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NUMERICAL METHODS FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS CONTAINING TWO PERTURBATION PARAMETERS[1]

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Abstract. A priori parameter explicit bounds on the derivatives of the solution of a two parameter singularly perturbed elliptic problem in two space dimensions are presented. These bounds are used to establish parameter uniform error bounds for a numerical method consisting of upwinding on a tensor product of two piecewise uniform meshes.

Key words: Elliptic, two parameters, a priori bounds

1. Introduction

When analysing the convergence behaviour of numerical approximations to the solution of a singularly perturbed differential equation involving two singular perturbation parameters (denoted here by ε and μ), it is worth noting that the error is a function of three parameters: the mesh parameter N (the number of mesh elements used in each coordinate direction) and the two perturbation parameters. Parameter-uniform numerical methods [2] are methods such that the pointwise error $E(N;\varepsilon,\mu)$ is bounded independently of both perturbation parameters. Parameter-uniform methods for two-parameter

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problems based on fitted operator methods on uniform meshes are given in [10, 11]. More recently, fitted piecewise–uniform meshes have been used to generate parameter-uniform methods for two-parameter ordinary differential equations [7, 8] and for singularly perturbed parabolic equations [6]. In this paper, we examine a two-parameter elliptic problem in two space dimensions.

Consider the following class of singularly perturbed elliptic problems posed on the unit square $\Omega = (0,1)^2$,

$$L_{\varepsilon,\mu}u = \varepsilon(u_{xx} + u_{yy}) + \mu(a_1u_x + a_2u_y) - bu = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u|_{\Gamma_B} = s_1(x), \quad u|_{\Gamma_T} = s_2(x), \quad u|_{\Gamma_L} = q_1(y), \quad u|_{\Gamma_R} = q_2(y), \tag{1.1b}$$

$$s_1(0) = q_1(0),\ s_2(0) = q_1(1),\ s_1(1) = q_2(0),\ s_2(1) = q_2(1), \tag{1.1c}$$

$$a_1(x,y) \geq \alpha_1 > 0,\ a_2(x,y) \geq \alpha_2 > 0,\ b(x,y) \geq 2\beta > 0, \tag{1.1d}$$

where $\Gamma_B, \Gamma_T, \Gamma_L$ and Γ_R are the edges of the boundary $\partial\Omega$ defined by

$$\Gamma_B = \{(x,0)|0 \leq x \leq 1\}, \quad \Gamma_T = \{(x,1)|0 \leq x \leq 1\},$$
$$\Gamma_L = \{(0,y)|0 \leq y \leq 1\}, \quad \Gamma_R = \{(1,y)|0 \leq y \leq 1\}.$$

Throughout this paper, we assume sufficient regularity and compatibility on the data so that the solution and its components are sufficiently smooth for the following analysis to be valid. With respect to regularity assume that

$$a_1,\ a_2,\ b,\ f \in C^{n,\lambda}(D),\ \lambda \in (0,1),\ s_1,\ s_2,\ q_1,\ q_2 \in C^m(J),$$

where D, J are open sets such that $\overline{\Omega} \subset D,\ [0,1] \subset J$ and n, m are sufficiently large for our analysis. In this paper, the norm $\|v\|_R = \max_{\vec{x}\in R}|v(\vec{x})|$ is the maximum pointwise norm. Throughout this paper C is a generic constant which is independent of the singular perturbation parameters ε, μ and the mesh parameters N, M.

Note that the differential equation (1.1a) contains two singular perturbation parameters $0 < \varepsilon \leq \varepsilon_0 = \mathcal{O}(1)$ and $0 \leq \mu \leq 1$. We let

$$\alpha = \min\{\alpha_1, \alpha_2\}, \quad \gamma < \min\left\{\frac{b}{2a_1}, \frac{b}{2a_2}\right\}. \tag{1.1e}$$

The analysis for this two-parameter problem naturally splits into two cases, $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ and $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$. In the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, the analysis is similar to the reaction-diffusion problem [1] when $\mu = 0$ and boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ appear in the neighbourhood of all four edges. For the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ the analysis is more intricate and boundary layers of width $\mathcal{O}(\frac{\varepsilon}{\mu})$ appear in the neighbourhood of the edges $x = 0,\ y = 0$ and boundary layers of width $\mathcal{O}(\mu)$ appear in the neighbourhood of $x = 1, y = 1$.

In this paper, we confine the discussion to the case of

$$\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}. \tag{1.1f}$$

2. Bounds on the Solution u and its Derivatives

In this section we establish *a priori* bounds on the derivatives of the solution of (1.1). These bounds are essential for the error analysis in subsequent sections. We begin by stating a continuous minimum principle for the differential operator in (1.1). The proof of this comparison principle is standard.

Lemma 1. *If $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $L_{\varepsilon,\mu} w|_{\Omega} \leq 0$ and $w|_{\partial\Omega} \geq 0$, then $w|_{\bar{\Omega}} \geq 0$.*

An immediate consequence of this Lemma is the stability bound:

$$||u|| \leq ||s||_{\Gamma_B \cup \Gamma_T} + ||q||_{\Gamma_L \cup \Gamma_R} + \frac{1}{2\beta}||f||.$$

The next lemma establishes parameter-explicit bounds on the derivatives of the solution. Within the realm of singularly perturbed problems, the proof is essentially classical except that here it is applied in the case of a two parameter problem.

