_i). \tag{4.12}$$

By means of the mean value Theorem and the continuity of $x^{(r-1)}$, one can show that

$$\|x^{(r-2)}(t) - x^{(r-2)}(t_i)\|_\infty \le C_1 h. \tag{4.13}$$

We know have

$$|x_n^{(r-2)}(t) - x_n^{(r-2)}(t_i)| \le \int_{t_i}^{t} |x_n^{(r-1)}(s)| ds \tag{4.14}$$

and since $x_n(t) \in {}_oW_2^r$, it follows that

$$|x_n^{(r-2)}(t) - x_n^{(r-2)}(t_i)| \le C_2 h. \tag{4.15}$$

Using Theorem 9, for large n we have

$$\|x^{(r-2)}(t_i) - x_n^{(r-2)}(t_i)\|_\infty \le \varepsilon. \tag{4.16}$$

Since ε is arbitrary and by combining Eqs. (4.12)–(4.16), for the chosen value of n, we must have

$$\|x^{(r-2)}(t) - x_n^{(r-2)}(t)\|_\infty \le C_3\ h. \tag{4.17}$$

We know have

$$\begin{aligned} x^{(j)}(t) - x_n^{(j)}(t) &= x^{(j)}(t_i) - x_n^{(j)}(t_i) \\ &+ \int_{t_i}^{t} (x^{(j+1)}(s) - x_n^{(j+1)}(s)) ds,\ 0 < j < r-2. \end{aligned} \tag{4.18}$$

By using Eqs. (4.17)–(4.18) and applying Theorem 9 for large n, it is straightforward to see that

$$\|x(t) - x_n(t)\|_\infty \le C\ h^{r-1}, \quad h = \max_{1 \le i \le n-1} (t_{i+1} - t_i),$$

and the proof is completed now. □

5 Test examples

In this section, some illustrative examples are considered to reveal the effectiveness and the accuracy of the proposed method for solving a class of VPs. All of the computations have been performed by using the Maple software package. Results obtained by method are compared with exact solution of each example and are found to be in good agreement.

Example 1. Consider the following VP: find the extremum of the functional

$$\begin{aligned} &\mathrm{J}[x] = \int_0^1 (\frac{1}{3}x^3(t) - (2 + (t^2 - t)^2))x(t) - \frac{1}{2}(x^{'}(t))^2) dt, \\ &x(0) = 0, \quad x(1) = 0. \end{aligned}$$

An exact solution of this problem is given as $x(t) = t^2 - t$.

Let $t_{i+1} = \frac{1}{2}\cos(\pi\frac{i}{n-1}) + \frac{1}{2},\ i = 0, 1, ..., n-1$ be a set of gird points of the interval [0,1]. An iterative sequence is constructed as

$$\begin{cases} z_n^{''}(t) = -x_{n-1}^2(t) + 2 + (t^2 - t)^2, \\ x_n(t) = P_n z_n(t), \end{cases}$$

where $P_n :\ {}_oW_2^3 \to \{\overline{\chi}_1(t), \overline{\chi}_2(t), ..., \overline{\chi}_n(t)\}$ is an orthogonal projection operator. Therefore considering the numerical computation, we define the n-term approximation $x_n(t)$ to $x(t)$ by

$$x_n(t) = P_n z_n(t) = \sum_{i=1}^{n} \mathbf{H}_i \overline{\chi}_i(t), \quad n = 1, 2, ...,$$

where

$$\begin{cases} \mathbf{H}_1 = \beta_{11}[-x_0^2(t_1) + 2 + (t_1^2 - t_1)^2]], \\ \mathbf{H}_2 = \sum_{k=1}^{2} \beta_{2k}[-x_{k-1}^2(t_k) + 2 + (t_k^2 - t_k)^2]], \\ \mathbf{H}_3 = \sum_{k=1}^{3} \beta_{3k}[-x_{k-1}^2(t_k)] + 2 + (t_k^2 - t_k)^2]], \\ \dots \end{cases}$$

We obtain the approximate solution for $n = 60, 70, 80$. The absolute values of the errors are given in Figure 1.

