Mathematical Modelling and Analysis
Volume 21 Number 4, July 2016, 569–584
http://dx.doi.org/10.3846/13926292.2016.1188427

Publisher: Taylor&Francis and VGTU
http://www.tandfonline.com/TMMA
ISSN: 1392-6292
eISSN: 1648-3510

A Main Class of Integral Inequalities with Applications

Mohammad Masjed-Jamei[a], **Edward Omey**[b] **and Sever S. Dragomir**[c]

[a] *Department of Mathematics, K. N. Toosi University of Technology*
P.O. Box 1618, 16315–1618 Tehran, Iran
[b] *KU Leuven, Faculty of Business and Economics - Campus Brussels*
Warmoesberg 26, 1000 Brussels, Belgium
[c] *Research Group in Mathematical Inequalities and Applications, School of Engineering and Science, Victoria University*
P.O. Box 14428, Melbourne City, 8001 MC Victoria, Australia
E-mail: mmjamei@kntu.ac.ir
E-mail(*corresp.*): edward.omey@kuleuven.be
E-mail: sever.dragomir@vu.edu.au

Received December 18, 2015; revised May 5, 2016; published online July 1, 2016

Abstract. In this paper, we define some integral transforms and obtain suitable bounds for them in order to introduce a main class of integral inequalities including Ostrowski and Ostrowski-Grüss inequalities and various kinds of new integral inequalities. In this sense, we also introduce a three point quadrature formula and obtain its error bounds.

Keywords: Ostrowski inequality, Ostrowski-Grüss inequality, integral transforms, error bounds, quadrature rules.

AMS Subject Classification: 46BXX; 26D10; 26D15; 26D20.

1 Introduction

Throughout the paper we will consider real and measurable functions which are defined on the finite interval $[a,b]$. For $p \geq 1$, let $L^p([a,b])$ denote the space of real functions with the bounded norm

$$\|f\|_p = (\int_a^b |f(t)|^p \, dt)^{1/p} < \infty$$

and $L^\infty([a,b])$ the space of bounded functions with the norm

$$\|f\|_\infty = \|f\| = \sup_{a \leq t \leq b} |f(t)| < \infty.$$

In many integral inequalities, the norms $\|.\|_p$ and/or $\|.\|$ are applied. For example, if $h \in L^1([a,b])$ and $g \in L^\infty([a,b])$, then we have

$$\left| \int_a^b g(t)h(t)dt \right| \leq \|h\|_1 \|g\|,$$

cf [17]. In 1934, Grüss [6] showed that if $\gamma_1 \leq g(x) \leq \Gamma_1$ and $\gamma_2 \leq h(x) \leq \Gamma_2$ for all $x \in [a,b]$ then

$$\left| \frac{1}{b-a} \int_a^b g(t)h(t)dt - \frac{1}{(b-a)^2} \int_a^b g(t)dt \int_a^b h(t)dt \right| \leq \frac{(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2)}{4}, \quad (1.1)$$

where $\Gamma_1, \Gamma_2, \gamma_1, \gamma_2$ are real numbers. The constant $1/4$ is the best possible number in the sense that it cannot be replaced by a smaller number. Another well-known inequality is due to Ostrowski [18]. In 1938 he proved that if f has a bounded derivative, then for all $x \in [a,b]$ we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \|f'\|_\infty . \quad (1.2)$$

In 1997, Dragomir and Wang [3] introduced a mixed type of inequalities (1.1) and (1.2) and named it the Ostrowski-Grüss inequality. In other words, if f is differentiable with $-\infty < \alpha_0 \leq f'(x) \leq \beta_0 < \infty$, $\forall x \in [a,b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}(x - \frac{a+b}{2}) \right| \leq \frac{(b-a)(\beta_0 - \alpha_0)}{4}. \quad (1.3)$$

Many improvements, extensions and generalizations of these inequalities have been presented in the literature up to now. We refer to the references [1, 2, 4, 5, 7, 8, 15, 16] at the end of this paper.

In this work, we present a general approach to obtain inequalities of the type (1.1) to (1.3). We introduce several families of integral transforms and obtain three types of inequalities, which cover many results available in the literature.

2 Inequalities for a class of integral transforms

2.1 Definitions

We first define four kinds of integral operators that are directly related to the results of this paper and give some specific examples for each of them.

2.1.1 The operator $L_K(.;x)$

Following the papers of Masjed-Jamei and Dragomir [9, 10, 11, 12, 13], let us introduce the integral transform $L_K(.;x)$ as follows. For fixed $x \in [a,b]$, if $K(t;x)$ denotes a kernel with $\|K\| < \infty$ then we define

$$L_K(f;x) = \int_a^b f(t)K(t;x)dt.$$

It is clear that if $\|K\| \equiv \sup_{a\le t\le b}|K(t;x)| < \infty$, then $|L_K(f;x)| \le \|K\|\,\|f\|_1$ provided that $f \in L_1([a,b])$. Also $L_K(f;x)$ is bilinear, i.e.

$$L_{A+B}(f;x) = L_A(f;x) + L_B(f;x)$$

and

$$L_K(f+g;x) = L_K(f;x) + L_K(g;x).$$

Many integral transforms are special cases of the above-mentioned operator. For example, the Laplace transform

$$\mathbf{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt, \;\; s>0,$$

the mean of f

$$\mathbf{M}(f) = \frac{1}{b-a}\int_a^b f(t)dt$$

and the Cesàro mean

$$C_\alpha(f)(x) = \frac{1}{x^\alpha}\int_0^x (x-t)^{\alpha-1} f(t)dt, \;\; \alpha \ge 1, \;\; 0 \le x \le 1$$

are all special cases of $L_K(f;x)$.

