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An Optimal Family of Eighth-Order Iterative Methods with an Inverse Interpolatory Rational Function Error Corrector for Nonlinear Equations

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Abstract. The main motivation of this study is to propose an optimal scheme with an inverse interpolatory rational function error corrector in a general way that can be applied to any existing optimal multi-point fourth-order iterative scheme whose first sub step employs Newton's method to further produce optimal eighth-order iterative schemes. In addition, we also discussed the theoretical and computational properties of our scheme. Variety of concrete numerical experiments and basins of attraction are extensively treated to confirm the theoretical development.

Keywords: nonlinear equations, simple roots, computational order of convergence, Newton's method, basins of attraction.

AMS Subject Classification: 65Hxx.

1 Introduction

The conceptualization and construction of higher-order multi-point solution techniques has always been a paramount importance in the field of numerical analysis that provides a more accurate and efficient approximate solution ξ of nonlinear equation of the form

$$f(x) = 0,$$

where $f : \mathbb{C} \to \mathbb{C}$ is an analytic function in a region containing ξ. This topic has attracted the attention of many researchers from the worldwide, since Traub [19] initiated the qualitative as well as the quantitative analysis of iterative methods. In 2012, Petković et al. [11] gathered up and updated the state of the art of multi-point methods.

Due to the advancement of digital computer, advanced computer arithmetics and symbolic computation, a special attention has been paid to the development of optimal eighth-order iterative methods, which nowadays converge very fast towards the required root. In addition,we can reach our desired accuracy within a very small number of iterations.

In the literature, we can easily find that a large number of optimal eighth-order methods have been proposed by various researchers in [5, 8, 9, 13, 15, 16, 17, 18, 20]. Most of them are extensions of Newton's method or Newton-like method at the expense of additional functional evaluations or increased number of sub-steps of the original methods. However, there are a few number of optimal schemes which are applicable to any iterative method of particular order to further obtain higher-order methods. According to our knowledge, very recently, Sharma et al. [13] proposed an optimal scheme which is applicable to every optimal fourth-order method whose first sub step should be Newton to further extend eighth-order convergence. However, they randomly consider the third sub step in their proposed scheme without the justification of this step.

Nowadays, a constructive development of optimal schemes of eighth-order which are applicable to every fourth-order iterative method/family of iterative methods with Newton's method applied to the first sub-step iteration rather than the usual development dependent on particular fourth-order methods becomes a more interesting and challenging task in the filed of numerical analysis.

In order to develop a new scheme, it is quite often to approximate functions. Several types of approximations are available in the literature, for example, by use of Functional approach, Sampling approach, Geometric approach, Weight function approach, Adomain approach, Composition approach and Rational function approach. Every approach has some advantages and disadvantages because it is dependent on the problem under consideration. The choice of suitable approximation approach can save considerable amount of computation. Rational function approach is one of the most important techniques in numerical analysis for approximating the function or finding the next approximation.

In general, the number of tangency conditions to be discussed in Section 2 are equal to the number of undetermined constants. Further, we will get an improved method with higher-order convergence as we increase the number of undetermined constants in the rational function(for the details, see Jarratt and Nudds [7]).

Therefore, in this manuscript we pursue to develop a generic optimal eighth-order scheme which will be applicable to every fourth-order optimal method or family of methods whose first sub -step employs Newton's method. The derivation of the proposed scheme is based on the concept of the rational approximations. The beauty of the proposed scheme is that it is applicable to every optimal scheme of fourth-order whose first sub-step employs Newton's method. The efficiency of the methods is tested on a number of numerical examples and it is found that our proposed methods perform better than existing optimal methods of the same order. Moreover, we also investigate the dynamics of the listed simple root finders in the complex plane using the basins of attraction which give important information about convergence and stability of the listed methods. The study of dynamics of these methods plays a role in understanding their convergence behavior of rather global character in terms of periodic, quasi-periodic or chaotic orbits.

2 Development of eighth-order optimal schemes

In this section, we will propose an optimal eighth-order family of iterative methods. We begin the development by considering a general fourth-order scheme in the following way:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = \phi_4(x_n,\ y_n). \end{cases} \tag{2.1}$$

We first apply the classical Newton's method to the third sub-step in order to get an eighth-order method as follows:

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},$$

which is not optimal in the sense of Kung-Traub conjecture [8] due to the additional use of $f'(z_n)$. In order to avoid the use of $f'(z_n)$, we introduce a function $\Omega(x)$ satisfying the tangency conditions

$$\Omega(x_n) = f(x_n),\ \Omega'(x_n) = f'(x_n),\ \Omega(y_n) = f(y_n),\ \Omega(z_n) = f(z_n). \tag{2.2}$$

We now choose a second-order rational function $\Omega(x)$ of the form

$$\Omega(x) = \Omega(x_n) + \frac{(x - x_n) + a_1}{a_2(x - x_n)^2 + a_3(x - x_n) + a_4}, \tag{2.3}$$

where parameters $a_i(1 \le i \le 4)$ will be determined in terms of $f(x_n),\ f'(x_n),$ $f(y_n),\ f(z_n)$ by means of the above tangency conditions (2.2). Then the third sub-step iteration will be replaced by

$$x_{n+1} = z_n - \frac{f(z_n)}{\Omega'(z_n)}, \tag{2.4}$$

which no longer requires $f'(z_n)$. Hence, equations (2.1) and (2.4) would yield an optimal eighth-order method. One should note that $\Omega(x)$ in (2.2) plays a important role leading to the development of an eighth-order optimal method.