Lemma 2. *If $f \in C^{1,\lambda}(\bar{\Omega})$, $s, q \in C^{3,\lambda}(0,1)$ are independent of ε and μ, and assuming sufficient compatibility of the boundary data at the corners, the derivatives of the solution of* (1.1) *satisfy the following bounds for all nonnegative integers k and m, where $1 \leq k+m \leq 3$*

$$\left\| \frac{\partial^{k+m} u}{\partial x^k \partial y^m} \right\| \leq C \left(\frac{1}{\sqrt{\varepsilon}} \right)^{k+m} (1 + ||u||),$$

where C is independent of the parameters ε and μ.

Proof. Note that we can write $\omega = u - g$, where ω satisfies an equation similar to (1.1) with homogeneous boundary conditions. We have

$$L_{\varepsilon,\mu}\omega = f - L_{\varepsilon,\mu} g = \hat{f} \quad \text{on} \ \ \Omega, \quad \omega \equiv 0 \quad \text{on} \ \ \partial\Omega.$$

Consider the transformation

$$\xi = \frac{(\mu + \sqrt{\varepsilon})x}{\varepsilon}, \quad \eta = \frac{(\mu + \sqrt{\varepsilon})y}{\varepsilon}.$$

The transformed domain $\tilde{\Omega}$ is given by $\tilde{\Omega} = (0, \frac{\mu+\sqrt{\varepsilon}}{\varepsilon})^2$. Applying this transformation, the above differential equation becomes

$$\tilde{\omega}_{\xi\xi} + \tilde{\omega}_{\eta\eta} + \frac{\mu}{\sqrt{\varepsilon}+\mu}\tilde{a_1}\tilde{\omega}_\xi + \frac{\mu}{\sqrt{\varepsilon}+\mu}\tilde{a_2}\tilde{\omega}_\eta - \frac{\varepsilon}{(\sqrt{\varepsilon}+\mu)^2}\tilde{b}\tilde{\omega} = \tilde{f}, \quad \text{on } \tilde{\Omega},$$

where $\tilde{\omega}(\xi, \eta) = \omega(x, y)$. The coefficients $\tilde{a_1}$,$\tilde{a_2}$, $\tilde{b}$ are defined similarly and

$$\tilde{f}(\xi, \eta) = \frac{\varepsilon}{(\sqrt{\varepsilon}+\mu)^2}\hat{f}(x, y).$$

For each $(\zeta_1, \zeta_2) \in \tilde{\Omega}$, we denote the rectangle $(\zeta - \delta, \zeta + \delta)^2 \cap \tilde{\Omega}$ by $\tilde{R}_\delta(\zeta_1, \zeta_2)$, where $\delta = \mathcal{O}(1)$. For all $(\xi, \eta) \in \tilde{\Omega}$ and $\tilde{R}_\delta$ we have (see [4, page 110] or [5, Theorem 3.1]) that

$$|\tilde{\omega}|_{1,\lambda,\tilde{R}_\delta} \leq C(\|\tilde{f}\|_{0,\lambda,\tilde{R}_{2\delta}} + \|\tilde{\omega}\|_{\tilde{R}_{2\delta}}),$$

and for $l = 0, 1$

$$|\tilde{\omega}|_{l+2,\lambda,\tilde{R}_{2\delta}} \leq C(\|\tilde{f}\|_{l,\lambda,\tilde{R}_{2\delta}} + \|\tilde{\omega}\|_{\tilde{R}_{2\delta}}),$$

where $|\cdot|_{l+2,\lambda,\tilde{R}_\delta}$ and $\|\cdot\|_{l+1,\lambda,\tilde{R}_\delta}$ are the standard semi-norms and norms in $C^{k,\lambda}$ (see, for example, [4, 5]). Since $|\omega|_{k,\Omega} \leq |\omega|_{k,\lambda,\Omega}$, we obtain

$$|\tilde{\omega}|_{1,\tilde{R}_\delta} \leq |\omega|_{1,\lambda,\tilde{R}_\delta} \leq C(\|\tilde{f}\|_{0,\lambda,\tilde{R}_{2\delta}} + \|\tilde{\omega}\|_{\tilde{R}_{2\delta}}),$$

and for $l = 0, 1$

$$|\tilde{\omega}|_{l+2,\tilde{R}_\delta} \leq |\omega|_{l+2,\lambda,\tilde{R}_\delta} \leq C(\|\tilde{f}\|_{l,\lambda,\tilde{R}_{2\delta}} + \|\tilde{\omega}\|_{\tilde{R}_{2\delta}}).$$