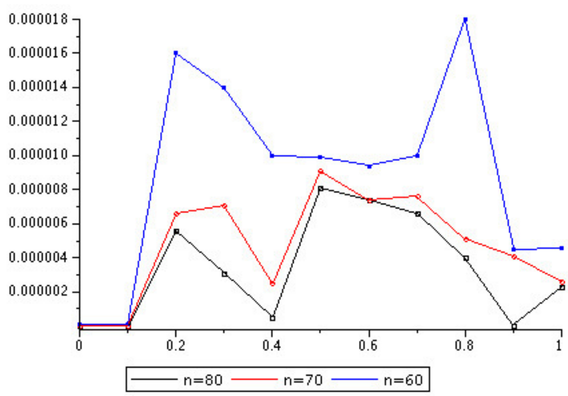

Figure 1. Absolute errors for $n = 60, 70, 80$ of Example 1.

Table 1 presents the numerical results for proposed method in the interval $[0, 1]$. From the Table 1 it can be seen that the approximate solutions are in good agreement with the exact solution. Absolute errors show that the present method gives approximate solution with a high degree of accuracy.

Example 2. Consider the following VP: find the extremum of the functional

$$\mathrm{J}[x] = \int_0^1 [(x^{'}(t) - b_1(t))^2 + z(t)]dt,$$

Table 1. Estimated and exact value of $x(t)$ for $n = 60, 70, 80$ of Example 1.

t	n=60	n=70	n=80	Exact
0.0	0.0000000663	0.0000000361	0.0000000164	0.0000000000
0.1	-0.0899999240	-0.0899999840	-0.0899999740	-0.0900000000
0.2	-0.1599840000	-0.1599934000	-0.1599944000	-0.1600000000
0.24	-0.1823853248	-0.1823931879	-0.1823952347	-0.1824000000
0.3	-0.2099860000	-0.2099929000	-0.2099969000	-0.2100000000
0.4	-0.2399900000	-0.2399975000	-0.2399995000	-0.2400000000
0.5	-0.2499901000	-0.2499909000	-0.2499919000	-0.2500000000
0.6	-0.2399906000	-0.2399926000	-0.2399926000	-0.2400000000
0.7	-0.2099900000	-0.2099924000	-0.2099934000	-0.2100000000
0.8	-0.1599820000	-0.1599949000	-0.1599960000	-0.1600000000
0.9	-0.0899955000	-0.0899959000	-0.0899999500	-0.0900000000
0.999	-0.0009934374	-0.0009951123	-0.0009955284	-0.0009990000
1.0	0.0004625102	0.0000026651	0.0000023141	0.0000000000
CPU time	380(s)	554(s)	682(s)	

where

$$z(t) = \int_0^t (x(s) - b_2(s))^2 ds,$$

subject to $x(0) = 0$ and $x^{'}(0) = 0$ and

$$b_1(t) = \frac{8}{7}t^7 - \frac{27}{2}t^{\frac{7}{2}} + \frac{9}{2}t, \quad b_2(t) = t^8 - 3t^{\frac{9}{2}} + \frac{9}{4}t^2.$$

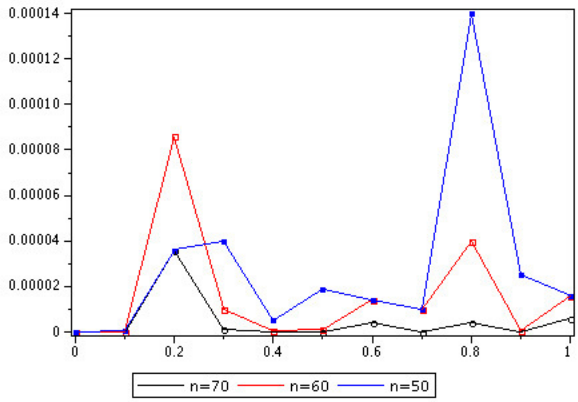

Figure 2. Absolute errors for $n = 50, 60, 70$ of Example 2.