2.1.2 The operator $F_K(.;x)$

Suppose that f is a differentiable function. We can introduce the integral operator

$$F_K(f;x) = L_K(f';x) = \int_a^b f'(t)K(t;x)dt.$$

In this case if $\|K\| < \infty$, then $|F_K(f;x)| \le \|K\|\,\|f'\|_1$, i.e. $F_K(.;x)$ is well defined if $f' \in L^1([a,b])$. For example:

- If $K(t;x) = 1$ for $a \le t \le b$, then $F_K(f;x) = f(b) - f(a)$.
- If $K(t;x) = t$ for $a \le t \le b$, then $F_K(f;x) = bf(b) - af(a) - \int_a^b f(t)dt$ or equivalently

  $$\int_a^b f(t)dt = bf(b) - af(a) - F_K(f;x).$$

  This means that $F_K(f;x)$ can be seen as an error value in the integral approximation $\int_a^b f(t)dt \approx bf(b) - af(a)$.
- If $K(t;x) = t - w$ for $a \le t \le b$, then

  $$F_K(f;x) = bf(b) - af(a) - w(f(b) - f(a)) - \int_a^b f(t)dt.$$

- If $K(t;x) = \begin{cases} t-A & a \le t \le x, \\ t-B & x < t \le b, \end{cases}$ then the relation

$$F_K(f;x) = (B-A)f(x) - \int_a^b f(t)dt + f(b)(b-B) - f(a)(a-A)$$

can be seen as an error value in the integral approximation

$$\int_a^b f(t)dt \approx f(b)(b-B) - f(a)(a-A) + (B-A)f(x).$$

- If $K(t;x) = e^{-xt}$ for $x, t \ge 0$, then

$$F_K(f;x) = -f(0) + x\int_0^\infty f(t)e^{-xt}dt = x\mathbf{L}(f)(x) - f(0).$$

2.1.3 The operator $F_{K^\circ}(.;x)$

Let us take $\int_a^b K(t;x)dt = k(x) < \infty$ and then define

$$K^\circ(t;x) = K(t;x) - \frac{1}{b-a}\int_a^b K(t;x)\,dt = K(t;x) - \frac{1}{b-a}k(x).$$

By noting the above definition, another linear operator can be defined as

$$F_{K^\circ}(f;x) = \int_a^b f'(t)K^\circ(t;x)dt = F_K(f;x) - k(x)\frac{f(b)-f(a)}{b-a}.$$

For example:

- If $K(t;x) = 1$ for $a \le t \le b$, then $F_{K^0}(f;x) = 0$ because we have $F_K(f;x) = f(b) - f(a)$ and $k(x) = b-a$.
- If $K(t;x) = t$ for $a \le t \le b$, then $k(x) = (b^2-a^2)/2$ and $K^\circ(t;x) = K(t;x) - (a+b)/2$. Therefore

$$F_{K^\circ}(f;x) = \frac{b-a}{2}(f(b)-f(a)) - \int_a^b f(t)dt,$$

which can be seen as an error value in the integral approximation

$$\int_a^b f(t)dt \approx (b-a)(f(b)-f(a))/2.$$

2.1.4 The operator corresponding to higher order derivatives

When f has a second or higher order derivative, we can define the operator

$$F_K^{(j)}(f;x) = L_K(f^{(j)};x) = \int_a^b f^{(j)}(t)K(t;x)dt.$$

It is clear that $F_K^{(1)}(.;x) = F_K(.;x)$. Moreover, corresponding to the kernel K° we have

$$\begin{aligned} F_{K^\circ}^{(j)}(f;x) &= \int_a^b f^{(j)}(t)K^\circ(t;x)dt \\ &= F_K^{(j)}(f;x) - \frac{k(x)}{b-a}(f^{(j-1)}(b) - f^{(j-1)}(a)). \end{aligned}$$

For example:

- If $K(t;x) = t^2$ for $a \le t \le b$, then $k(x) = (b^3 - a^3)/3$ and

$$F_K^{(2)}(f;x) = b^2 f'(b) - a^2 f'(a) - 2(bf(b) - af(a)) + 2\int_a^b f(t)dt.$$

- If $K(t;x) = t$ for $a \le t \le b$, then $k(x) = (b^2 - a^2)/2$ and

$$F_K^{(2)}(f;x) = bf'(b) - af'(a) - (f(b) - f(a)).$$

2.2 Main results

In the following theorem, we present two-sided bounds for the operator $F_K(f;x)$ under various conditions on f'. Note that in parts a. and b. of the theorem we use the norm $\|K\| = \sup_{a\le t\le b}|K(t;x)|$.