Nevertheless, we in this paper seek a different third sub-step iteration of the form:

$$x_{n+1} = x_n - W_f(x_n, y_n, z_n), \tag{2.5}$$

where W_f can be regarded as an error correction term to be called naturally as "error corrector". This kind of form of a third sub-step iteration is more useful in the error analysis as well as in the study of dynamics via basins of attraction. One way of finding such a third sub-step iteration with a feasible error corrector is to apply the Inverse Function Theorem [1] to (2.3). That is to say, since $\Omega'(\xi) \neq 0$ (otherwise, $f(x)$ would possess ξ as a multiple root), there exists a unique function $\tau(x)$ satisfying $\Omega(\tau(x)) = x$ in some neighborhood of $\tau(\xi)$. As a result, we use such an inverse function $\tau(x)$ to develop the desired third sub-step iteration in the form of (2.5) rather than directly using $\Omega(x)$ in (2.3).

After applying the Inverse Function Theorem to (2.3), we obtain the desired third sub-step iteration as the following rational function by means of $\tau(x)$ as follows:

$$x = x_n + \frac{\tau(x) - \tau(x_n) + a_1}{a_2\big(\tau(x) - \tau(x_n)\big)^2 + a_3\big(\tau(x) - \tau(x_n)\big) + a_4}, \tag{2.6}$$

where $a_i (1 \leq i \leq 4)$ are free disposable parameters and can be determined by imposing the following tangency conditions

$$\tau(x_n) = f(x_n), \;\; \tau'(x_n) = f'(x_n), \;\; \tau(y_n) = f(y_n), \;\; \tau(z_n) = f(z_n). \tag{2.7}$$

One should note that the rational function on the right side of (2.6) is regarded as an error corrector. Indeed, the desired third sub-step iteration (2.6) is obtained using the inverse interpolatory function approach meeting the tangency constraints (2.7). Clearly, the third sub-step iteration (2.6) looks more suitable than (2.4) in the error analysis. It remains us to determine parameters $a_i (1 \leq i \leq 4)$ in (2.6).

By applying the first two tangency conditions, we obtain

$$a_1 = 0, \quad a_4 = f'(x_n).$$

Again, with the help of last two tangency conditions and the above values of a_1 and a_4, we have the following two independent relations

$$a_2\big(f(y_n) - f(x_n)\big)^2 + a_3\big(f(y_n) - f(x_n)\big) + f'(x_n) - f[x_n, y_n] = 0,$$
$$a_2\big(f(z_n) - f(x_n)\big)^2 + a_3\big(f(z_n) - f(x_n)\big) + f'(x_n) - f[x_n, z_n] = 0,$$

which further yield

$$\begin{aligned} a_2 &= \big(f'(x_n)(f(z_n) - f(y_n)) + f[x_n, z_n](f(y_n) - f(x_n)) + f[x_n, y_n] \\ &\qquad \times (f(x_n) - f(z_n))\big) / \big(f(y_n) - f(x_n))(f(y_n) - f(z_n))(f(x_n) - f(z_n)\big), \\ a_3 &= \frac{-a_2(f(y_n) - f(x_n))^2 - f'(x_n) + f[x_n, y_n]}{f(y_n) - f(x_n)}, \end{aligned} \tag{2.8}$$

where $f[\cdot, \, \cdot]$ is a forward divided difference of order one.

In order to find the next approximation x_{n+1}, we assume that the above rational function (2.6) meets the $x-$axis at $x = x_{n+1}$. Then, we obtain

$$\tau(x_{n+1}) = 0,$$

which further yields

$$x_{n+1} = x_n - \frac{f(x_n)}{a_2 f(x_n)^2 - a_3 f(x_n) + f'(x_n)}. \tag{2.9}$$

Finally, by using expressions (2.6) and (2.9), we get

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = \phi_4(x_n,\ y_n), \\ x_{n+1} = x_n - \dfrac{f(x_n)}{a_2 f(x_n)^2 - a_3 f(x_n) + f'(x_n)}, \end{cases} \tag{2.10}$$

where a_2 and a_3 are defined earlier in expression (2.8). In the following Theorem 1, we demonstrate that the order of convergence will reach the optimal eighth-order convergence without using any more functional evaluations. It is interesting to observe that only a single coefficient A_0 in $\phi_4(x_n,\ y_n)$ contributes to its role in the construction of the desired asymptotic error constant as can be seen in Theroem 1.

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{C}$ have a simple zero ξ and be an analytic function in the region containing ξ. Further, assume that $\phi_4(x_n,\ y_n)$ is any optimal scheme of order four with A_0 as its asymptotic error constant and initial guess $x = x_0$ is sufficiently close to ξ for guaranteed convergence. Then, the iterative scheme defined by* (2.10) *has an optimal eighth-order convergence and satisfies the following error equation*

$$e_{n+1} = A_0 c_2 \left(2c_2^3 - 3c_3 c_2 + c_4\right) e_n^8 + O(e_n^9),$$

where $e_n = x_n - \xi$ and $c_j = \frac{f^{(j)}(\xi)}{j! f'(\xi)}$ for $j = 2,\ 3,\ \ldots,\ 8$.

Proof. The Taylor's series expansion of the function $f(x_n)$ and $f'(x_n)$ around $x = \xi$ with the assumption $f'(\xi) \neq 0$ leads us to:

$$f(x_n) = f'(\xi)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^7 + O(e_n^9)] \tag{2.11}$$

$$f'(x_n) = f'(\xi)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + 9c_9 e_n^8 + O(e_n^9)], \tag{2.12}$$

respectively.