Transforming back to the original variables this implies for all $(x, y) \in \Omega$ and $R_\delta = R_\delta(x, y)$

$$\left(\frac{\varepsilon}{\mu + \sqrt{\varepsilon}}\right)|\omega|_{1,R_{2\delta}} \leq C\left(\frac{\varepsilon}{(\mu + \sqrt{\varepsilon})^2}\left(\frac{\varepsilon}{\mu + \sqrt{\varepsilon}}\right)^\lambda \|\hat{f}\|_{0,\lambda,R_{2\delta}} + \|\omega\|_{R_{2\delta}}\right),$$

and for $l = 0, 1$

$$\left(\frac{\varepsilon}{\mu + \sqrt{\varepsilon}}\right)^{l+2}|\omega|_{l+2,R_\delta} \leq C\frac{\varepsilon}{(\mu + \sqrt{\varepsilon})^2}$$

$$\times\left(\sum_{v=0}^{l}\left(\frac{\varepsilon}{\mu + \sqrt{\varepsilon}}\right)^v|\hat{f}|_{v,R_{2\delta}} + \left(\frac{\varepsilon}{\mu + \sqrt{\varepsilon}}\right)^{l+\lambda}|\hat{f}|_{l,\lambda,R_{2\delta}}\right) + C\|\omega\|_{R_{2\delta}}.$$

Replacing $\hat{f}$ by $f - L_{\varepsilon,\mu}g$ yields the required result. ∎

Remark 1. The proof in Lemma 2 is applicable in the case where the positivity constraint (1.1d) is relaxed to

$$a_1(x, y) \geq 0, \quad a_2(x, y) \geq 0, \quad b(x, y) \geq 2\beta > 0.$$

3. Regular Component

In order to obtain more informative parameter explicit error bounds on the derivatives of the solution of (1.1), the solution is decomposed into a sum of regular and layer components. The extension of idea from [9] is used to define the regular solution, which avoids imposing overly artificial compatibility conditions. We show that there exists a function v such that $L_{\varepsilon,\mu}v = f$ and when its boundary conditions are chosen appropriately, the function v and its derivatives up to second order are bounded independently of the small parameters.

Define the zero order differential operator L_0 to be

$$L_0 z = -bz.$$

Consider the extended domain $\Omega^* = (-d, 1+d) \times (-d, 1+d) \supset \overline{\Omega}$, $d > 0$. The extended differential operators $L^*_{\varepsilon,\mu}$ and L^*_0 coincide with the operators $L_{\varepsilon,\mu}$ and L_0 respectively on Ω. Below, we implicitly define smooth extensions a^*_1, a^*_2, b^* and f^* of the functions a_1, a_2, b and f to Ω^* so that they coincide with the functions a_1, a_2, b and f in $\overline{\Omega}$. These extensions are constructed so that $a^*_1 \geq 0,\ a^*_2 \geq 0,\ b^* \geq 2\beta > 0$ at all points in Ω^* and

$$f^* = a^*_1 = a^*_2 = 0, \quad b^* = 2\beta, \quad (x,y) \in \Omega^* \setminus D,$$

where D is an open set such that $\overline{\Omega} \subset D \subset \Omega^*$.

Consider the differential equation $L^*_{\varepsilon,\mu} v^* = f^*$ on Ω^* and decompose v^* as follows

$$v^*(x,y) = v^*_0(x,y) + \sqrt{\varepsilon} v^*_1(x,y) + \varepsilon v^*_2(x,y),$$

where

$$L^*_0 v^*_0 = f^*, \quad \sqrt{\varepsilon} L^*_0 v^*_1 = (L^*_0 - L^*_{\varepsilon,\mu}) v^*_0,$$
$$\varepsilon L^*_{\varepsilon,\mu} v^*_2 = \sqrt{\varepsilon}(L^*_0 - L^*_{\varepsilon,\mu}) v^*_1, \quad v^*_2|_{\partial\Omega^*} = 0.$$

Note that v^*_0 and v^*_1 satisfy zero order differential equations and hence there are no issues of compatibility. The term v^*_2 is the solution of an elliptic problem on the extended domain Ω^*. The extensions b^*, f^* are such that the function $g^* \equiv (L^*_0 - L^*_{\varepsilon,\mu}) v^*_1$ is zero at the four corners of the extended domain and $g^* \in C^{1,\lambda}(\bar{\Omega}^*)$. In this way the term $v_2 \in C^{3,\lambda}(\bar{\Omega}^*)$ is sufficiently regular for our purposes [3].

Given $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we see that the functions v^*_0 and v^*_1 and their derivatives are bounded independently of both small parameters. Since v^*_2 satisfies a similar equation to (1.1) we can use Lemma 1 and Lemma 2 along with $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ to obtain for $0 \leq k+m \leq 3$

$$\left\| \frac{\partial^{k+m} v^*_2}{\partial x^k \partial y^m} \right\| \leq C\Big(\frac{1}{\sqrt{\varepsilon}}\Big)^{k+m}, \quad \text{if} \quad \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}.$$

Define the regular component v to be the solution of the elliptic problem

$$L_{\varepsilon,\mu} v = f,\ (x,y) \in \Omega, \quad \text{and} \quad v = v^*, (x,y) \in \partial\Omega.$$

Assuming sufficient smoothness of the coefficients, we can establish the following bounds on the first three derivatives of the regular component v

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial y^m} \right\| \leq C\Big(1 + \varepsilon^{(2-k-m)/2}\Big), \ \ 0 \leq k+m \leq 3, \quad \text{if} \ \ \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}.$$