Theorem 1. *a. Suppose that $f'(t) \le \beta(t)$ for any $a \le t \le b$. Then*

$$m_{1,a} \le F_K(f;x) \le M_{1,a},$$

where

$$m_{1,a} = L_K(\beta;x) - \|K\|\left(\int_a^b \beta(t)dt + f(a) - f(b)\right),$$

and

$$M_{1,a} = L_K(\beta;x) + \|K\|\left(\int_a^b \beta(t)dt + f(a) - f(b)\right).$$

b. Suppose that $\alpha(t) \le f'(t)$ for any $a \le t \le b$. Then

$$m_{1,b} \le F_K(f;x) \le M_{1,b},$$

where

$$m_{1,b} = L_K(\alpha;x) - \|K\|\left(\int_a^b \alpha(t)dt + f(b) - f(a)\right),$$

and

$$M_{1,b} = L_K(\alpha;x) + \|K\|\left(\int_a^b \alpha(t)dt + f(b) - f(a)\right).$$

c. Suppose that $\alpha(t) \le f'(t) \le \beta(t)$ for any $a \le t \le b$. Then

$$m_{1,c} \le F_K(f;x) \le M_{1,c},$$

where

$$m_{1,c} = \int_a^b \alpha(t)\frac{K(t;x)+|K(t;x)|}{2}dt + \int_a^b \beta(t)\frac{K(t;x)-|K(t;x)|}{2}dt,$$

and

$$M_{1,c} = \int_a^b \alpha(t)\frac{K(t;x)-|K(t;x)|}{2}dt + \int_a^b \beta(t)\frac{K(t;x)+|K(t;x)|}{2}dt.$$

Proof. First we prove part a. Since

$$F_K(f;x) - L_K(\beta;x) = \int_a^b (f'(t)-\beta(t))K(t;x)dt,$$

so

$$\begin{aligned} |F_K(f;x) - L_K(\beta;x)| &\leq \int_a^b (\beta(t)-f'(t))\,|K(t;x)|\,dt \\ &\leq \|K\|\left(\int_a^b \beta(t)dt + f(a) - f(b)\right), \end{aligned}$$

where $\|K\| = \sup_{a\le t\le b}|K(t;x)|$.
As the proof of part b. is similar, we withdraw it. To prove part c, first we have

$$F_K(f;x) - L_K\left(\frac{\alpha(t)+\beta(t)}{2};x\right) = \int_a^b (f'(t) - \frac{\alpha(t)+\beta(t)}{2})K(t;x)dt,$$

whence

$$\left|F_K(f;x) - L_K\left(\frac{\alpha(t)+\beta(t)}{2};x\right)\right| \leq \int_a^b \left|f'(t) - \frac{\alpha(t)+\beta(t)}{2}\right| |K(t;x)|\,dt.$$

But the condition $\alpha(t) \leq f'(t) \leq \beta(t)$ implies that

$$\left|f'(t) - \frac{\alpha(t)+\beta(t)}{2}\right| \leq \frac{\beta(t)-\alpha(t)}{2}.$$

Therefore

$$\left|F_K(f;x) - L_K\left(\frac{\alpha(t)+\beta(t)}{2};x\right)\right| \leq \int_a^b \frac{\beta(t)-\alpha(t)}{2}\,|K(t;x)|\,dt,$$

and it follows that

$$F_K(f;x) \leq \int_a^b \frac{\alpha(t)+\beta(t)}{2}K(t;x)dt + \int_a^b \frac{\beta(t)-\alpha(t)}{2}\,|K(t;x)|\,dt \equiv M_{1,c}.$$

Similarly, one can obtain

$$F_K(f;x) \geq \int_a^b \frac{\alpha(t)+\beta(t)}{2}K(t;x)dt - \int_a^b \frac{\beta(t)-\alpha(t)}{2}\,|K(t;x)|\,dt \equiv m_{1,c},$$

which proves the theorem. □

When the functions $\alpha(t)$ and $\beta(t)$ are constant numbers, the following Corollary is derived.

Corollary 1. a. Suppose that $f'(t) \le \beta_0$ for any $a \le t \le b$. Then by recalling that $k(x) = \int_a^b K(t;x)\,dt$ we have

$$|F_K(f;x) - \beta_0 k(x)| \le \|K\| \left(\beta_0(b-a) + f(a) - f(b)\right).$$

b. Suppose that $\alpha_0 \le f'(t)$ for any $a \le t \le b$. Then

$$|F_K(f;x) - \alpha_0 k(x)| \le \|K\| \left(\alpha_0(b-a) + f(b) - f(a)\right).$$

c. Suppose that $\alpha_0 \le f'(t) \le \beta_0$ for any $a \le t \le b$. Then

$$\left|F_K(f;x) - \frac{\alpha_0+\beta_0}{2} k(x)\right| \le \frac{\beta_0-\alpha_0}{2} \int_a^b |K(t;x)|\,dt.$$

Corollary 2. (i) If $K \ge 0$ in Theorem 1.c., then

$$L_K(\beta;x) \le F_K(f;x) \le L_K(\alpha;x).$$

(ii) If $|f'(t)| \le \theta(t)$ for any $a \le t \le b$ then $|F_K(f;x)| \le L_{|K|}(\theta;x)$.