By using the above expressions (2.11) and (2.12) in the first sub step, we have

$$\begin{aligned} y_n - \xi = {} & c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3 + (4c_2^3 - 7c_3 c_2 + 3c_4) e_n^4 + (20 c_3 c_2^2 - 8c_2^4 - 6c_3^2 \\ & - 10 c_4 c_2 + 4c_5) e_n^5 + (16 c_2^5 - 52 c_3 c_2^3 + 28 c_4 c_2^2 + (33 c_3^2 - 13 c_5) c_2 - 17 c_3 c_4 \\ & + 5c_6) e_n^6 - 2\big(16 c_2^6 - 64 c_3 c_2^4 + 36 c_4 c_2^3 + 9(7c_3^2 - 2c_5) c_2^2 + (8c_6 - 46 c_3 c_4) c_2 \\ & - 9c_3^3 + 6c_4^2 + 11 c_3 c_5 - 3c_7\big) e_n^7 + \Big(64 c_2^7 - 304 c_3 c_2^5 + 176 c_4 c_2^4 + (408 c_3^2 \\ & - 92 c_5) c_2^3 + (44 c_6 - 348 c_3 c_4) c_2^2 + (-135 c_3^3 + 118 c_5 c_3 + 64 c_4^2 - 19 c_7) c_2 \\ & + 75 c_3^2 c_4 - 31 c_4 c_5 - 27 c_3 c_6 + 7c_8\Big) e_n^8 + O(e_n^9). \end{aligned}$$

The following expansion of $f(y_n)$ about a point $x = \xi$ with the help of Taylor series yields

$$\begin{aligned} f(y_n) =& f'(\xi)\Big[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_3c_2 + 3c_4)e_n^4 - 2(6c_2^4 - 12c_3c_2^2 \\ &+ 5c_4c_2 + 3c_3^2 - 2c_5)e_n^5 + \{28c_2^5 - 73c_3c_2^3 + 34c_4c_2^2 + (37c_3^2 - 13c_5)c_2 \\ &- 17c_3c_4 + 5c_6\}e_n^6 + H_1e_n^7 + H_2e_n^8 + O(e_n^9)\Big], \end{aligned}$$

where H_1 and H_2 are constant functions of $c_2, c_3, \ldots, c_8$.

Since $\phi_4(x_n,\ y_n)$ is an optimal fourth-order optimal scheme, the scheme will satisfy the following error equation

$$z_n - \xi = A_0e_n^4 + A_1e_n^5 + A_2e_n^6 + A_3e_n^7 + A_4e_n^8 + O(e_n^9),$$

where $A_0 \neq 0$.

Now, we can expand the function $f(z_n)$ about a point $z = \xi$ with the help of Taylor series expansion, which is given as follows

$$f(z_n)=f'(\xi)\left[A_0e_n^4+A_1e_n^5 + A_2e_n^6 + A_3e_n^7 + (A_0^2c_2 + A_4)e_n^8 + O(e_n^9)\right]. \quad (2.13)$$

With the help of expressions (2.11) – (2.13), we have

$$\frac{f(x_n)}{a_2f(x_n)^2-a_3f(x_n)+f'(x_n)}=e_n-A_0c_2\left(2c_2^3 - 3c_3c_2 + c_4\right)e_n^8 + O(e_n^9). \quad (2.14)$$

Finally, by inserting above expression (2.14), in the last sub step of the proposed scheme (2.10), we get

$$e_{n+1} =A_0c_2(2c_2^3 - 3c_3c_2 + c_4)e_n^8 + O(e_n^9). \quad (2.15)$$

This reveals that the proposed scheme (2.10) reaches an optimal eighth-order convergence in the sense of Kung-Traub conjecture. This completes the proof. □

Remark 1. In general, one naturally expects that the asymptotic error constant of the proposed scheme (2.10) may be dependent on $c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and A_0, A_1, A_2, A_3, A_4. Nevertheless, it is undoubtedly interesting to observe that the asymptotic error constant shown in (2.15) appears as a simple expression, being dependent only on c_2, c_3, c_4 and A_0. This simplicity clearly reflects that our current approach using the inverse interpolatory function with the tangency conditions plays a key role in the development of an optimal family of eighth-order methods.

3 Numerical experiments

In this section, we shall check the effectiveness and validity of our theoretical results which we have proposed in Section 2. For this purpose, we shall consider a concrete variety of nonlinear equations, which are given as follows:

$$f_1(x) = \exp\left(-x^2 + x + 2\right) + x^3 - \cos(x + 1) + 1;\ [18] \quad \xi = -1,$$

$f_2(x) = \sin^{-1}(x^2 - 1) - \dfrac{x}{2} + 1;$ [4] $\xi = 0.59481096839836917752,$
$f_3(x) = \log\left(x^2 + x + 2\right) - x + 1;$ [11] $\xi = 4.15259073675715827499,$
$f_4(x) = \cos(x) - x;$ [5] $\xi = 0.73908513321516064165,$
$f_5(x) = x^5 + x^4 + 4x^2 - 15;$ [16] $\xi = 1.34742809896830498151,$
$f_6(x) = x\exp\left(x^2\right) - \sin^2(x) + 3\cos(x) + 5;$ [9] $\xi = -1.20764782713091892701.$

We shall verify the theoretical order of convergence of the proposed methods on the basis of the results obtained from $|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}|$ and computational order of convergence. In Table 1, we displayed the number of iteration indexes (n), approximated zeros (x_n), absolute residual error of the corresponding function ($|f(x_n)|$), error in the consecutive iterations $|x_{n+1} - x_n|$, $|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}|$, the asymptotic error constant $\eta = \lim_{n\to\infty} |\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}|$ and ρ. In order to calculate the computational order of convergence ρ, we use the following formula

$$\rho = \log|(x_{n+1} - x_n)/\eta| / \log|x_n - x_{n-1}|, \quad n = 1, 2, 3.$$

We calculate the computational order of convergence, asymptotic error constant and other constants up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error.

