ON FULLY DISCRETE GALERKIN APPROXIMATIONS FOR THE CAHN-HILLIARD EQUATION

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Abstract. Standard Galerkin approximations, using smooth splines to solutions of the nonlinear evolutionary Cahn-Hilliard equation are analysed. The existence, uniqueness and convergence of the fully discrete Crank-Nicolson scheme are discussed. At last a linearized Galerkin approximation is presented, which is also second order accurate in time fully discrete scheme.

Key words: Cahn-Hilliard equation, Galerkin scheme, convergence, linearization

1. Introduction

Consider the Cahn-Hilliard equation

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 \phi(u)}{\partial x^2}, \quad x \in \Omega =]0,1[, \quad t \in (0,T], \tag{1.1}$$

with boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0,1} = 0, \quad \left.\frac{\partial^3 u}{\partial x^3}\right|_{x=0,1} = 0, \quad t \in (0,T], \tag{1.2}$$

and an initial condition

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \tag{1.3}$$

where $\phi(u) = \gamma(u^3 - \beta^2 u)$ and γ, β are constants with $\gamma > 0$.

The Cahn-Hilliard equation (1.1) arises as a phenomenological mode for phase separation in cooling binary solutions such as alloys, glasses and polymer

mixtures, see Cahn and Hilliard [4], Novick-Cohen et al.[7] and the references cited therein. Here $u(x,t)$ is a perturbation of the concentration of one of the phases.

Global existence and uniqueness of the solution for (1.1) have been shown by Elliott and Zheng [11] and Yin [18]. A continuous in time Morley finite element Galerkin approximation for (1.1) is presented and an optimal-order error estimates in L^2 is derived, see Elliott et al.[10]. A semi discrete finite element method (with quadrature) for (1.1) was first introduced and analysed by Elliott et al. [9]. Mixed finite element methods have been applied by Dean et al. [8]. A finite difference schemes are developed in Choo et al. [5, 6], Furihata [12, 13], Sun [16].

The plan of the paper is as follows. In section 2, after explaining notation, the numerical scheme is described in detail. The existence and uniqueness of the approximate solution are shown in section 3. Optimal rate of convergence estimates for the numerical scheme is proved in section 4. In the last section, a linearized Galerkin method is presented which is also second-order convergent.

Throughout this article, C denotes generic constant, not necessarily the same at different occurrence.

2. Numerical Method

Let r and l be integers with $r \geq 4$ and $1 \leq l \leq r-2$. We consider a family of partitions $0 = x_0 < x_1 < x_2 < ... < x_I = 1$ of $[0,1]$ into subintervals $J_i = (x_{i-1}, x_i)$ and set

$$h = \max_{1 \leq i \leq I} (x_i - x_{i-1}).$$

Throughout the paper, we use D to denote $\dfrac{\partial}{\partial x}$. The norms of $L^2(\Omega)$, $L^\infty(\Omega)$ and $H^s(\Omega)$ are denoted by $\|.\|, \|.\|_\infty$ and $\|.\|_s$. The semi-norm $\|D^s v\|$ is denoted by $|v|_s$, $(v, w) = \displaystyle\int_\Omega vw\, dx$ denotes the inner product of $L^2(\Omega)$. Let S_h be a space of the piecewise polynomial splines:

$$S_h = \left\{ \chi \in C^l(\Omega), \quad \chi\big|_{J_i} \in P_{r-1}(J_i), \quad i = 1, ..., I; \quad D\chi(0) = D\chi(1) = 0 \right\},$$

where $P_{r-1}(J_i)$ denotes the set of polynomials on J_i of degree less or equal to $r-1$. Let $S_h \subset \tilde{H}^2(\Omega)$, where

$$\tilde{H}^2(\Omega) = \left\{ u \in H^2(\Omega), \quad \frac{\partial u}{\partial x} = 0, \quad x \in \partial\Omega \right\},$$

and denote

$$\tilde{S}_h = S_h \cap \{\chi, (\chi, 1) = 0\}.$$

A natural Galerkin approximation to (1.1) is: find $u_h \in S_h$ satisfying

$$\left(\frac{\partial u_h}{\partial t}, \chi \right) + (D^2 u_h, D^2\chi) = (\phi(u_h), D^2\chi), \quad \forall \chi \in S_h, \qquad (2.1)$$

with

$$u_h(0) = u_{0h}, \qquad (2.2)$$

where $u_{0h} \in S_h$ is an appropriate approximation to u_0.

We introduce the so called elliptic projection $P_h : \tilde{H}^2(\Omega) \to \tilde{S}_h$ defined by the following problem (see,e.g Elliott et al. [11]): for $v \in \tilde{H}^2(\Omega)$, the function $P_h v$ is the unique solution of

$$\begin{cases} (D^2(P_h v - v), D^2 \chi) = 0, & \forall \chi \in \tilde{S}_h, \\ (P_h v - v, 1) = 0. \end{cases} \qquad (2.3)$$

The existence of a unique $P_h v$ satisfying (2.3) follows from the Lax-Milgram Theorem and the Friedrichs-Poincaré inequality

$$\|w\|_2 \le C(|w|_2 + |(w,1)|), \quad \forall w \in \tilde{H}^2(\Omega).$$

We begin with the following results due to Elliott and Zheng [11]