Theorem 1 can be also formulated for the operators $F_{K^\circ}(.;x)$ and $F_K^{(k)}(.;x)$. Corollary 3 formulates a result for $F_{K^\circ}(.;x)$ and Corollary 4 a result for the case where f has a second derivative.

Corollary 3. Suppose that $\alpha_0 \le f'(t) \le \beta_0$ for any $a \le t \le b$. Then we have

$$\left|F_K(f;x) - \frac{1}{b-a} k(x)(f(b)-f(a)) - \frac{\alpha_0+\beta_0}{2} k(x)\right| \le \frac{\beta_0-\alpha_0}{2} \int_a^b |K^\circ(t;x)|\,dt.$$

Corollary 4. Suppose that $\alpha(t) \le f''(t) \le \beta(t)$ for any $a \le t \le b$. Then

$$m_{1,c} \le F_K^{(2)}(f;x) \le M_{1,c},$$

where

$$F_K^{(2)}(f;x) = \int_a^b f''(t)K(t;x)\,dt,$$

and $m_{1,c}$ and $M_{1,c}$ are given as in Theorem 1.c.

Note that an important advantage of the presented theorem and its corollaries is that the bounds m_1 and M_1 are formulated in terms of α, β and K and not in terms of f, f' or some norms of f or f'. In many papers, see e.g. [1, 2, 3, 4, 7, 8, 11] the bounds are formulated in terms of $\|f\|$ or $\|f'\|$, which are usually hard to calculate.

2.3 Special cases

In this section, we apply the results of the previous section for a number of special cases. We follow the approach used in [10] and consider the general kernel

$$K(t;x) = \begin{cases} u(t), & a \le t \le x, \\ v(t), & x < t \le b, \end{cases} \tag{2.1}$$

in which u, v are arbitrary integrable functions such that $u \in C^1([a,x])$ and $v \in C^1((x,b])$. By using partial integration, it is easy to verify that the following relation holds for the kernel (2.1):

$$\begin{aligned} F_K(f;x) &= (u(x)-v(x))f(x) + v(b)f(b) - u(a)f(a) \\ &\quad - \int_a^x u'(t)f(t)dt - \int_x^b v'(t)f(t)dt. \end{aligned}$$

Moreover we have

$$k(x) = \int_a^b K(t;x)dt = \int_a^x u(t)dt + \int_x^b v(t)dt.$$

Hence, by considering the kernel $K^\circ(t;x) = K(t;x) - k(x)/(b-a)$, we find that

$$\begin{aligned} F_{K^\circ}(f;x) &= (u(x)-v(x))f(x) + v(b)f(b) - u(a)f(a) - \int_a^x u'(t)f(t)dt \\ &\quad - \int_x^b v'(t)f(t)dt - \frac{f(b)-f(a)}{b-a}\left(\int_a^x u(t)\,dt + \int_x^b v(t)\,dt\right). \end{aligned}$$

Corollary 5. Suppose that $\alpha(t) \le f'(t) \le \beta(t)$ for any $a \le t \le b$. Then

$$m_{1,c} \le F_K(f;x) \le M_{1,c},$$

and

$$m_1^0 \le F_K(f;x) - k(x)(f(b)-f(a))/(b-a) \le M_1^0,$$

where $m_{1,c}, M_{1,c}$ are given in Theorem 1.c and

$$m_1^0 = \int_a^b \alpha(t)\frac{K^\circ(t;x) + |K^\circ(t;x)|}{2}dt + \int_a^b \beta(t)\frac{K^\circ(t;x) - |K^\circ(t;x)|}{2}dt,$$

and

$$M_1^0 = \int_a^b \alpha(t)\frac{K^\circ(t;x) - |K^\circ(t;x)|}{2}dt + \int_a^b \beta(t)\frac{K^\circ(t;x) + |K^\circ(t;x)|}{2}dt.$$

For special choices of u and v in the kernel (2.1), let us consider some known examples.

Example 1. By taking the kernel (2.1) as

$$K(t;x) = \begin{cases} t-a, & a \le t \le x, \\ t-b, & x < t \le b, \end{cases}$$

we have

$$F_K(f;x) = (b-a)f(x) - \int_a^b f(t)dt \quad \text{and} \quad k(x) = (b-a)(x - \frac{a+b}{2}).$$

As an alternative to inequality (1.2), Theorem 1.c gives

$$m_1 \le (b-a)f(x) - \int_a^b f(t)\, dt \le M_1,$$

where

$$m_1 = \int_a^x \alpha(t)(t-a)dt + \int_x^b \beta(t)(t-b)dt,$$

and

$$M_2 = \int_a^x \beta(t)(t-a)dt + \int_x^b \alpha(t)(t-b)dt,$$

which is a result of [12].