Table 1. Convergence behavior of methods $PM1_8$ and $PM2_8$ on $f_1(x)$ –$f_6(x)$

Cases	$f(x)$	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1} - x_n\rvert$	$\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\rvert$	η	ρ
$PM1_8$	f_1	0	−0.8	1.3	2.0(−1)			
		1	−0.99999997763	1.3(−7)	2.2(−8)	0.0087394782	0.0034013	7.41365
		2	−1.00000000000	1.3(−63)	2.1(−64)	0.0034012941		8.00000
		3	−1.00000000000	8.8(−512)	1.5(−512)	0.0034012933		8.00000
$PM1_8$	f_2	0	1	5.0(−1)	4.1(−1)			
		1	0.5948090837283	2.0(−6)	1.9(−6)	0.00259392681	0.00008837	4.25920
		2	0.5948109683984	1.5(−50)	1.4(−50)	0.00008836552		8.0000
		3	0.5948109683984	1.4(−403)	1.4(−403)	0.00008836711		8.00000
$PM1_8$	f_3	0	3.2	5.4(−1)	9.5(−1)			
		1	4.152590944848	1.3(−7)	2.1(−7)	3.0690368(−7)	7.9649(−8)	−19.77
		2	4.152590736757	1.7(−61)	2.8(−61)	7.9649402(−8)		8.0000
		3	4.152590736757	1.8(−492)	3.0(−492)	7.9649424(−8)		8.00000
$PM2_8$	f_4	0	0.5	3.8(−1)	2.4(−1)			
		1	0.73908514888	2.6(−8)	1.6(−8)	0.00146696579	0.00055131	7.31607
		2	0.73908513322	3.3(−66)	2.0(−66)	0.00055130498		8.00000
		3	0.73908513322	2.3(−529)	1.4(−529)	0.00055130501		8.00000
$PM2_8$	f_5	0	1.2	4.7	1.5			
		1	1.347429011193	3.4(−5)	9.1(−7)	4.087317765	2.1745982	7.67037
		2	1.347428098968	3.9(−47)	1.0(−48)	2.174589401		8.00000
		3	1.347428098968	1.1(−382)	3.0(−384)	2.174598218		8.00000
$PM2_8$	f_6	0	−1.3	2.2	9.2(−2)			
		1	−1.20764783189	9.7(−8)	4.8(−9)	0.8989533433	0.13905535	7.21652
		2	−1.20764782713	7.4(−67)	3.6(−68)	0.1390553883		8.0000
		3	−1.20764782713	8.8(−540)	4.3(−514)	0.1390553493		8.00000

It is straight forward to say that our methods have the small asymptotic error constant which confirm the theoretical results.

We consider also the following test functions:

$$f_7(x) = \tan^{-1} x; \quad [2], \quad x_0 = 0.5, \;\; \xi = 0,$$
$$f_8(x) = x^3 + \sin x - 1; \quad [11] \quad x_0 = 0.4,$$
$$\xi = 0.70569369763018399372425969776447061,$$
$$f_9(x) = x^3 - 30x + 5; \quad [2] \quad x_0 = -0.4,$$
$$\xi = 0.16682141791816451151054900720209898,$$
$$f_{10}(x) = 10xe^{-x^2} - 1; \quad [11] \quad x_0 = 1.1,$$
$$\xi = 1.6796306104284499406749203388379704,$$
$$f_{11}(z) = z^4 + (5+2i)z + \sqrt{5}i + 1; \; [11] \quad z_0 = 0.5 + 1.6i,$$
$$\xi = 0.7674379412974 \cdots + 1.7131311525356 \ldots i$$

Table 2. Comparison of residual error on the test examples $f_7(x)$ –$f_{11}(x)$

$f(x)$	$\lvert f(x_n)\rvert$	CM_8	LM_8	SM_8	TM_8	SA_8	$PM1_8$	$PM2_8$
	$\lvert f(x_1)\rvert$	1.2(−5)	1.2(−4)	4.0(−5)	1.2(−4)	4.0(−5)	3.0(−6)	5.6(−6)
f_7	$\lvert f(x_2)\rvert$	1.1(−46)	9.4(−38)	1.8(−50)	3.4(−35)	7.0(−42)	1.7(−63)	1.7(−60)
	$\lvert f(x_3)\rvert$	1.1(−415)	1.5(−335)	3.0(−549)	1.6(−312)	1.0(−372)	2.7(−693)	3.8(−660)
	$\lvert f(x_1)\rvert$	9.4(−6)	3.4(−5)	1.2(−3)	6.4(−4)	4.7(−6)	3.2(−7)	2.9(−5)
f_8	$\lvert f(x_2)\rvert$	2.5(−44)	1.9(−39)	7.0(−28)	4.8(−28)	2.1(−47)	2.4(−57)	1.9(−40)
	$\lvert f(x_3)\rvert$	6.0(−353)	2.0(−313)	8.6(−222)	5.4(−221)	3.6(−378)	2.7(−458)	6.8(−322)
	$\lvert f(x_1)\rvert$	3.2(−7)	2.8(−7)	1.0(−7)	3.0(−7)	2.6(−7)	1.0(−9)	1.1(−9)
f_9	$\lvert f(x_2)\rvert$	4.1(−69)	1.4(−69)	9.9(−74)	2.2(−69)	6.6(−70)	8.4(−91)	1.1(−90)
	$\lvert f(x_3)\rvert$	3.1(−564)	5.0(−568)	6.9(−602)	1.7(−566)	1.1(−570)	1.9(−739)	2.3(−738)
	$\lvert f(x_1)\rvert$	3.3(−3)	2.1(−3)	6.0(−4)	2.1(−3)	3.0(−3)	1.6(−4)	1.6(−4)
f_{10}	$\lvert f(x_2)\rvert$	3.1(−23)	2.0(−24)	2.1(−31)	1.4(−23)	2.5(−24)	1.7(−34)	9.9(−34)
	$\lvert f(x_3)\rvert$	1.5(−183)	1.7(−192)	5.0(−251)	4.8(−185)	5.9(−193)	3.2(−274)	2.0(−267)
	$\lvert f(x_1)\rvert$	7.6(−3)	2.4(−2)	4.8(−2)	6.3(−1)	1.7(−4)	1.3(−3)	1.7(−2)
f_{11}	$\lvert f(x_2)\rvert$	1.2(−26)	2.8(−22)	8.0(−21)	4.5(−10)	1.9(−41)	2.5(−33)	1.1(−23)
	$\lvert f(x_3)\rvert$	3.5(−217)	1.0(−181)	5.1(−171)	3.4(−83)	5.6(−337)	3.1(−271)	3.0(−193)