4. Layer Components

Associated with the left edge Γ_L, we define a boundary layer function w_L. Consider the extended domain $\Omega^{**} = (0,1) \times (-d, 1+d),\ 0.5 > d > 0$. We define w_L^* to be the solution of

$$L^{**}_{\varepsilon,\mu} w_L^* = 0, \quad (x,y) \in \Omega^{**}, \tag{4.1a}$$

$$w_L^*|_{\Gamma_L} = (u-v)^*, \quad w_L^*(1,y) = 0, \quad y \in [-d, 1+d], \tag{4.1b}$$

$$w_L^*(x,-d) = w_L^*(x,1+d) = 0, \quad x \in [0,1]. \tag{4.1c}$$

Lemma 3. *When $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, the boundary layer function w_L^* satisfies the following bounds*

$$|w_L^*(x,y)| \le Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}, \quad \left\|\frac{\partial^i w_L^*}{\partial y^i}\right\| \le C(1 + \sqrt{\varepsilon}^{\ 1-i}), \qquad i = 1,2,3.$$

Proof. The boundary function $(u-v)(0,y)$ is extended so that $(u-v)^*(0,y) = 0$ for $y < -\frac{d}{2}$ and $y > 1 + \frac{d}{2}$. By the comparison principle, it follows that

$$|w_L^*(x,y)| \le Ce^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x}, \quad (x,y) \in \overline{\Omega}^{**}.$$

Note that the crude derivative bounds given in Lemma 2 also apply in the case when $a_1(x,y) \ge 0,\ a_2(x,y) \ge 0$. Using the same argument on the extended domain we can show that these crude bounds on the derivatives also apply to w_L^*. In the direction orthogonal to the layer we sharpen these bounds. We first obtain a bound on w_L^* to reflect the fact that it is zero on the edges Γ_T^{**} and Γ_B^{**}. Note that the coefficient a_2 is extended to the domain Ω^{**} so that $\|a_2^*\|_{\Omega^{**}} \le C_1(d+y)(1+d-y)$. Note also that

$$L^{**}_{\varepsilon,\mu}\left((d+y)(1+d-y)\right) = -2\varepsilon + \mu(1-2y)a_2^* - b^*(d+y)(1+d-y).$$

Assuming that μ is sufficiently small (so that $4C_1\mu < \beta$), it follows that

$$|w_L^*(x,y)| \le C(d+y)(1+d-y), \quad (x,y) \in \bar{\Omega}^{**}.$$

From the above bound on $|w_L^*(x,y)|$ and the fact that $w_L^*(x,-d) = w_L^*(x,1+d) = 0$, we obtain

$$\left|\frac{\partial w_L^*}{\partial y}(0,y)\right| \le C, \qquad \frac{\partial w_L^*}{\partial y}(1,y) = 0,$$

$$\left|\frac{\partial w_L^*}{\partial y}(x,-d)\right| \le C, \qquad \left|\frac{\partial w_L^*}{\partial y}(x,1+d)\right| \le C.$$

Differentiate the equation $L^{**}_{\varepsilon,\mu} w_L^* = 0$ with respect to y to obtain

$$L^{**}_{\varepsilon,\mu}\frac{\partial w_L^*}{\partial y} = -\mu\frac{\partial a_1^*}{\partial y}\frac{\partial w_L^*}{\partial x} - \mu\frac{\partial a_2^*}{\partial y}\frac{\partial w_L^*}{\partial y} + \frac{\partial b^*}{\partial y}w_L^* = \tilde{f}.$$

Using the crude derivative bounds from Lemma 2 and $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we have $\|\tilde{f}\| \leq C$ and therefore

$$\left\|\frac{\partial w_L^*}{\partial y}\right\| \leq C.$$

This argument can be extended to produce the higher derivative bounds. Using (4.1a), (4.1c) and the fact that $a_2^*(x,-d) = a_2^*(x,1+d) = 0$ we see that

$$\frac{\partial^2 w_L^*}{\partial y^2}(x,-d) = \frac{\partial^2 w_L^*}{\partial y^2}(x,1+d) = 0,$$

$$\left|\frac{\partial^2 w_L^*}{\partial y^2}(0,y)\right| \leq C, \qquad \frac{\partial^2 w_L^*}{\partial y^2}(1,y) = 0.$$

Using Taylor expansions and the bounds on the regular component v we have the bound

$$\left|\frac{\partial^2 w_L^*}{\partial y^2}(0,y)\right| \leq C\frac{(d+y)(1+d-y)}{\sqrt{\varepsilon}}.$$

Differentiate (4.1) twice with respect to y to obtain

$$L_{\varepsilon,\mu}^{**}\frac{\partial^2 w_L^*}{\partial y^2} = -2\mu\frac{\partial a_1^*}{\partial y}\frac{\partial^2 w_L^*}{\partial x\,\partial y} - 2\mu\frac{\partial a_2^*}{\partial y}\frac{\partial^2 w_L^*}{\partial y^2} + (2\frac{\partial b^*}{\partial y} - \mu\frac{\partial^2 a_2^*}{\partial y^2})\frac{\partial w_L^*}{\partial y}$$
$$- \mu\frac{\partial^2 a_1^*}{\partial y^2}\frac{\partial w_L^*}{\partial x} + \frac{\partial^2 b^*}{\partial y^2}w_L^* = \tilde{f}_1, \; (x,y) \in \Omega^{**}.$$

