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On Boundary Value Problems for φ-Laplacian on the Semi-Infinite Interval

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Abstract. The Dirichlet problem and the problem with functional boundary condition for φ-Laplacian on the semi-infinite interval are studied as well as solutions between the lower and upper functions.

Keywords: boundary value problem, φ-Laplacian, lower and upper functions.

AMS Subject Classification: 34B15.

1 Introduction

The Dirichlet problem on the bounded interval for

$$\begin{aligned} &(\varphi(t,x,x'))' = f(t,x,x'), \quad t \in I = [a,b], \\ &x(a) = A, \quad x(b) = B, \end{aligned}$$

is well studied [1,2,5,10]. The method of lower and upper functions is often used for proving the existence of a solution of this problem [13]. Let α and β be a lower and an upper functions. Then under additional conditions of the Nagumo or Schrader type one can prove the existence of a solution x of the Dirichlet problem satisfying the estimates $\alpha \le x \le \beta$. In [11] it is proved that there exists a generalized solution of the Dirichlet problem provided that there only exist lower and upper functions. A generalized solution has good properties. The set of generalized solutions of the φ-Laplacian between a lower and an upper function is compact (in the sense of [13]) and has the minimum and maximum generalized solutions, and the Dirichlet problem is solvable without additional conditions of the Nagumo or Schrader type [18]. The derivative of a generalized solution for the φ-Laplacian can be equal to $+\infty$ and $-\infty$. But if we have the additional conditions of the Nagumo or Schrader type then a generalized solution is a solution (in the sense of Def. 1). A historical survey of the basic information on the theory of the φ-Laplacian can be found in $[1,2,3,4,5,6,10,15]$.

2 Results

Consider the boundary value problem

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad t \in I = [a,+\infty), \tag{2.1}$$

$$x(a) = A \in [\alpha(a), \beta(a)], \quad \alpha \le x \le \beta, \tag{2.2}$$

where $a \in R$, α is a lower function, β is an upper function and their definitions will be given later.

Let for each compact interval $J = [a,b]$, $b \in (a,+\infty)$ and for all $x, x' \in R$ it will be $\varphi_J(t,x,x') = \varphi(t,x,x')$, $f_J(t,x,x') = f(t,x,x')$. Let us assume that for all $(t,x,x') \in J \times R \times R$ and for each compact interval J the function $\varphi_J : J \times R^2 \to R$ satisfies the conditions: $\varphi_J \in C(J \times R^2, R)$ and for fixed t and x is strictly increasing on x'. The function $f_J : J \times R^2 \to R$ satisfies the Caratheodory conditions: the function $f_J(t,\cdot,\cdot)$ is measurable on J for fixed $x, x' \in R$, the function $f_J(\cdot,x,x')$ is continuous on R^2 for fixed $t \in J$ and for each compact set $P \subset R^2$ there exists a function $g \in L(J,R)$ such that the inequality $|f_J(t,x,x')| \le g(t)$ holds for all $(t,x,x') \in J \times P$.

Definition 1. The function $x \in C^1(I,R)$ is a solution of the equation (2.1), if the function $\varphi_J(t,x(t),x'(t))$ is absolute continuous and the equation (2.1) fulfils almost everywhere for each J. The set of solutions of the boundary value problem (2.1)–(2.2) will be denoted by S.

To prove the main theorem, we need the definitions, presented below, of generalized upper and lower functions and a generalized solution as well as Theorem 1 (see [11]– [14]).

Definition 2. The class $BB^+(I,R)$ consists of functions $\alpha\colon I \to R$ satisfying the following conditions: for each $t \in (a,b]$, there exists a left derivative $\alpha'_l(t)$ and a limit $\lim\limits_{\tau\to t-} \alpha'_l(\tau)$, and moreover $\alpha'_l(t) \ge \lim\limits_{\tau\to t-} \alpha'_l(\tau)$; for each $t \in [a,b)$; there exists a right derivative $\alpha'_r(t)$ and a limit $\lim\limits_{\tau\to t+} \alpha'_r(\tau)$, and moreover, $\alpha'_r(t) \le \lim\limits_{\tau\to t+} \alpha'_r(\tau)$, and $\alpha'_l(t) \le \alpha'_r(t)$ for each $t \in (a,b)$.