As we mentioned in the above paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the value of x_n and ρ up to 15 and 6 significant digits, respectively. In addition, we also display $\left|\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\right|$ and η up to 10 significant digits. Moreover, absolute residual error in the function $|f(x_n)|$ and error in the consecutive iterations $|x_{n+1} - x_n|$ are displayed up to 2 significant digits with exponent power which are mentioned in Tables 1, 2 and 3. Furthermore, the approximated zeros up to 30 significant digits are also displayed in Table 1 although minimum 1000 significant digits are available with us.

Now, we want to see the comparison of our methods with the other existing optimal methods of the same order. Therefore, we consider some other nonlinear functions which are displayed above.

We consider the following methods.

(i) Let us consider an optimal family of fourth-order methods proposed by Behl et al. [3]. Then, we obtain the following optimal family of eighth-order methods with the help of our proposed scheme (2.10), which is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad z_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ \qquad \times \left[\dfrac{(b_1^2 + b_1 b_2 - b_2^2) f(x_n) f(y_n) - b_1(b_1 - b_2)\{f(x_n)\}^2}{\big(b_1 f(x_n) - b_2 f(y_n)\big)\big((2b_1 - b_2) f(y_n) - (b_1 - b_2) f(x_n)\big)}\right], \\ x_{n+1} = x_n - \dfrac{f(x_n)}{a_2 f(x_n)^2 - a_3 f(x_n) + f'(x_n)}, \end{cases}$$

where $b_1,\ b_2 \in \mathbb{R}$ such that b_1 neither equal to 0 nor b_2. For a computational point of view, let us consider $b_1 = 1$ and $b_2 = \frac{1}{10}$ in the above scheme, denoted by $(PM1_8)$.

(ii) Next, we shall choose another optimal family of fourth-order methods proposed by Behl et al. in [2]. Then, we further yield another new optimal family of eighth-order methods, which is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \dfrac{f(x_n)}{f'(x_n)}\left[1 + \dfrac{f(y_n)}{f(x_n) + f(y_n)} + (\alpha + 2)\left(\dfrac{f(y_n)}{f(x_n) + f(y_n)}\right)^2 \right. \\ \qquad \left. + \dfrac{H'''(1)}{6}\left(\dfrac{f(y_n)}{f(x_n) + f(y_n)}\right)^3\right], \\ x_{n+1} = x_n - \dfrac{f(x_n)}{a_2 f(x_n)^2 - a_3 f(x_n) + f'(x_n)}, \end{cases}$$

where $\alpha, H'''(1) \in \mathbb{R}$. Let us consider $\alpha = -1$ and $H'''(1) = -9$ in the above scheme, denoted by $(PM2_8)$.

Now, we will compare them with the optimal families of eighth-order methods which were proposed by Cordero et al. in [5], Li and Wang in [9], Soleymani et al. in [16], Thukral in [17] and Sharma and Arora [14], which are respectively

Table 3. Comparison of error in the consecutive iterations on the test examples $f_7(x)$ – $f_{11}(x)$

f	$\lvert x_{n+1}-x_n \rvert$	CM_8	LM_8	SM_8	TM_8	SA_8	$PM1_8$	$PM2_8$
f_7	$\lvert x_2-x_1 \rvert$	1.2(−5)	1.2(−4)	4.0(−5)	2.2(−4)	4.0(−5)	3.0(−6)	5.6(−6)
	$\lvert x_3-x_2 \rvert$	1.1(−46)	9.4(−38)	1.8(−50)	3.4(−35)	7.0(−42)	1.7(−63)	1.7(−60)
	$\lvert x_4-x_3 \rvert$	1.1(−415)	1.5(−335)	3.0(−549)	1.6(−312)	1.0(−372)	2.7(−693)	3.8(−660)
f_8	$\lvert x_2-x_1 \rvert$	4.2(−6)	1.5(−5)	5.3(−4)	2.8(−4)	2.1(−6)	1.4(−7)	1.3(−5)
	$\lvert x_3-x_2 \rvert$	1.1(−44)	8.5(−40)	3.1(−28)	2.1(−28)	9.3(−48)	1.1(−57)	8.5(−41)
	$\lvert x_4-x_3 \rvert$	2.7(−353)	8.8(−314)	3.8(−222)	2.4(−221)	1.6(−378)	1.2(−458)	3.0(−322)
f_9	$\lvert x_2-x_1 \rvert$	1.1(−8)	9.5(−9)	3.4(−9)	1.0(−8)	8.8(−9)	3.4(−11)	3.5(−11)
	$\lvert x_3-x_2 \rvert$	1.4(−70)	4.7(−71)	3.3(−75)	7.3(−71)	2.2(−71)	2.8(−92)	3.8(−92)
	$\lvert x_4-x_3 \rvert$	1.0(−565)	1.7(−569)	2.3(−603)	5.8(−568)	3.5(−572)	6.4(−741)	7.7(−740)
f_{10}	$\lvert x_2-x_1 \rvert$	1.2(−3)	7.5(−4)	2.2(−4)	7.7(−4)	1.1(−3)	1.6(−4)	5.8(−5)
	$\lvert x_3-x_2 \rvert$	1.1(−23)	7.3(−25)	7.6(−32)	5.1(−24)	9.1(−25)	1.7(−34)	3.6(−34)
	$\lvert x_4-x_3 \rvert$	5.6(−184)	6.3(−193)	1.8(−251)	1.8(−185)	2.1(−193)	1.2(−274)	7.4(−268)
f_{11}	$\lvert x_2-x_1 \rvert$	3.6(−4)	1.1(−3)	2.3(−4)	2.9(−2)	8.1(−6)	6.4(−5)	8.0(−4)
	$\lvert x_3-x_2 \rvert$	5.5(−28)	1.3(−23)	3.8(−22)	2.1(−11)	9.2(−43)	1.2(−34)	5.1(−25)
	$\lvert x_4-x_3 \rvert$	1.7(−218)	4.9(−183)	2.4(−172)	1.6(−84)	2.7(−338)	1.5(−272)	1.4(−194)