By construction the extensions a_1^*, a_2^* and b^* are such that

$$\left|\frac{\partial^k a_i^*}{\partial y^k}\right|, \left|\frac{\partial b^*}{\partial y}\right| \leq C(d+y)(1+d-y), \; k = 0,1,2 \text{ and } i = 1,2.$$

We deduce that $\|\tilde{f}_1\| \leq \dfrac{C}{\sqrt{\varepsilon}}(d+y)(1+d-y)$. Using the conditions

$$C_1\mu(1+2d) - b^* < 0, \quad |a_2^*| \leq C_1(d+y)(1+d-y),$$

we obtain

$$\left|\frac{\partial^2 w_L^*}{\partial y^2}\right| \leq \frac{C}{\sqrt{\varepsilon}}(d+y)(1+d-y).$$

Using this bound we obtain

$$\left|\frac{\partial^3 w_L^*}{\partial y^3}(x,-d)\right| \leq \frac{C}{\sqrt{\varepsilon}}, \quad \left|\frac{\partial^3 w_L^*}{\partial y^3}(x,1+d)\right| \leq \frac{C}{\sqrt{\varepsilon}}$$

$$\left|\frac{\partial^3 w_L^*}{\partial y^3}(0,y)\right| \leq \frac{C}{\sqrt{\varepsilon}}, \quad \frac{\partial^3 w_L^*}{\partial y^3}(1,y) = 0.$$

Differentiate (4.1) three times with respect to y to obtain

$$L_{\varepsilon,\mu}^{**}\frac{\partial^3 w_L^*}{\partial y^3} = \tilde{f}_2.$$

By suitable extensions $\|\tilde{f}_2\| \leq \dfrac{C}{\varepsilon}$ and then

$$\left\|\frac{\partial^3 w_L^*}{\partial y^3}\right\| \leq \frac{C}{\varepsilon}.$$

To finish, note that in the case where $4C_1\mu > \beta$ and $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$ then $\varepsilon \geq C$ and we are in the non-singularly perturbed case in which all the derivatives are bounded independently of ε. ■

Define the boundary layer function w_L by

$$L_{\varepsilon,\mu} w_L = 0,\ (x,y) \in \Omega, \quad w_L\Big|_{\Gamma_L} = u - v,\ w_L\Big|_{\Gamma_R} = 0,\ w_L\Big|_{\Gamma_T \cup \Gamma_B} = w_L^*.$$

For the boundary layer function w_T associated with the top edge Γ_T , the extended domain is taken to be $(x, y) \in [-d, 1 + d] \times [0, 1]$ and

$$L_{\varepsilon,\mu} w_T = 0,\ (x,y) \in \Omega, \quad w_T\Big|_{\Gamma_T} = u - v,\ w_T\Big|_{\Gamma_B} = 0,\ w_T\Big|_{\Gamma_L \cup \Gamma_R} = w_T^*.$$

Lemma 4. *When $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, the boundary layer function w_T^* satisfies the following bounds*

$$\left|w_T^*(x,y)\right| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)}, \quad \left\|\frac{\partial^i w_T^*}{\partial x^i}\right\| \leq C(1 + \sqrt{\varepsilon}^{\ 1-i}), \quad i = 1, 2, 3.$$

Proof. The proof follows the same lines as the proof of the previous lemma. However, note that

$$\begin{aligned} L_{\varepsilon,\mu}^{**} e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)} &= \Big(\gamma\alpha + \frac{\mu}{\sqrt{\varepsilon}} a_2^* \sqrt{\alpha\gamma} - b^*\Big) e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)} \\ &\leq \Big(\gamma\alpha + \gamma a_2^* - b^*\Big) e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)}. \end{aligned}$$

Note that on the original domain $\gamma\alpha + \gamma a_2 - b \leq 2\gamma a_2 - b \leq 0$. The extensions are constructed to maintain this sign pattern on the extended domain. Also, a_1 can be suitably extended so that

$$L_{\varepsilon,\mu}^{**}\big((x + d)(1 + d - x)\big) = \big(-2\varepsilon + \mu(1 - 2x)a_1^* - b^*(x + d)(1 + d - x)\big) \leq 0.$$

■

Define the boundary layer functions associated with the other two edges w_R and w_B analogously to w_T and w_L. Associated with the corner $\Gamma_{LB} = \Gamma_L \cap \Gamma_B$ define a corner layer function w_{LB} such that

$$\begin{aligned} &L_{\varepsilon,\mu} w_{LB} = 0,\ \ (x,y) \in \Omega, \\ &w_{LB} = -w_B,\ \ (x,y) \in \Gamma_L, \quad w_{LB} = -w_L,\ \ (x,y) \in \Gamma_B, \\ &w_{LB} = 0,\ \ (x,y) \in \Gamma_R, \quad w_{LB} = 0,\ \ (x,y) \in \Gamma_T. \end{aligned}$$

Note that at the corner $(0,0)$, $w_L(x,0)$ is equal to $w_L(0,y) = (u-v)(0,y)$ which is equal to $(u-v)(x,0) = w_B(x,0)$ which in turn is equal to $w_B(0,y)$. Note also that $u-v, w_L, w_B \in C^{3,\lambda}(\bar{\Omega})$ and

$$L_{\varepsilon,\mu} w_L = L_{\varepsilon,\mu} w_B = L_{\varepsilon,\mu}(u-v) = 0.$$