In this example, the extremal is $x(t) = t^8 - 3t^{\frac{9}{2}} + \frac{9}{4}t^2$. To apply the proposed method, we consider uniform grid points

$$t_{i+1} = \frac{1}{2}\cos(\pi\frac{i}{n-1}) + \frac{1}{2}, \ i = 0, 1, ..., n-1$$

Table 2. Estimated and exact value of $x(t)$ for $n = 50, 60, 70$ of Example 2.

t	n=50	n=60	n=70	Exact
0.0	0.0000000367	0.0000000164	0.0000000681	0.0000000000
0.1	0.0224057016	0.0224052076	0.0224051676	0.0224051416
0.2	0.0878919347	0.0879419347	0.0878919347	0.0878559347
0.3	0.1892959519	0.1892659519	0.1892569519	0.1892559518
0.4	0.3120877751	0.3120832751	0.3120828251	0.3120827751
0.5	0.4338427285	0.4338246285	0.4338239285	0.4338237285
0.6	0.5256469750	0.5256469750	0.5256369750	0.5256329749
0.7	0.5575117929	0.5575117929	0.5575018929	0.5575017928
0.8	0.5088400277	0.5087400277	0.5087040277	0.5087000276
0.9	0.3856988744	0.3856743744	0.3856739744	0.3856738744
1.0	0.2500162410	0.2500162546	0.2500066211	0.2500000000
CPU time	560(s)	731(s)	894(s)	

on $[0, 1]$. Using the proposed method, we calculate the approximate solution $x_n(t)$ for $n = 50, 60, 70$ in ${}_oW_2^3$. Figure 2 gives the absolute errors for proposed method in the interval $[0, 1]$. In Table 2, the value of $x(t)$ using the proposed method for $n = 50, 60, 70$ is compared with the exact solution. From Figure 2 and Table 2 it can be seen that the approximate solutions obtained by proposed method are in prefect agreement with the exact solution.

Example 3. Consider the following VP: find the extremum of the functional

$$\mathrm{J}[x] = \int_0^1 [(\frac{1}{2}x'^2(t) + x(t) - c_1(t))^2 + z(t)]dt,$$

where

$$z(t) = \int_0^t (x'(s) - c_2(s))^2 ds,$$

subject to $x(0) = 0$ and $x(1) = 0$ and

$$c_1(t) = \frac{1}{2}(2t-1)^2 e^{2t^2-2t} + 1 - e^{t^2-t}, \quad c_2(t) = -(2t-1)e^{t^2-t}.$$

An exact solution of this problem is given as $x(t) = 1 - e^{t^2-t}$.

We consider grid points $t_{i+1} = \frac{1}{2}\cos(\pi\frac{i}{n-1}) + \frac{1}{2},\ i = 0, 1, ..., n-1$ on $[0, 1]$. By employing the proposed method, we obtain the approximate solution for $n = 80, 90, 100$ in ${}_oW_2^4$. The absolute values of the errors for $n = 80, 90, 100$ are given in Figure 3. Table 3 presents the numerical results for proposed method in the interval $[0, 1]$. From Table 3 and Figure 3, it is clear that the approximate solutions are in good agreement with the exact solution.

6 Conclusions

This paper describes a semi-analytical method in reproducing kernel space for finding the extremum of VPs depending on indefinite integrals. By using this

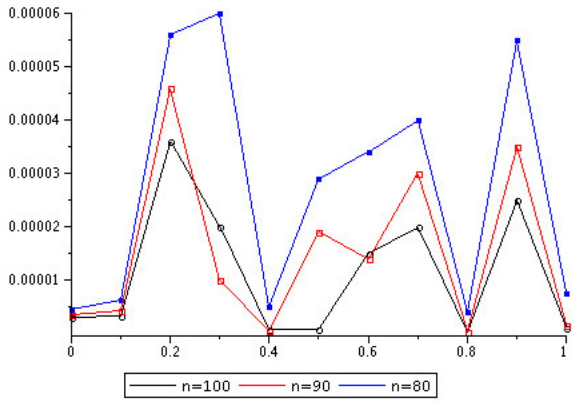

Figure 3. Absolute errors for $n = 80, 90, 100$ of Example 3.