defined as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad z_n = x_n - \dfrac{f(x_n)-f(y_n)}{f(x_n)-2\,f(y_n)}\dfrac{f(x_n)}{f'(x_n)}, \\ u_n = z_n - \dfrac{f(z_n)\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} + \frac{f(z_n)}{2(f(y_n)-2f(z_n))}\right)^2}{f'(x_n)}, \\ x_{n+1} = u_n - \dfrac{3(b_2+b_3)f(z_n)\,(u_n-z_n)}{f'(x_n)\,(b_2\,(-x_n+y_n)+b_1\,(u_n-z_n)+b_3\,(-x_n+z_n))}, \end{cases} \tag{3.1}$$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \dfrac{f(x_n)f(y_n)}{f'(x_n)f(x_n)-2f'(x_n)f(y_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)\left(\frac{(f(x_n)-f(y_n))^2}{(f(x_n)-2f(y_n))^2} + \frac{f(z_n)}{f(y_n)-af(z_n)} + \frac{4f(z_n)}{f(x_n)+bf(z_n)}\right)}{f'(x_n)}, \end{cases} \tag{3.2}$$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \dfrac{(2f(x_n)-f(y_n))\,f(y_n)}{f'(x_n)\,(2f(x_n)-5f(y_n))}, \\ x_{n+1} = z_n - \dfrac{f(z_n)\left(\frac{f(z_n)^2}{f(x_n)^2} - \frac{3f(y_n)^3}{2f(x_n)^3} - \frac{31f(y_n)^4}{4f(x_n)^4} - \frac{f(y_n)^2+f(z_n)^2}{f'(x_n)^2}\right)}{-f'(x_n)+2f[z_n,x_n]} \\ \qquad - \dfrac{f(z_n)\left(\frac{f(y_n)^2+f(z_n)^2}{f'(x_n)^2} + \frac{f(y_n)^2+f(y_n)f(z_n)+f(z_n)^2}{f(y_n)^2}\right)}{-f'(x_n)+2f[z_n,x_n]}, \end{cases} \tag{3.3}$$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad z_n = x_n - \dfrac{f(x_n)^2 + f(y_n)^2}{f'(x_n)f(x_n) - f'(x_n)f(y_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)\left(\frac{4f(z_n)}{f(x_n)} - \frac{2f(y_n)^2}{f(x_n)^2} - \frac{6f(y_n)^3}{f(x_n)^3} + \frac{\left(f(x_n)^2+f(y_n)^2\right)^2}{f(x_n)^2(f(x_n)-f(y_n))^2} + \frac{f(z_n)}{f(y_n)}\right)}{f'(x_n)} \end{cases} \tag{3.4}$$

and

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \dfrac{f(y_n)}{2f[y_n, x_n] - f'(x_n)}, \\ x_{n+1} = z_n - \dfrac{f[z_n, y_n]}{f[z_n, x_n]} \dfrac{f(z_n)}{2f[z_n, y_n] - f[z_n, x_n]}. \end{cases} \tag{3.5}$$

We shall call the above expression (3.1) (for $b_1 = 1,\ b_2 = 1,\ b_3 = 2$), expression (3.2) (for $a = 0, b = 0$), expression (3.3), expression (3.4) and expression (3.1), by CM_8, LM_8 SM_8, TM_8 and SA_8, respectively.

For better comparisons of our proposed methods with other existing ones, we have given two types of comparison tables in each test function. First one is related to absolute residual error in the corresponding function ($|f(x_n)|$) displayed in Table 2. The other one is related to absolute error between the two consecutive iterations $|x_{n+1} - x_n|$ in Table 3. Further, we consider the approximated zero of test functions when the exact zero is not available, which is corrected up to 1000 significant digits to calculate $|x_n - \xi|$. For the computer programming, all computations have been performed using the programming package *Mathematica* 9 with multiple precision arithmetic. Further, the notation of $a(\pm b)$ implies $a \times 10^{(\pm b)}$ in the following Tables 2–3.