Hence we have sufficient compatibility for $w_{LB} \in C^{3,\lambda}(\bar{\Omega})$. By using the comparison principle and the obvious barrier function, the following bound on w_{LB} holds

$$\Big| w_{LB}(x,y) \Big| \le C e^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}x} e^{-\sqrt{\frac{\gamma\alpha}{\varepsilon}}y}.$$

Associated with the corner $\Gamma_{RT} = \Gamma_R \cap \Gamma_T$ define a corner layer function w_{RT} such that

$$\begin{aligned} &L_{\varepsilon,\mu} w_{RT} = 0, \quad (x,y) \in \Omega, \\ &w_{RT} = -w_R, \quad (x,y) \in \Gamma_T, \quad w_{RT} = -w_T, \quad (x,y) \in \Gamma_R, \\ &w_{RT} = 0, \quad (x,y) \in \Gamma_L, \quad w_{RT} = 0, \quad (x,y) \in \Gamma_B. \end{aligned}$$

Noting that

$$\begin{aligned} L_{\varepsilon,\mu} & e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \\ &= \Big(\frac{\gamma\alpha}{2} + \frac{\mu a_1 \sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} + \frac{\mu a_2 \sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} - b \Big) e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \\ &\le \Big(\frac{\gamma\alpha}{2} + \frac{\gamma}{2}(a_1 + a_2) - b \Big) e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \\ &\le \Big(\gamma(a_1 + a_2) - b \Big) e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)} \le 0 \end{aligned}$$

one can establish the bound

$$\Big| w_{RT}(x,y) \Big| \le C e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)}.$$

Analogous bounds hold for the other two corners. In summary we state the main result of this paper:

Theorem 1. *The solution* u *of* (1.1) *can be decomposed into the following sum of components*

$$u = v + w_L + w_R + w_T + w_B + w_{LB} + w_{LT} + w_{RB} + w_{RT}$$

where $L_{\varepsilon,\mu} v = f$*, and the layer and corner layer functions are each solutions of the homogeneous equation* $L_{\varepsilon,\mu} w = 0$*. Boundary conditions for these functions can be specified so that the bounds on the components and their derivatives given below hold:*

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial y^m} \right\| \leq C\big(1+\varepsilon^{\frac{2-k-m}{2}}\big), \qquad 0 \leq k+m \leq 3,$$

$$|w_L(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}, \qquad |w_B(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y},$$

$$|w_R(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-x)}, \qquad |w_T(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}(1-y)},$$

$$|w_{LB}(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y}, \qquad |w_{LT}(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}x}e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)},$$

$$|w_{RB}(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)}e^{-\frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}y}, \quad |w_{RT}(x,y)| \leq Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)}e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-y)},$$

$$\left\| \frac{\partial^k w_L}{\partial y^k} \right\| \leq C\left(1+\sqrt{\varepsilon}^{\;1-k}\right), \qquad \left\| \frac{\partial^k w_R}{\partial y^k} \right\| \leq C\left(1+\sqrt{\varepsilon}^{\;1-k}\right), \; 1 \leq k \leq 3,$$

$$\left\| \frac{\partial^k w_B}{\partial x^k} \right\| \leq C\left(1+\sqrt{\varepsilon}^{\;1-k}\right), \qquad \left\| \frac{\partial^k w_T}{\partial x^k} \right\| \leq C\left(1+\sqrt{\varepsilon}^{\;1-k}\right), \; 1 \leq k \leq 3.$$

For all the layer components, we also have that

$$\left\| \frac{\partial^{k+m} w}{\partial x^k \partial y^m} \right\| \leq C\varepsilon^{\frac{-k-m}{2}}, \quad 1 \leq k+m \leq 3.$$

5. Numerical Method

Consider the following upwind finite difference scheme

$$L^{N,M}U(x_i,y_j) = \varepsilon\delta_x^2 U + \varepsilon\delta_y^2 U + \mu a_1 D_x^+ U + \mu a_2 D_y^+ U - bU = f,$$

where D^+ is the forward difference operator and δ^2 is the standard second order centered difference operator. We apply the above finite difference operator on the tensor product mesh $\Omega^{N,M} = \Omega^N \times \Omega^M$, where Ω^N (Ω^M) is a piecewise uniform mesh [2] that places a uniform mesh containing $\mathcal{O}(N)$ mesh points in each of the three subregions $[0,\sigma], [\sigma, 1-\sigma], [1-\sigma, 1]$. The transition points σ_x, σ_y are taken to be

$$\sigma_x = \min\left\{\frac{1}{4}, \; 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln N\right\}, \quad \sigma_y = \min\left\{\frac{1}{4}, \; 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln M\right\}.$$

Remark 2. Note that if $a_1 = a_2 = 1, \; \mu = 0$ then $\gamma = \beta$ and the above numerical method coincides with the method analysed in [1] for the reaction–diffusion problem and shown to be parameter-uniform of second order (up to logarithmic factors).