Table 3. Estimated and exact value of $x(t)$ for $n = 80, 90, 100$ of Example 3.

t	n=80	n=90	n=100	Exact
0.0	0.0000045823	0.0000035134	0.0000031109	0.0000000000
0.1	0.0860626147	0.0860646147	0.0860656147	0.0860688147
0.2	0.1478002110	0.1478102110	0.1478202110	0.1478562110
0.3	0.1893557540	0.1894057540	0.1893957540	0.1894157540
0.4	0.2133671389	0.2133716389	0.2133728389	0.2133721389
0.5	0.2212282169	0.2212182169	0.2212001169	0.2211992169
0.6	0.2134061389	0.2133861389	0.2133871389	0.2133721389
0.7	0.1894557540	0.1894457540	0.1894357540	0.1894157540
0.8	0.1478602110	0.1478566110	0.1478565110	0.1478562110
0.9	0.0861238147	0.0861038147	0.0860938147	0.0860688147
1.0	0.0000076504	0.0000017210	0.0000010631	0.0000000000
CPU time	950(s)	1127(s)	1269(s)	

method, we introduced an iterative sequence which converges uniformly to exact solution. Moreover, it is found that the bound for the error in $W_2^r[0,1]$ is $O(h^{r-1})$. The applicability and accuracy of the method were examined on some test examples by calculating the the discrete maximum error.

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References

[1] R. Almedia, R. Khosravian-Arab and M. Shamsi. A generalized fractional variational problem depending on indefinite integrals: Euler-Lagrange

equation and numerical solution. *J. Vib. Cont.*, **19**(14):2177–2186, 2012. http://dx.doi.org/10.1177/1077546312458818.

[2] N. Aronszajn. Theory of reproducing kernels. *Transactions of the American Mathematical Society*, **68**:337–404, 1950. http://dx.doi.org/10.1090/S0002-9947-1950-0051437-7.

[3] O. Abu Arqub and M. Al-Smadi. Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations. *Appl. Math. Comput.*, **243**:911–922, 2014. http://dx.doi.org/10.1016/j.amc.2014.06.063.

[4] O. Abu Arqub, M. Al-Smadi, S. Momani and T. Hayat. Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method. *Soft Computing.*, pp. 1–20, 2015. http://dx.doi.org/10.1007/s00500-015-1707-4.

[5] A. Berlinet and C.T. Agnan. *Reproducing Kernel Hilbert Space in Probability and Statistics.* Kluwer Academic Publishers, Boston, Mass, USA, 2004. http://dx.doi.org/10.1007/978-1-4419-9096-9.

[6] P. Bouboulis and M. Mavroforakis. Reproducing kernel Hilbert spaces and fractal interpolation. *J. Comput. Appl. Math.*, **235**(12):3425–3434, 2011. http://dx.doi.org/10.1016/j.cam.2011.02.003.

[7] M. Cui and F. Geng. A computational method for solving one-dimensional variable-coefficient Burgers equation. *Appl. Math. Comput.*, **188**(2):1389–1401, 2007. http://dx.doi.org/10.1016/j.amc.2006.11.005.

[8] M. Cui and F. Geng. Solving singular two-point boundary value problem in reproducing kernel space. *J. Comput. Appl. Math.*, **205**(1):6–15, 2007. http://dx.doi.org/10.1016/j.cam.2006.04.037.

[9] M. Cui and Y. Lin. *Nonlinear Numerical Analysis in the Reproducing Kernel Space.* Nova Science, New York, NY, USA, 2008.

[10] M. Cui and L. Yang. A new method of solving the coefficient inverse problem of differential equation. *Science in China A.*, **50**(4):561–572, 2007. http://dx.doi.org/10.1007/s11425-007-0013-8.

[11] A. Daniel. *Reproducing Kernel Spaces and Applications.* Springer, Basel, Switzerland, 2003.

[12] L.E. Elgolic. *Calculus of Variations.* Pergamon press, Oxford, 1962.

[13] I.M. Gelfand and S.V. Fomin. *Calculus of Variations.* Prentice-Hall, New Jersey, 1963.

[14] F. Geng. Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method. *Appl. Math. Comput.*, **215**(6):2095–2102, 2009. http://dx.doi.org/10.1016/j.amc.2009.08.002.