The class $BB^-(I,R)$ consists of functions $\beta\colon I \to R$ satisfying the following conditions: for each $t \in (a,b]$ there exists a left derivative $\beta'_l(t)$ and a limit $\lim\limits_{\tau\to t-} \beta'_l(\tau)$, and moreover $\beta'_l(t) \le \lim\limits_{\tau\to t-} \beta'_l(\tau)$; for each $t \in [a,b)$; there exists a right derivative $\beta'_r(t)$ and a limit $\lim\limits_{\tau\to t+} \beta'_r(\tau)$, and moreover, $\beta'_r(t) \ge \lim\limits_{\tau\to t+} \beta'_r(\tau)$, and $\beta'_l(t) \ge \beta'_r(t)$ for each $t \in (a,b)$.

Definition 3. We say that a bounded function $\alpha \in BB^+(I,R)$ is a *generalized lower function* and write $\alpha \in AG(I,R)$ if, for any interval $[c,d] \subset I$ on which α satisfies the Lipschitz condition, the inequality

$$\varphi(t_2,\alpha(t_2),\alpha'(t_2)) - \varphi(t_1,\alpha(t_1),\alpha'(t_1)) \ge \int_{t_1}^{t_2} f(s,\alpha(s),\alpha'(s))\, ds$$

holds for arbitrary points $t_1 \in (c,d)$ and $t_2 \in (t_1,d)$ at which the derivative exists.

We say that a bounded function $\beta \in BB^-(I,R)$ is a *generalized upper function* and write $\beta \in BG(I,R)$ if, for any interval $[c,d] \subset I$ on which β satisfies the Lipschitz condition, the inequality

$$\varphi(t_2,\beta(t_2),\beta'(t_2)) - \varphi(t_1,\beta(t_1),\beta'(t_1)) \le \int_{t_1}^{t_2} f(s,\beta(s),\beta'(s))\,ds$$

holds for arbitrary points $t_1 \in (c,d)$ and $t_2 \in (t_1,d)$ at which the derivative exists.

A function $x\colon I \to R$ is called a *generalized solution* if $x \in AG(I,R) \cap BG(I,R)$. A set of generalized solutions will be denoted $SG(I,R)$.

At each point, a generalized solution has a derivative x' which may be equal to $-\infty$ or $+\infty$ and is continuous on $[-\infty,+\infty]$. If a derivative x' does not take the values $-\infty$ and $+\infty$ on some interval, then x is a solution on that interval.

The following assertion was proved in [17].

Theorem 1. *Let $\alpha \in AG(J,R)$, $\beta \in BG(J,R)$ and $\alpha \le \beta$. Then there exists a* generalized solution *of Dirichle's problem*

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad x(a) = A, \quad x(b) = B, \quad \alpha \le x \le \beta \tag{2.3}$$

for all $A \in [\alpha(a),\beta(a)]$ and $B \in [\alpha(b),\beta(b)]$.

For solvability of the boundary value problem (2.1)–(2.2) some conditions on α and β and additional compactness conditions are needed. The Nagumo condition [16] for φ-Laplacian and the Schrader's condition [18] are the sufficient conditions of compactness. We use the following compactness condition.

Definition 4. We shall say that the conditions of compactness are fulfilled on interval J if for every $A \in [\alpha(a),\beta(a)]$ and $B \in [\alpha(b),\beta(b)]$ any generalized solution of the Dirichlet's problem (2.3) is a solution.

It is clear that this condition is weaker then the Schrader's condition. If a generalized solution has not infinite derivatives then it is usual solution. The conditions by Nagumo and Schrader forbid solutions with infinite derivatives. The above condition of compactness allows to improve the Schrader condition (for details one can consult the work [12]). If the Nagumo function can be found then a solution of boundary value problem cannot have infinite derivatives. The Schrader condition simply forbid infinite derivatives. Which condition to use depends on a problem to be studied.

Remark 1. The Dirichlet's problem (2.3) has a solution if $\alpha \in AG(J,R)$, $\beta \in BG(J,R)$, $\alpha \le \beta$ and the conditions of compactness are fulfilled.