It is worthy to note from Table 2 that the minimum residual errors belong to our methods in all the test problems except the last one. So, we can say that our methods give a more accurate approximate zero of the involved function as compare to other existing methods. In addition, from Table 3 our methods have minimum error in the consecutive iterations corresponding to the test functions f_7-f_{11}, except the last test problem. Hence, we confirm that our methods converge faster towards required zero of the corresponding function as compared to other existing methods.

4 Attractor basins in the complex plane

In order to see the comparison of iterative methods, one can take into account their convergence orders, the numerical stability, CPU time, minimum number of iterations required to attain the desired accuracy, computational order of convergence, asymptotic error constants, absolute residual error in the function by using the same number of functional evaluations, etc. However, the main drawback of these types of comparisons is that they generally start with one initial guess which is chosen at random regardlessly of many other possible choices of initial guesses.

Therefore, we here investigate the dynamics of the listed simple root finders in the complex plane using basins of attraction which gives important information about convergence and stability of the method. To start with, let us recall some basic concepts which are related to basins of attractions. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a rational map on the Riemann sphere. The orbit of a point $z \in \mathbb{C}$ under ϕ is defined

$$\{z,\ \phi(z),\ \phi^2(z),\ \ldots,\ \phi^n(z),\ \ldots\},$$

which consists of successive images of z by the rational map ϕ. The dynamic behavior of the orbit of a point of ϕ would be characterize by its asymptotic behavior. We first introduce some notions of a point in the orbit under ϕ: a point $z_0 \in \mathbb{C}$ is known as a *fixed point* of ϕ, if $\phi(z_0) = z_0$. In addition, z_0 is known as a *periodic point of period* $m > 1$, if $\phi^m(z_0) = z_0$, where m is smallest such integer. Further, if z_0 is a periodic point of period m then it is a fixed point for ϕ^m. Moreover, there are mainly four types of fixed points of a map ϕ, which are based on the magnitude of the derivative.

A fixed point z_0 is known as: (a) *attracting point* if $|\phi'(z_0)| < 1$, (b) *repelling point* if $|\phi'(z_0)| > 1$, (c) *neutral or parabolic point* if $|\phi'(z_0)| = 1$, (d) *super-attracting point* if $|\phi'(z_0)| = 0$.

Basins of attraction. If ξ is a root of $f(x)$, then the basin of attraction of ξ, is the collection of those initial approximations x_0 which converge to ξ. It is mathematically defined as follows:

$$B(\xi) = \{z_0 \in \mathbb{C} :\ \phi^n(z_0) \to \xi \text{ as } n \to \infty\}.$$

Arthur Cayley was the first person who considered the concept of the basins of attraction for Newton's method in 1879. Initially, he considered this concept for the quadratic polynomial. After some time, he also considered cubic polynomials, but was unable to find an obvious division for the basins of attraction as he earlier defined for the quadratic equations. In the early of 20th century, the French mathematicians Gaston Julia and Pierre Fatou started to understand the nature of complex cubic polynomials. The Julia set of a nonlinear map $\phi(z)$, called $J(\phi)$, is the closure of the set of its repelling fixed points and establishes the borders between the basins of attraction. On the other hand, the complement of $J(\phi)$ is known as the Fatou set $F(\phi)$. In simple words, the basins of attraction of any fixed point belongs to the Fatou set $F(\phi)$ and the boundaries of these basins of attraction belong to the Julia set $J(\phi)$. For the details of these concepts please see [6, 10, 12]. The aim herein is to use the basins of attraction as another way for characterizing initial approximations converging to the desired root ξ for the listed iteration algorithms. That is to say, the basins of attraction play a role representing a valuable dynamics of the iteration schemes under consideration.

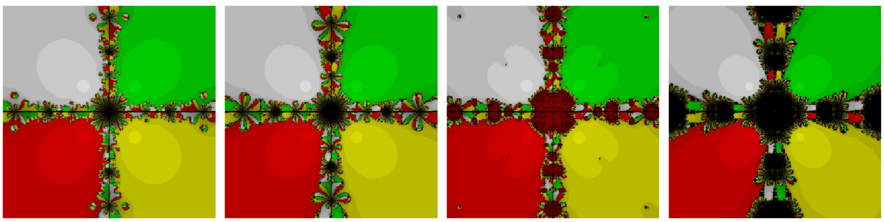
In order to achieve a vivid description from a dynamical point of view, we consider a rectangle $D = [-3,\ 3] \times [-3,\ 3] \in \mathbb{C}$ with a 400×400 grid, and we assign a color to each point $z_0 \in D$ according to the simple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each method is considered to be 200.

In this way, we distinguish the attraction basins by their colors for different methods. For concrete examples of dynamics of the listed methods behind the basins of attraction, we present several test problems described below.

Test problem 1. Let $p_1(z) = (z^4 + 1)$, and it has simple zeros

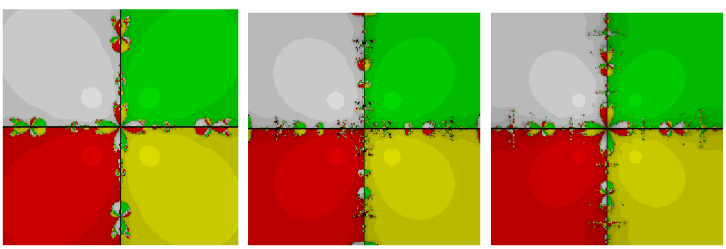
$$\{-0.707107 - 0.707107i, \ \ -0.707107 + 0.707107i, \ \ 0.707107 - 0.707107i, \\ 0.707107 + 0.707107i\}.$$

It is straight forward to see from Figures 1 − 2 that our methods, namely $PM1_8$, $PM2_8$ and method SA_8 are the best methods in terms of less chaotic behavior to obtain the solutions when we compare them with among listed methods. Further, they also have the largest basins for the solution and are faster in comparison to all the mentioned methods except SA_8.