Example 2. If we take the kernel (2.1) as

$$K(t;x) = \begin{cases} t-A, & a \le t \le x, \\ t-B, & x < t \le b, \end{cases}$$

then we obtain

$$\begin{aligned} F_K(f;x) &= (B-A)f(x) - \int_a^b f(t)dt + f(b)(b-B) - f(a)(a-A), \\ k(x) &= \int_a^x (t-A)dt + \int_x^b (t-B)dt \\ &= (B-A)(x - \frac{A+B}{2}) - \frac{(a-A)^2 - (b-B)^2}{2}. \end{aligned}$$

Now different choices of A, B lead to different interesting results. For instance,
a) If $A = B = 0$ then

$$\begin{aligned} F_K(f;x) &= bf(b) - af(a) - \int_a^b f(t)dt, \quad k(x) = \frac{b^2-a^2}{2}, \\ F_{K^\circ}(f;x) &= bf(b) - af(a) - \int_a^b f(t)dt - \frac{b+a}{2}(f(b) - f(a)). \end{aligned}$$

b) If $A = a + \theta$ and $B = b - \delta$, then $B - A = b - a - \delta - \theta$ and

$$f(b)(b-B) - f(a)(a-A) = \delta f(b) + \theta f(a).$$

Hence, we see that

$$F_K(f;x) = (b-a-\delta-\theta)f(x) - \int_a^b f(t)dt + \theta f(a) + \delta f(b),$$

$$F_{K^\circ}(f;x) = F_K(f;x) - \frac{a+b}{2}(f(b)-f(a)).$$

This means that

$$\int_a^b f(t)dt = (b-a-\delta-\theta)f(x) - \frac{a+b}{2}(f(b)-f(a)) + \theta f(a) + \delta f(b) + F_{K^\circ}(f;x),$$

where Corollary 6 can be employed to find bounds for the approximate error $F_{K^\circ}(f;x)$.

c) If we choose A, B as

$$A = \frac{bf(b)-af(a)}{f(b)-f(a)} - w\frac{f(b)}{f(b)-f(a)} \quad \text{and} \quad B = A + w,$$

then $B - A = w$ and

$$f(b)(b-B) - f(a)(a-A) = bf(b) - af(a) - f(b)w + (f(a)-f(b))A = 0.$$

Hence we find

$$F_K(f;x) = wf(x) - \int_a^b f(t)dt.$$

d) Finally, if we choose A, B as

$$A = \lambda(x-b) + c \quad \text{and} \quad B = \lambda(x-a) + c,$$

then $B - A = \lambda(b-a)$ and

$$\begin{aligned} &f(b)(b-B) - f(a)(a-A) \\ &\quad = bf(b) - af(a) - f(b)(\lambda(x-a)+c) + f(a)(\lambda(x-b)+c) \\ &\quad = f(b)(b-c+a\lambda) - f(a)(a-c+b\lambda) - \lambda(f(b)-f(a))x. \end{aligned}$$

3 Application in quadrature rules

In this section, we study in detail a general three point quadrature formula as

$$\int_a^b f(t)dt \cong \lambda_1 f(a) + (b-a-(\lambda_1+\lambda_2))f(c) + \lambda_2 f(b), \tag{3.1}$$

in which λ_1, λ_2 are two free parameters and $c \in [a,b]$.

According to previous sections, the error value of the approximation (3.1) can be written as

$$\lambda_1 f(a) + (b-a-(\lambda_1+\lambda_2))f(c) + \lambda_2 f(b) - \int_a^b f(t)dt = F_K(f;c), \tag{3.2}$$

where, cf. Example 2,

$$K(t;c) = \begin{cases} t-a-\lambda_1 & a \le t \le c, \\ t-b+\lambda_2 & c < t \le b. \end{cases}$$

Corollary 6. Suppose that $\alpha(t) \le f'(t) \le \beta(t)$ for any $a \le t \le b$. Then the residue in (3.2) can be bounded as

$$m_1^* \le F_K(f;c) \le M_1^*,$$

where

$$m_1^* = \int_a^b \alpha(t)\frac{K(t;c)+|K(t;c)|}{2}dt + \int_a^b \beta(t)\frac{K(t;c)-|K(t;c)|}{2}dt,$$
$$M_1^* = \int_a^b \alpha(t)\frac{K(t;c)-|K(t;c)|}{2}dt + \int_a^b \beta(t)\frac{K(t;x)+|K(t;c)|}{2}dt.$$

On the other hand, m_1^* (and M_1^*) can be rewritten as

$$m_1^* = I(i) + I(ii) + II(i) + II(ii),$$

where

$$I(i) = \int_a^c \alpha(t)\frac{t-a-\lambda_1+|t-a-\lambda_1|}{2}dt,$$
$$I(ii) = \int_c^b \alpha(t)\frac{t-b+\lambda_2+|t-b+\lambda_2|}{2}dt,$$
$$II(i) = \int_a^c \beta(t)\frac{t-a-\lambda_1-|t-a-\lambda_1|}{2}dt,$$
$$II(ii) = \int_c^b \beta(t)\frac{t-b+\lambda_2-|t-b+\lambda_2|}{2}dt.$$