From the pointwise bounds on the layer components and for this choice of transition point, it follows that

$$\left\| w_L(x_i,y_j) \right\| \leq CN^{-2}, \; x_i \geq \sigma_x, \quad \text{when } \sigma_x < \frac{1}{4}.$$

The discrete solution is decomposed into the sum

$$U = V + W_L + W_R + W_T + W_B + W_{LB} + W_{LT} + W_{RB} + W_{RT},$$

where

$$L^{N,M}V = f, \quad V\Big|_{\Gamma^{N,M}} = v\Big|_{\Gamma^{N,M}},$$
$$L^{N,M}W_L = 0, \quad W_L\Big|_{\Gamma^{N,M}} = w_L\Big|_{\Gamma^{N,M}},$$

and the other layer functions are defined similarly. The maximum pointwise error $\|u - U\|$ is estimated by bounding each of the error components $\|v - V\|, \|w_L - W_L\|, \|w_R - W_R\| \ldots$ separately. The error $\|v - V\|$ is bounded using a classical truncation error and comparison principle argument. Using a standard truncation error argument

$$|L^{N,M}(V - v)(x_i, y_j)| \leq C_1 N^{-1} \left(\varepsilon\|v_{xxx}\| + \mu\|v_{xx}\|\right) + C_2 M^{-1} \left(\varepsilon\|v_{yyy}\| + \mu\|v_{yy}\|\right) \leq C(N^{-1} + M^{-1})\sqrt{\varepsilon}.$$

Thus at each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$ the regular component of the error satisfies the following parameter-uniform estimate

$$|(V - v)(x_i, y_j)| \leq C(N^{-1} + M^{-1})\sqrt{\varepsilon}.$$

Lemma 5. *At each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$, the left singular component of the error satisfies the bound*

$$|(W_L - w_L)(x_i, y_j)| \leq C(N^{-1} \ln N + M^{-1}).$$

Proof. Using the truncation error bounds

$$|L^{N,M}(W_L - w_L)(x_i, y_j)| \leq C_1(h_{i+1} + h_i)\left(\varepsilon\|w_{Lxxx}\| + \mu\|w_{Lxx}\|\right) + C_2(k_{j+1} + k_j)\left(\varepsilon\|w_{Lyyy}\| + \mu\|w_{Lyy}\|\right)$$

and since $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we obtain

$$|L^{N,M}(W_L - w_L)(x_i, y_j)| \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i) + C_2 M^{-1}. \tag{5.1}$$

The following bounds on the discrete boundary layer function W_L

$$|W_L(x_i, y_j)| \leq C \prod_{s=1}^{i} \left(1 + \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} h_s\right)^{-1} = \Psi_i,$$

are established using the discrete comparison principle and the fact that

$$\varepsilon\delta^2 \leq \frac{\gamma\alpha}{2}\Psi_{i+1}, \quad \mu D^+\Psi_i = -\frac{\mu\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\Psi_{i+1}, \quad b\Psi_i \geq b\Psi_{i+1}.$$

In the region $[\sigma_x, 1) \times (0, 1)$

$$|W_L(x_i,y_j)| \leq C\Psi_{N/4} \leq CN^{-1}, \quad \sigma_x < 1/4,$$

which leads to

$$|(W_L - w_L)(x_i,y_j)| \leq CN^{-1}, \quad (x_i,y_j) \in [\sigma_x, 1) \times (0,1).$$

Note that, for $x_i < \sigma_x$, the truncation error is

$$|L^{N,M}(w_L - W_L)| \leq CN^{-1}\ln N\Big(\varepsilon^{3/2}\Big\|\frac{\partial^3 w_L}{\partial x^3}\Big\| + \mu\Big\|\frac{\partial^2 w_L}{\partial x^2}\Big\|\Big) \\ +CM^{-1}\Big(\varepsilon\Big\|\frac{\partial^3 w_L}{\partial y^3}\Big\| + \mu\Big\|\frac{\partial^2 w_L}{\partial y^2}\Big\|\Big).$$

In the layer region $(0,\sigma)\times(0,1)$ and when $\sigma < \frac{1}{4}$, $h_i = h_{i+1} = \dfrac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}N^{-1}\ln N$ one can use (5.1) to obtain

$$|L^{N,M}(W_L - w_L)| \leq C(N^{-1}\ln N + M^{-1}), \quad x_i < \sigma_x.$$

Use an appropriately chosen barrier function and the discrete minimum principle to obtain the required result in this region. When either $\sigma_x = 1/4$ or $\sigma_y = 1/4$, a classical truncation error with discrete comparison principle is used to establish the error bound. ■

Analogous bounds hold for the error components $|(W_B - w_B)|$, $|(W_R - w_R)|$ and $|(W_T - w_T)|$.