Theorem 2. *The boundary value problem (2.1)–(2.2) has a generalized solution if $\alpha \in AG(I,R)$ and $\beta \in BG(I,R)$.*

Proof. Suppose a sequence $b_i \in (a,+\infty), i = 1,2,\ldots$ is increasing and tends to $+\infty$, $B_i \in [\alpha(b_i),\beta(b_i)]$ and x_i is a generalized solution of the Dirichlet's problem

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad x(a) = A, \quad x(b_i) = B_i,$$

such that $\alpha(t) \le x(t) \le \beta(t)$, $t \in [a, b_i]$.

Without loss of generality we can assume that the sequence x_i converges in all rational points to a function x lying between α and β. Note that without loss of generality for any compact interval J from boundedness of x and from the Lagrange's formula follows that we can found the interval $[c, d] \subset J$, for which

$$\sup\{|\ x_i'(t)\ |: i \in \{1, 2, ...\}, t \in [c, d]\} < +\infty.$$

It is evident that x satisfies the Lipschitz's condition on $[c, d]$. Consequently, we may redefine x up to continuity on the whole interval $[c, d]$. Thus redefined function x will satisfy the Lipschitz's condition. The sequence $x_i(t)$ converges to $x(t)$ for all $t \in [c, d]$. It follows from the Lipschitz's condition.

It is evident that the derivatives $x_i'(t)$ converge to the derivatives $x'(t)$ for all $t \in [c, d]$. Consequently, $x(t)$ is a solution of the equation (2.1) on an interval $[c, d]$. Continuing construction of the function $x(t)$ to the right and to the left, we obtain the existence of a solution of the equation (2.1) on the maximal interval (c_1, d_1) or $[a, d_1)$. If $c_1 > a$, then $\lim\limits_{t \to c_1+} x'(t)$ equals $-\infty$ or $+\infty$. Similarly, if $d_1 < +\infty$, then $\lim\limits_{t \to d_1-} x'(t)$ equals $-\infty$ or $+\infty$. Continuing this construction, we will find an open in I the set J_1, where the function $x(t)$ is defined such that $x(t)$ is a solution of the equation (2.1) on subintervals of J_1.

The set $J_2 = I \setminus J_1$ is closed and nowhere dense. For $t \in J_2$ the limit $\lim\limits_{i \to \infty} x_i'(t)$ equals $-\infty$ or $+\infty$. Indeed, supposing the contrary and arguing as above we have that $t \in J_1$. Let us define $x(t)$ on irrational points of J_2. If $\tau \in J_2 \setminus \{a\}$ then $x(\tau) = \lim\limits_{t \to \tau-} x(t)$ or if $a \in J_2$ then $x(a) = \lim\limits_{t \to a+} x(t)$. Since $x(t)$ is monotonous near points of J_2 such limits exist. Acting as previously we have that for $t \in J_2$ $x'(t) = \lim\limits_{i \to \infty} x_i'(t)$ and $\lim\limits_{\tau \to t} x'(\tau) = x'(t)$. Consequently, $x(t)$ is a generalized solution of equation (2.1). □

Let us show how Theorem 2 may be used to prove the existence of a solution to the Thomas-Fermi boundary value problem (see [17], p. 376)

$$x'' = t^{-0.5} x^{1.5}, \quad x(0) = 1, \quad \lim_{t \to +\infty} x(t) = 0.$$

Let $\alpha = 0$ and $\beta = 1$. By Theorem 2 there exists a solution x of the boundary value problem

$$x'' = t^{-0.5} x^{1.5}, \quad x(0) = 1, \quad \alpha \le x \le \beta.$$

Let us show, that this solution is a solution of the Thomas-Fermi boundary value problem. If $x(t_0) = 0$ for some $t_0 \in (0, \infty)$, then $x'(t_0) = 0$, and from uniqueness of a solution of the Caushy problem we have $x(t) \equiv 0$, but that contradicts the condition $x(0) = 1$. Therefore $x(t) > 0 \quad t \in [0, \infty)$. If $x'(t_0) \ge 0, \quad t_0 \in [0, \infty)$, then from $x'' \ge t^{-0.5} x^{1.5}(t_0), \quad t \in [t_0, +\infty)$ it follows, that $x(t) \to +\infty$ for $t \to +\infty$, and this contradicts the estimate $x \le \beta$. Consequently, $x' < 0$, and there exists a limit $\lim\limits_{t \to +\infty} x(t) = B \ge 0$. If $B > 0$, then from $x''(t) > t^{-0.5}B^{1.5}, \quad t \in [t_0, +\infty)$ it follows, that $x(t) \to +\infty$ for $t \to +\infty$, and that contradicts the inequality $x \leq \beta$. Consequently, $B = 0$, and x is the solution of the Thomas-Fermi boundary value problem.