[15] F. Geng. A novel method for solving a class of singularly perturbed boundary value problems based on reproducing kernel method. *Appl. Math. Comput.*, **218**(8):4211–4215, 2011. http://dx.doi.org/10.1016/j.amc.2011.09.052.

[16] F. Geng and M. Cui. Solving a nonlinear system of second order boundary value problems. *J. Math. Anal. Appl.*, **327**(2):1167–1181, 2007. http://dx.doi.org/10.1016/j.jmaa.2006.05.011.

[17] F. Geng and M. Cui. Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space. *Appl. Math. Comput.*, **192**(2):389–398, 2007. http://dx.doi.org/10.1016/j.amc.2007.03.016.

[18] F. Geng and M. Cui. A reproducing kernel method for solving nonlocal fractional boundary value problems. *Appl. Math. Lett.*, **25**(5):818–823, 2012. http://dx.doi.org/10.1016/j.aml.2011.10.025.

[19] F. Geng and X.M. Li. A new method for Riccati differential equations based on reproducing kernel and quasilinearization methods. *Abs. Appl. Anal.*, p. 8 pages, 2012. http://dx.doi.org/10.1155/2012/603748.

[20] M. Ghasemi, M. Fardi and R. Khoshsiar Ghaziani. Numerical solution of nonlinear delay differential equations of fractional order in reproducing kernel Hilbert space. *Appl. Math. Comput.*, **268**:815–831, 2015. http://dx.doi.org/10.1016/j.amc.2015.06.012.

[21] I. Gyongy. Existence and uniqueness results for semilinear stochastic partial differential equations. *Stoch. Process Appl.*, **73**(2):271–299, 1998. http://dx.doi.org/10.1016/S0304-4149(97)00103-8.

[22] W. Jiang and Z. Chen. A collocation method based on reproducing kernel for a modified anomalous subdiffusion equation. *Numer. Meth. Part. Diff. Equ.*, **30**(1):289–300, 2014. http://dx.doi.org/10.1002/num.21809.

[23] M.L. Krasnov, G.I. Makarenko and A.I. Kiselev. *Problems and Exercises in the Calculus of Variations.* MIR Publishers, Moscow, 1975.

[24] M. Mohammadi and R. Mokhtari. A reproducing kernel method for solving a class of nonlinear systems of PDEs. *Math. Model. Anal.*, **19**(2):180–198, 2014. http://dx.doi.org/10.3846/13926292.2014.909897.

[25] S. Momani, O. Abu Arqub, T. Hayat and H. Al-Sulami. A computational method for solving periodic boundary value problems for integro-differential equations of Fredholm-Volterra type. *Appl. Math. Comput.*, **240**:229–239, 2014. http://dx.doi.org/10.1016/j.amc.2014.04.057.

[26] V.M. Tikhomirov. *Store about Maxima and Minima, American Mathematica Society.* Providence, RI, 1990.

[27] J. Wang and G. Warnecke. Existence and uniqueness of solutions for a non-uniformly parabolic equation. *J. Diff. Equ.*, **183**(1):1–16, 2003. http://dx.doi.org/10.1016/S0022-0396(02)00059-1.

[28] L. Yang and M. Cui. New algorithm for a class of nonlinear integro-differential equations in the reproducing kernel space. *Appl. Math. Comput.*, **174**(2):942–960, 2006. http://dx.doi.org/10.1016/j.amc.2005.05.026.

[29] Y. Yang, M. Du, F. Tan, Z. Li and T. Nie. Using reproducing kernel for solving a class of fractional partial differential equation with non-classical conditions. *Appl. Math. Comput.*, **219**(11):5918–5925, 2013. http://dx.doi.org/10.1016/j.amc.2012.12.009.

[30] Y.F. Zho, M.G. Cui and Y.Z. Lin. A computational method for nonlinear $2m$-th order boundary value problems. *Math. Model. Anal.*, **15**(4):571–586, 2010. http://dx.doi.org/10.3846/1392-6292.2010.15.571-586.

[31] D.X. Zhou. Capacity of reproducing kernel spaces in learning theory. *IEEE Transactions on Information Theory*, **49**(7):1743–1752, 2003. http://dx.doi.org/10.1109/TIT.2003.813564.