Now consider the following particular cases.
a) If $\lambda_1 = \lambda_2 = 0$ and $c = (a+b)/2$ in (3.1), then (3.2) changes to

$$\int_a^b f(t)dt = (b-a)f(c) - F_K(f;c),$$

such that $m_{1,1}^* \le F_K(f;c) \le M_{1,1}^*$ and

$$\begin{aligned}
I(i) &= \int_0^{c-a} \alpha(z+a)\frac{z+|z|}{2}dz = \int_0^{(b-a)/2} \alpha(z+a)zdz,\\
I(ii) &= \int_{c-b}^0 \alpha(z+b)\frac{z+|z|}{2}dz = 0,\\
II(i) &= \int_0^{c-a} \beta(z+a)\frac{z-|z|}{2}dz = 0,\\
II(ii) &= \int_{c-b}^0 \beta(z+b)\frac{z-|z|}{2}dt = -\int_0^{(b-a)/2} \beta(b-z)zdz.
\end{aligned}$$

In this way, if $\alpha(t) = \alpha_0 + \alpha_1 t$ and $\beta(t) = \beta_0 + \beta_1 t$ are considered, then we obtain

$$I(i) = \frac{(b-a)^2}{4}(\frac{\alpha_0 + a\alpha_1}{2} + \alpha_1\frac{b-a}{6}),$$

and

$$II(ii) = \frac{(b-a)^2}{4}\left(-\frac{\beta_0+\beta_1 b}{2} + \beta_1\frac{b-a}{6}\right),$$

which eventually yields

$$m^*_{1,1} = \frac{(b-a)^2}{4}\left((\alpha_1+\beta_1)\frac{b-a}{6} + \frac{\alpha_0+a\alpha_1-(\beta_0+\beta_1 b)}{2}\right).$$

In a similar way we can find that

$$M^*_{1,1} = \frac{(b-a)^2}{4}\left((\alpha_1+\beta_1)\frac{b-a}{6} + \frac{-(\alpha_0+a\alpha_1)+\beta_0+\beta_1 b}{2}\right).$$

The final result is a new bound for the error of midpoint rule as

$$m^*_{1,1} \le (b-a)f\left(\frac{a+b}{2}\right) - \int_a^b f(t)dt \le M^*_{1,1}.$$

b) If $\lambda_1=\lambda_2=\lambda=(b-a)/2$ and $c=(a+b)/2$ in (3.1), then (3.2) changes to

$$F_K(f;c) = \frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(t)dt, \tag{3.3}$$

such that $m^*_{1,2} \le F_K(f;c) \le M^*_{1,2}$ and

$$\begin{aligned}
I(i) &= \int_a^c \alpha(t)\frac{t-(a+b)/2+|t-(a+b)/2|}{2}dt,\\
I(ii) &= \int_c^b \alpha(t)\frac{t-(a+b)/2+|t-(a+b)/2|}{2}dt,\\
II(i) &= \int_a^c \beta(t)\frac{t-(a+b)/2-|t-(a+b)/2|}{2}dt,\\
II(ii) &= \int_c^b \beta(t)\frac{t-(a+b)/2-|t-(a+b)/2|}{2}dt.
\end{aligned}$$

Therefore

$$m^*_{1,2} = \int_{(a-b)/2}^{(b-a)/2}\left(\alpha(z+c)\frac{z+|z|}{2} + \beta(z+c)\frac{z-|z|}{2}\right)dz,$$

and in a similar way we can find that

$$M^*_{1,2} = \int_{(a-b)/2}^{(b-a)/2}\left(\alpha(z+c)\frac{z-|z|}{2} + \beta(z+c)\frac{z+|z|}{2}\right)dz.$$

The above results give new bounds for the trapezoidal rule (3.3) as follows

$$m^*_{1,2} \le \frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(t)dt \le M^*_{1,2}.$$

For instance, if $\alpha(t) = \alpha_0$ and $\beta(t) = \beta_0$ then

$$m^*_{1,2} = \frac{(b-a)^2}{8}(\alpha_0 - \beta_0) \quad \text{and} \quad M^*_{1,2} = \frac{(b-a)^2}{8}(\beta_0 - \alpha_0),$$

leading to the well known inequality

$$\left| \frac{b-a}{2}(f(a) + f(b)) - \int_a^b f(t)dt \right| \leq \frac{(b-a)^2}{8}(\beta_0 - \alpha_0).$$

c) If $0 < \lambda_1 = \lambda_2 = \lambda < (b-a)/2$ and $c = (a+b)/2$ in (3.1), then (3.2) changes to a generalized Simpson rule, cf. [9, 14, 19] as

$$\lambda(f(a) + f(b)) + (b - a - 2\lambda)f(\frac{a+b}{2}) - \int_a^b f(t)dt = F_K(f; c),$$