Lemma 6. *At each mesh point $(x_i,y_j) \in \Omega^{\bar{N},M}$, the left, bottom and right singular components of the error satisfies the following estimates*

$$|(W_B - w_B)(x_i,y_j)| \leq C(N^{-1} + M^{-1}\ln M),$$
$$|(W_R - w_R)(x_i,y_j)| \leq C(N^{-1}\ln N + M^{-1}),$$
$$|(W_T - w_T)(x_i,y_j)| \leq C(N^{-1} + M^{-1}\ln M).$$

Lemma 7. *At each mesh point $(x_i,y_j) \in \bar{\Omega}^{N,M}$, the bottom-left corner singular component of the error satisfies the following estimate*

$$|(W_{LB} - w_{LB})(x_i,y_j)| \leq C(N^{-1}\ln N + M^{-1}\ln M).$$

Proof. Note the truncation error bounds

$$|L^{N,M}(W_{LB} - w_{LB})(x_i,y_j)| \leq C_1(h_{i+1}+h_i)\left(\varepsilon\|w_{LBxxx}\| + \mu\|w_{LBxx}\|\right) \\ + C_2(k_{j+1}+k_j)\left(\varepsilon||w_{LByyy}|| + \mu||w_{LByy}||\right) \\ \leq \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1}+h_i) + \frac{C_2}{\sqrt{\varepsilon}}(k_{j+1}+k_j).$$

Consider the region $\Omega^{N,M}\backslash(0,\sigma)\times(0,\sigma)$. Note that

$$|W_{LB}(x_{\frac{N}{4}}, y_{\frac{N}{4}})| \le C \prod_{s=1}^{\frac{N}{4}} \left(1 + \frac{4\sigma_x}{N}\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1} \prod_{r=1}^{\frac{M}{4}} \left(1 + \frac{4\sigma_y}{M}\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}\right)^{-1}$$

In an analogous fashion to the bound on W_L, when $\sigma_x < 1/4,\ \sigma_y < 1/4$ we have

$$|W_{LB}(x_i, y_j)| \le CN^{-1}M^{-1}, \qquad x_1 \ge \sigma_x \text{ and } y_j \ge \sigma_y$$

and

$$|w_{LB}(x_i, y_j)| \le CN^{-2}M^{-2}, \qquad x_1 \ge \sigma_x \text{ and } y_j \ge \sigma_y.$$

In the region $\Omega^{N,M}\backslash(0,\sigma_x) \times (0,\sigma_y)$ one can establish

$$|(W_{LB} - w_{LB})(x_i, y_j)| \le C(N^{-1} + M^{-1}), \quad \sigma_x < 1/4,\ \sigma_y < 1/4.$$

In the corner region $(0,\sigma_x) \times (0,\sigma_y)$, the mesh sizes are such that

$$h_i = h_{i+1} = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} N^{-1} \ln N, \quad k_j = k_{j+1} = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}} M^{-1} \ln M.$$

Using the truncation error bound we obtain

$$|L^{N,M}(W_{LB} - w_{LB})(x_i, y_j)| \le C(N^{-1}\ln N + M^{-1}\ln M),$$

The proof is completed as in the case of Lemma 6. ∎

Similar bounds hold for the error components $|(W_{RB} - w_{RB})|$, $|(W_{RT} - w_{RT})|$ and $|(W_{LT} - w_{LT})|$ and we conclude with the following result.

Lemma 8. *Let u be the solution of the differential equation* (1.1) *and U be the discrete solution defined above. Then at each mesh point $(x_i, y_j) \in \bar{\Omega}^{N,M}$*

$$|(U - u)(x_i, y_j)| \le CN^{-1}\ln N + CM^{-1}\ln M$$

where C is a constant independent of ε, μ and N.

References

[1] C. Clavero, J.L. Gracia and E. O'Riordan. A parameter robust numerical method for a two dimensional reaction-diffusion problem. *Mathematics of Computation*, **74**, 1743–1758, 2005.

[2] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan and G.I. Shishkin. *Robust computational techniques for boundary layers.* Chapman and Hall/CRC Press, Boca Raton, 2000.

[3] H. Han and R.B. Kellogg. Differentiability properties of solutions of the equation $-\varepsilon^2\triangle u + ru = f(x, y)$ in a square. *SIAM J. Math. Anal.*, **21**, 394–408, 1990.

[4] O.A. Ladyzhenskaya and N.N. Ural'tseva. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York and London, 1968.

[5] T. Linß and M. Stynes. Asymptotic analysis and Shishkin-type decomposition for an elliptic convection-diffusion problem. *J. Math. Anal. and Applications*, **261**, 604–632, 2001.

[6] E. O'Riordan, M.L. Pickett and G.I. Shishkin. Parameter-uniform finite difference schemes for singularly perturbed parabolic diffusion-convection-reaction problems. *Mathematics of Computation.* (to appear)

[7] E. O'Riordan, M.L. Pickett and G.I. Shishkin. Singularly perturbed problems modeling reaction-convection-diffusion processes. *Computational Methods in Applied Mathematics*, **3**(3), 424–442, 2003.

[8] H.-G. Roos and Z. Uzelac. The SDFEM for a convection diffusion problem with two small parameters. *Computational Methods in Applied Mathematics*, **3**(3), 1–16, 2003.

[9] G.I. Shishkin. *Discrete approximation of singularly perturbed elliptic and parabolic equations.* Russian Academy of Sciences, Ural Section, Ekaterinburg, 1992. (In Russian)

[10] G.I. Shishkin and V.A. Titov. A difference scheme for a differential equation with two small parameters at the derivatives. *Chisl. Metody Meh. Sploshn. Sredy*, **7**(2), 145–155, 1976. (In Russian)

[11] V.A. Titov and G.I. Shishkin. A numerical solution of a parabolic equation with small parameters multiplying the derivatives with respect to the space variables. *Trudy Inst. Mat. i Meh. Ural Nauchn. Centr Akad. Nauk SSSR*, **21**, 38–43, 1976. (In Russian)