(a) CM_8 (b) LM_8 (c) SM_8 (d) TM_8

Figure 1. The methods for test problem 1.



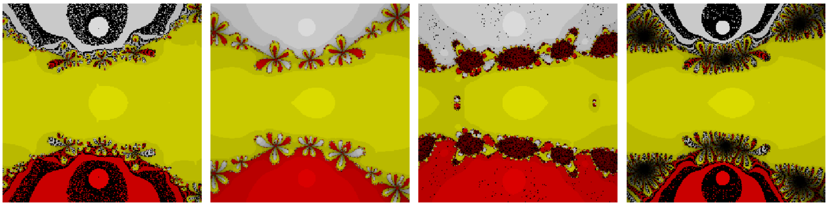
(a) SA_8 (b) $PM1_8$ (c) $PM2_8$

Figure 2. The methods for test problem 1.

Test problem 2. Let $p_2(z) = (z^3 + 2z - 1)$, and it has simple zeros

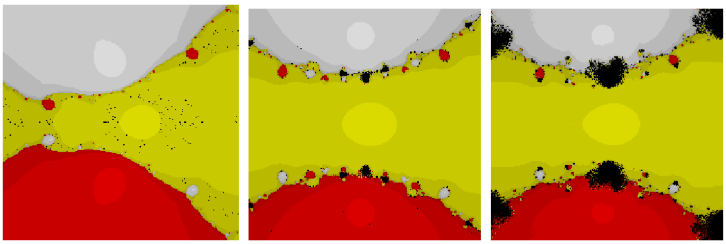
$$\{-0.0992186 - 2.24266i, \ \ -0.0992186 + 2.24266i, \ \ 0.198437\}.$$

Based on Figures 3 − 4, it is observed that method $PM1_8$ and SA_8 are the best methods because they have almost zero non convergent points, larger and brighter basins of attraction in comparison to the methods namely, CM_8, SM_8 and TM_8. In addition, with no doubts method LM_8 has less number of divergent points as compared to our method $PM1_8$ but larger basins of



(a) CM_8 (b) LM_8 (c) SM_8 (d) TM_8

Figure 3. The methods for test problem 2.



(a) SA_8 (b) $PM1_8$ (c) $PM2_8$

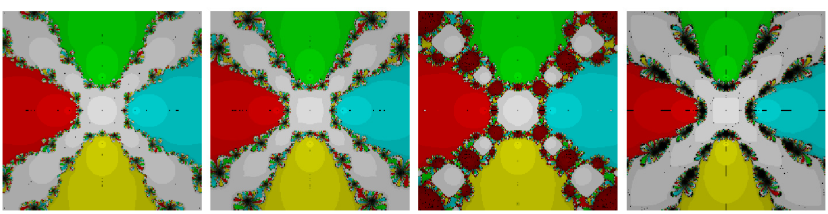
Figure 4. The methods for test problem 2.

attraction belongs to our method $PM1_8$. Hence, methods $PM1_8$ and SA_8 are the best methods among the listed methods.

Test problem 3. Let $p_3(z) = (z^5 - z)$, has simple zeros

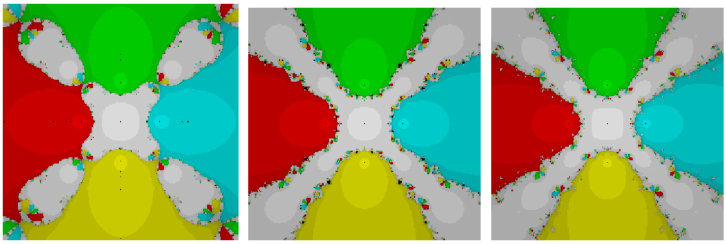
$$\{-1, \ 0, \ -i, \ i, \ 1\}.$$

Figures 5 – 6 confirm that methods $PM1_8$ and $PM2_8$ have almost zero divergent points as compared to the other mentioned methods except method SA_8. In addition, larger and brighter basins of attraction belongs to our methods in comparison to other methods namely, CM_8, LM_8, SM_8 and TM_8. Moreover, our methods don't show chaotic behavior on the boundaries as LM_8, SM_8 and SA_8.



(a) CM_8 (b) LM_8 (c) SM_8 (d) TM_8

Figure 5. The methods for test problem 3.



(a) SA_8 (b) $PM1_8$ (c) $PM2_8$

Figure 6. The methods for test problem 3.

5 Concluding remarks

In the earlier studies, several scholars proposed higher-order derivative free extensions of any particular existing methods like Ostrowski's method or King's method, etc. However, we contribute further to the development of the theory of iteration processes and propose a new optimal eighth-order scheme in a general way. The beauty of the proposed scheme is that it is capable to produce several new interesting eighth-order schemes from any optimal fourth-order scheme whose first sub step employs Newton's method. The derivation of the proposed scheme is based on inverse interpolatory approach. The proposed scheme is optimal in the sense of classical Kung-Traub conjecture. We also compare our methods with the existing robust methods of the same order on a series of numerical examples. The results in Table 2 and 3 overwhelmingly support that the minimum residual errors and minimum error in the consecutive iterations belongs to our methods. In addition, Table 2 also confirms that the simple asymptotic error constants belongs to our methods. Moreover, the study of dynamics of our methods also reflects that our proposed methods are comparable to or superior over the listed methods in most of current test problems. Such comparable or superior performance of our methods may be due to the inherent structure of our method with simple asymptotic error constants and inverse interpolatory approach. The future work based on the inverse interpolatory rational approach shall be devoted to a development of a new optimal higher-order scheme.

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