such that $m^*_{1,3} \leq F_K(f; c) \leq M^*_{1,3}$ and

$$\begin{aligned}
I(i) &= \int_{-\lambda}^{(b-a)/2-\lambda} \alpha(z + a + \lambda)\frac{z + |z|}{2}dz = \int_0^{(b-a)/2-\lambda} \alpha(z + a + \lambda)zdz, \\
I(ii) &= \int_{(a-b)/2-\lambda}^{\lambda} \alpha(z + b - \lambda)\frac{z + |z|}{2}dz = \int_0^{\lambda} \alpha(z + b - \lambda)zdz, \\
II(i) &= \int_{-\lambda}^{(b-a)/2-\lambda} \beta(z + a + \lambda)\frac{z - |z|}{2}dz = \int_{-\lambda}^{0} \beta(z + a + \lambda)zdz, \\
II(ii) &= \int_{(a-b)/2-\lambda}^{\lambda} \beta(z + b - \lambda)\frac{z - |z|}{2}dz = \int_{(a-b)/2-\lambda}^{0} \beta(z + b - \lambda)zdz.
\end{aligned}$$

For instance, if $\alpha(t) = \alpha_0$ and $\beta(t) = \beta_0$ then we have

$$\begin{aligned}
I(i) &= \alpha_0\frac{(b-a)/2 - \lambda)^2}{2}, \; I(ii) = \alpha_0\frac{\lambda^2}{2}, \\
II(i) &= -\beta_0\frac{\lambda^2}{2} \quad \text{and} \quad II(ii) = -\beta_0\frac{(a-b)/2 - \lambda)^2}{2},
\end{aligned}$$

which yields

$$\begin{aligned}
m^*_{1,3} &= (\alpha_0 - \beta_0)(\lambda^2 + \frac{(b-a)^2}{8} - \frac{(b-a)\lambda}{2}), \\
M^*_{1,3} &= (\beta_0 - \alpha_0)(\lambda^2 + \frac{(b-a)^2}{8} - \frac{(b-a)\lambda}{2}).
\end{aligned}$$

In this case, we can conclude that

$$\begin{aligned}
&\left|\lambda(f(a) + f(b)) + (b - a - 2\lambda)f(\frac{a+b}{2}) - \int_a^b f(t)dt\right| \\
&\leq (\beta_0 - \alpha_0)(\lambda^2 + \frac{(b-a)^2}{8} - \frac{(b-a)\lambda}{2}).
\end{aligned}$$

For $\lambda = (b-a)/6$ we find back a result of [14] in the form

$$\left| \frac{b-a}{6}(f(a)+f(b)+4f(\frac{a+b}{2})) - \int_a^b f(t)dt \right| \le \frac{5}{72}(\beta_0-\alpha_0)(b-a)^2,$$

and for $\lambda = (b-a)/4$, the following sharper bound [9] is derived

$$\left| \frac{b-a}{4}(f(a)+f(b)+2f(\frac{a+b}{2})) - \int_a^b f(t)dt \right| \le \frac{1}{16}(\beta_0-\alpha_0)(b-a)^2. \quad (3.4)$$

3.1 Numerical experiments

In this section, we present some numerical evidence that illustrates our given error bounds. For example, as we pointed out, if in the three point quadrature (3.1) we take $\lambda_1 = \lambda_2 = (b-a)/4$, $c = (a+b)/2$ and $\alpha_0 \le f'(t) \le \beta_0$ for any $t \in [a,b]$, then according to (3.4), the quadrature rule

$$\int_a^b f(t)\,dt \cong \frac{b-a}{4}\left(f(a) + 2\,f(\frac{a+b}{2}) + f(b)\right), \quad (3.5)$$

has a minimum error bound equal to

$$E_0 = (b-a)^2\,(\beta_0-\alpha_0)/16.$$

Clearly E_0 shows whenever the values $(b-a)^2$ and $\beta_0-\alpha_0$ are simultaneously small, the quadrature (3.5) is more accurate as Table 1 illustrates it.

Table 1. Numerical results for $\int_0^x t^{5/2}e^{-t}\,dt$ by using the quadrature (3.5).

x	Exact value	quadrature (3.5)	|Error|	bound E_0
0.1	0.000083603050612	0.000098121349521	0.000014518298909	0.000042920207297
0.2	0.000875499898740	0.001018429762975	0.000142929864234	0.000421069652405
0.3	0.003350778159436	0.003863958075520	0.000513179916083	0.001506394582400
0.4	0.008494639897631	0.009712342132216	0.001217702234585	0.003561160019326
0.5	0.017186588186474	0.019486941820256	0.002300353633783	0.006701280351256
0.6	0.030153102984455	0.033911410892320	0.003758307907865	0.010904011616469
0.7	0.047951752295790	0.053501364793226	0.005549612497436	0.016032067221619
0.8	0.070972179129481	0.078574829297206	0.007602650167725	0.021862927150639
0.9	0.099446467041305	0.109272148157979	0.009825681116674	0.028117956150800