Let us note that this reasoning is true if we change the condition $x(0) = 1$ to the condition $x(0) = A \in (0, 1)$.

Theorem 3. *If $\alpha \in AG(I, R), \quad \beta \in BG(I, R)$ and the condition of compactness $S = SG$ fulfils then the boundary value problem (2.1)–(2.2) has a solution.*

Remark 2. In conditions of Theorem 3 it follows from properties of lower and upper functions the existence of the upper and lower solutions $s^*, s_* \in S$ such that $s_* \leq x \leq s^*$ for all $x \in S$ (remark 1 in [11]).

Let us consider the Dirichle's problem. First formulate the following three conditions C_1, C_2 and C_3.

C_1. All the functions $x \in S$ as well as α and β have finite limits as $t \to +\infty$.

C_2. For any $B \in [\alpha(+\infty), \beta(+\infty)]$ and $\varepsilon > 0$ there exist $T \in (a, +\infty)$ and $\delta > 0$ such that for every $x \in S$ the condition $\mid B - x(+\infty) \mid < \delta$ implies that $\mid B - x(t) \mid < \epsilon$ for $t \geq T$.

C_3. For any $B \in (\alpha(+\infty), \beta(+\infty))$ there exists $T \in (a, +\infty)$ such that for every compact interval J and $x \in SG(J, R)$ the conditions $[a, T] \subset J, \quad x(a) = A$ and $x(b) = B$ imply the existence of $s \in SG$ such that $s = x$ on interval J.

Theorem 4. *The Dirichlet's problem*

$$\begin{aligned} &(\varphi(t, x, x'))' = f(t, x, x'), \quad x(a) = A, \\ &x(+\infty) = B \in [\alpha(+\infty), \beta(+\infty)], \quad \alpha \leq x \leq \beta \end{aligned} \tag{2.4}$$

has a solution if the conditions C_1, C_2, C_3 hold and the compactness condition $S = SG$ is fulfilled.

Proof. Fix $B \in (\alpha(+\infty), \beta(+\infty))$. Find T from the condition C_3. Let the sequence $b_i \in (T, +\infty), i = 1, 2, ...$ increase and tend to $+\infty$ and x_i be a generalized solution of the Dirichlet's problem

$$(\varphi(t, x, x'))' = f(t, x, x'), x(a) = A, x(b_i) = B, \alpha(t) \leq x(t) \leq \beta(t), t \in [a, b_i].$$

Consequently, the generalized solutions $s_i \in SG$ can be found such that $s_i(t) = x_i(t), \quad t \in [a, b_i]$. It follows from the compactness condition that $s_i \in S$. Without loss of generality we may consider that the sequence s_i on each compact interval uniformly converges to $s \in SG$ together with the derivatives. Let us consider the sequence $s_i(+\infty)$. Without loss of generality we can assume that $\lim_{i\to\infty} s_i(+\infty) = B_0$. If $B_0 = B$, then from the condition C_2 we obtain the uniform convergence s_i to s. Consider the case $B_0 \neq B$. Then for B_0 and $\varepsilon = 2^{-1}|B_0 - B|$ from the condition C_2 we obtain the contradiction. Consider the case $B = \alpha(+\infty)$. Let $B_i \in (\alpha(+\infty), \beta(+\infty)), \quad i = 1, 2, ...,$ $\lim_{i\to\infty} B_i = \alpha(+\infty)$ and s_i be a solution of the Dirichlet's problem (2.4) for $B = B_i$. Without loss of generality we shall assume that s_i converges to $s \in S$ and $s(+\infty) = \alpha(+\infty)$. Similarly the rest cases can be considered. $\square$

Let us consider the solvability of the boundary value problem

$$\begin{aligned} &(\varphi(t,x,x'))' = f(t,x,x'), \quad t \in I \quad Hx = 0, \\ &x(+\infty) = B \in [\alpha(+\infty), \beta(+\infty)], \quad \alpha \le x \le \beta, \end{aligned} \tag{2.5}$$

where H is a continuous functional. We will need the following conditions to formulate the theorem:

C_4. There exist $A_* \in [\alpha(a), \beta(a))$ and $A^* \in (A_*, \beta(a)]$ such that the maximal solution α_* of the Dirichlet's problem

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad x(a) = A_*, \quad x(+\infty) = B, \quad \alpha \le x \le \beta$$

and the minimal solution β_* of the Dirichlet's problem

$$(\varphi(t,x,x'))' = f(t,x,x'), \quad x(a) = A^*, \quad x(+\infty) = B, \quad \alpha_* \le x \le \beta$$

satisfy the inequality $H\alpha_* H\beta_* \le 0$.