In this table, we have considered a special case of the incomplete gamma function as $\Gamma(x;7/2) = \int_0^x t^{5/2}e^{-t}dt$ for different values of $x = 0.1\,(0.1)\,0.9$. By noting the value E_0, the aforesaid table shows whenever x is smaller the bound E_0 is also smaller and we have therefore a better result for the quadrature (3.5). In this direction note that if $f(t) = t^{5/2}\mathrm{e}^{-t}$, then we respectively have

$$f'(t) = t^{3/2}\mathrm{e}^{-t}\,(-t+5/2) \ge 0, \ \ \forall t \in [0, 0.9]$$

and

$$f''(t) = \frac{1}{4}t^{1/2}\mathrm{e}^{-t}\left(4t^2 - 20t + 15\right) \ge 0, \ \ \forall t \in [0, 0.9].$$

References

[1] G.A. Anastassiou. Ostrowski type inequalities. *Proceedings of the American Mathematical Society*, **123**(12):3775–3781, 1995. http://dx.doi.org/10.1090/S0002-9939-1995-1283537-3.

[2] X.-L. Cheng. Improvement of some Ostrowski-Grüss type inequalities. *Computers & Mathematics with Applications*, **42**(1-2):109–114, 2001. http://dx.doi.org/10.1016/S0898-1221(01)00135-3.

[3] S.S. Dragomir and S. Wang. An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules. *Computers & Mathematics with Applications*, **33**(11):15–20, 1997. http://dx.doi.org/10.1016/S0898-1221(97)00084-9.

[4] S.S. Dragomir and S. Wang. Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules. *Applied Mathematics Letters*, **11**(1):105–109, 1998. http://dx.doi.org/10.1016/S0893-9659(97)00142-0.

[5] I. Fedotov and S.S. Dragomir. An inequality of Ostrowski type and its applications for Simpson's rule and special means. *Mathematical Inequalities & Applications*, **2**(4):491–499, 1999. http://dx.doi.org/10.7153/mia-02-39.

[6] G. Grüss. Über das maximum des absoluten betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$. *Math. Z.*, **39**:215–226, 1935.

[7] V.N. Huy and Q.-A. Ngo. New bounds for the Ostrowski-like type inequalities. *Bulletin of the Korean Mathematical Society*, **48**:95–104, 2011. http://dx.doi.org/10.4134/BKMS.2011.48.1.095.

[8] Z. Liu. Some Ostrowski-Grüss type inequalities and applications. *Computers & Mathematics with Applications*, **53**(1):73–79, 2007. http://dx.doi.org/10.1016/j.camwa.2006.12.021.

[9] M. Masjed-Jamei. A linear constructive approximation for integrable functions and a parametric quadrature model based on a generalization of Ostrowski-Grüss type inequalities. *Electronic Transactions on Numerical Analysis*, **38**:218–232, 2011.

[10] M. Masjed-Jamei. A certain class of weighted approximations for integrable functions and applications. *Numerical Functional Analysis and Optimization*, **34**(11):1224–1244, 2013. http://dx.doi.org/10.1080/01630563.2013.774413.

[11] M. Masjed-Jamei and S.S. Dragomir. A new generalization of the Ostrowski inequality and applications. *Filomat*, **25**(1):115–123, 2011. http://dx.doi.org/10.2298/FIL1101115M.

[12] M. Masjed-Jamei and S.S. Dragomir. An analogue of the Ostrowski inequality and applications. *Filomat*, **28**:373–381, 2014. http://dx.doi.org/10.2298/FIL1402373M.

[13] M. Masjed-Jamei and S.S. Dragomir. A generalization of the Ostrowski-Grüss inequality. *Analysis and Applications*, **12**(02):117–130, 2014. http://dx.doi.org/10.1142/S0219530513500309.

[14] M. Matić. Improvement of some inequalities of Euler-Grüss type. *Computers & Mathematics with Applications*, **46**(8-9):1325–1336, 2003. http://dx.doi.org/10.1016/S0898-1221(03)90222-7.

[15] M. Matić, Pečarić and N. Ujević. Improvement and further generalization of inequalities of Ostrowski-Grüss type. *Computers & Mathematics with Applications*, **39**(3-4):161–175, 2000. http://dx.doi.org/10.1016/S0898-1221(99)00342-9.

[16] M. Matić, J. Pečarić and N. Ujević. Generalization of an inequality of Ostrowski type and some related results. *Indian Journal of Mathematics*, **44**(2):189–209, 2002.

[17] D.S. Mitrinović, J.E. Pečarić and A.M. Fink. *Classical and New Inequalities in Analysis*, volume 61 of *Mathematics and Its Applications (East European Series)*. Springer Netherlands, 1993. http://dx.doi.org/10.1007/978-94-017-1043-5.

[18] A. Ostrowski. Über die absolutabweichung einer differentiiebaren funktion van ihrem integralmittelwert. *Commentarii Mathematici Helvetici*, **10**(1):226–227, 1938. http://dx.doi.org/10.1007/BF01214290.

[19] N. Ujević. New error bounds for the Simpson's quadrature rule and applications. *Computers & Mathematics with Applications*, **53**(1):64–72, 2007. http://dx.doi.org/10.1016/j.camwa.2006.12.008.