Theorem 5. *Suppose that the conditions C_1, C_2, C_3, C_4 are fulfilled, the Cauchy problem between α_* and β_* has a unique solution and the Shrader's condition is satisfied, that is, any generalized solution lying between α_* and β_*, is a solution. Then the boundary value problem (2.5) has a solution.*

Proof. Let S_B be a set of solutions $x : I \to R$, lying between α_* and β_*. Let us take an increasing sequence $b_i \in (a, +\infty)$, $\quad i = 1, 2, ...,$ which converges to $+\infty$. Let $x_i : [a, b_i] \to R$ be a minimal solution of the Dirichlet's problem

$$\begin{aligned} &(\varphi(t,x,x'))' = f(t,x,x'), \quad x(a) = A^*, \quad x(b_i) = \alpha_*(b_i), \\ &\alpha_*(t) \le x(t) \le \beta_*(t), \quad t \in [a, b_i] \end{aligned}$$

and $x_i(t) = \alpha_*(t), \quad t \in (b_i, +\infty)$. Let further $x_{i\gamma} : [a, b_i] \to R$ be a solution of the Cauchy problem

$$\begin{aligned} &(\varphi(t,x,x'))' = f(t,x,x'), \quad x(b_i) = \alpha_*(b_i), \\ &x'(b_i) = \gamma \in [x_i'(b_i), \alpha_*'(b_i)] \quad t \in [a, b_i], \end{aligned}$$

$x_{i\gamma} = \alpha_*(t), \quad t \in (b_i, +\infty)$ and $M_i = \{x_{i\gamma} : \gamma \in [x_i'(b_i), \alpha_*'(b_i)]\}, \quad i = 1, 2, ...$ From properties of α_* and β_* it follows that $\alpha_*(t) < x_{i\gamma}(t) < \beta_*(t), \quad \gamma \in (x_i'(b_i), \alpha_*'(b_i)), \quad t \in [a, b_i)$. It is clear that M_i is a continuum. To prove the convergence $x_i \to \beta_*$ it is sufficient to show that any convergent subsequence $x_{i*} \to x \in S_B$ converges to β_*. It follows from $x(a) = A^*$ and the minimality of β_* that $x = \beta_*$. Therefore $\{\alpha_*, \beta_*\} \subset LiM_i \subset LsM_i$, where Li is a lower limit and Ls is an upper limit (see [8], p. 343), LiM_i is a continuum (see. [9], p. 180) and α_*, β_* are lying in the same component of connectedness of the space $S_B \bigcup \{\bigcup_i M_i\}$ with $\rho(x,y) = \sup(\mid x(t) - y(t) \mid : t \in I)$. Then the existence of a solution of the boundary value problem (2.5) follows from the inequality $H\alpha_* H\beta_* \le 0$. $\square$

Using the equation of Thomas-Fermi as example, let us show another approach to the boundary value problem (2.5). Let $\alpha = 0$ and $\beta = 1$. It is known from the above considerations that all such solutions tend to zero as $t \to +\infty$. Let us show that a solution of equation with the condition $x(0) = A$ is unique. Let us assume the contrary. Suppose there exist x_1 and x_2 such that the conditions $x_1(0) = x_2(0) = A$ are fulfilled and the difference $u = x_2 - x_1$ has a positive maximum at $t_* \in (0, +\infty)$. Then $u''(t_*) \leq 0$, but $u''(t_*) = t_*^{-0.5}(x_2^{1.5} - x_1^{1.5}) > 0$. Then the continuous dependence of solutions of A and connectedness of a set of solutions follows.

3 Conclusion

In the references above mostly boundary value problems for φ-Laplacian equation on a finite interval were considered. However, it is known [7] that problems with a condition at infinity often arise in mathematical physics. In this article the results on existence of a solution for the φ-Laplacian equation on a semi-finite interval are given.

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