Proposition 1. *With P_h defined by (2.3), we have for $v \in H^r(\Omega) \cap \tilde{H}^2(\Omega)$,*

$$\sum_{j=0}^{2} h^j |v - P_h v|_j \le Ch^r \|v\|_r, \qquad (2.4)$$

and if $v \in \tilde{H}^2(\Omega)$, then

$$\|v - P_h v\|_\infty \le Ch^r \|v\|_{W^r_\infty(\Omega)}. \qquad (2.5)$$

Theorem 1. *Suppose that the solution $u(t)$ of (1.1) is sufficiently regular for a given $T > 0$ and that the solution of (2.1) satisfies*

$$\|u_h(t)\|_\infty \le C_T, \quad 0 \le t \le T. \qquad (2.6)$$

If $u_{0h} = P_h u_0$, then

$$\|u(t) - u_h(t)\|_\infty \le C_T(u)h^r, \quad \forall t \in [0,T]. \qquad (2.7)$$

Remark 1. The assumption (2.6) is not a restriction. By a standard argument and using error estimates (2.7) we may justify (2.6) a posteriori for any $T > 0$. (See Thomée [17], pp. 213 – 214). We shall need the same assumption in sections 3, 4 and 5.

The Crank-Nicolson fully discrete approximation is defined in the following way: find a sequence $\{U^n\}_{n=0}^{N} \subset S_h$ satisfying

$$\begin{cases} (\partial_t U^n, \chi) + (D^2 U^{n-\frac{1}{2}}, D^2 \chi) = (\phi(U^{n-\frac{1}{2}}), D^2 \chi), & \forall \chi \in S_h, \\ U^0 = u_{0h}. \end{cases} \qquad (2.8)$$

where $u_{0h} \in S_h$ is a suitable approximation to u_0, U^n is the approximation in S_h of $u(t)$ at $t = t^n = nk$ and $k = \dfrac{T}{N}$ denotes the size of the time discretization. In (2.8) we have used the notation

$$\partial_t U^n = \frac{1}{k}(U^n - U^{n-1}), \quad U^{n-\frac{1}{2}} = \frac{1}{2}(U^n + U^{n-1}).$$

For continuous function $u(t)$, we write $u^{n-\frac{1}{2}} = u(t^{n-\frac{1}{2}})$.

In this article, the main goal is to approximate the solutions of the Cahn-Hilliard equation by fully discrete finite element scheme. The standard Galerkin method is used for approximation in space and Crank-Nicolson-type second order accurate discretization is used for approximation of time derivatives. We will prove that the scheme is convergent.

3. Existence and Uniqueness

3.1. Existence

In this section we shall prove the existence of a sequence $\{U^n\}_{n=0}^N$ satisfying problem (2.8). For this, we shall use the following variant of the well-known fixed point theorem of Brouwer [2, 3]

Lemma 1. *Let H be a finite dimensional space with inner product $(.,.)_H$, and norm $\|.\|_H$. Let the map $g : H \to H$ be continuous. Suppose there exists $\alpha > 0$ such that $(g(Z), Z)_H \geq 0$ for all Z with $\|Z\|_H = \alpha$. Then there exists $Z^* \in H$, such that $g(Z^*) = 0$ and $\|Z^*\| \leq \alpha$.*

We shall need the auxiliary estimates:

Lemma 2. *For $v \in S_h$, we have*

$$|v|_1^2 \leq \frac{1}{\gamma\beta^2}|v|_2^2 + \frac{\gamma\beta^2}{4}\|v\|^2. \tag{3.1}$$

Proof. For $v \in S_h$, we prove immediately

$$-(D^2 v, v) = |v|_1^2,$$

from which, we have

$$|v|_1^2 \leq |v|_2\|v\|. \tag{3.2}$$

Using the inequality $ab \leq \dfrac{a^2}{\gamma\beta^2} + \dfrac{\gamma\beta^2}{4}b^2$, we prove the lemma. ∎

Lemma 3. *For $v \in S_h$ there holds*

$$(\phi(v), D^2 v) \leq \gamma\beta^2 |v|_1^2. \tag{3.3}$$

Proof. For $v \in S_h$, we have

$$(\phi(v), D^2 v) = -(D\phi(v), Dv) = -(\phi'(v)Dv, Dv),$$

using the definition of ϕ we find $\phi'(v) = \gamma(3u^2 - \beta^2) \geq -\gamma\beta^2$, which completes the proof. ∎

The proof of existence of $\{U^n\}_{n=0}^N$ proceeds in an inductive way. Obviously U^0 exists. Moreover, assume that $\{U^j\}_{j=0}^{n-1}$ exists. For $Z \in S_h$, define $g : S_h \longrightarrow S_h$ by

$$(g(Z), \chi) := (Z - U^{n-1}, \chi) + \frac{k}{2}\big((D^2 Z, D^2 \chi) - (\phi(Z), D^2 \chi)\big), \quad \forall \chi \in S_h. \quad (3.4)$$

Such a map exists by the Riesz representation theorem, g is obviously continuous. Taking $\chi = Z$ in (3.4) and using (3.3), we obtain

$$(g(Z), Z) \geq \|Z\|^2 - (U^{n-1}, Z) + \frac{k}{2}|Z|_2^2 - \frac{k\gamma\beta^2}{2}|Z|_1^2.$$

By Lemma 2, we find

$$(g(Z), Z) \geq \|Z\|^2 - \|U^{n-1}\|\|Z\| - \frac{k\gamma^2\beta^4}{8}\|Z\|^2.$$

Therefore,

$$(g(Z), Z) \geq \|Z\|\big((1 - \frac{k\gamma^2\beta^4}{8})\|Z\| - \|U^{n-1}\|\big).$$

Hence for $k < \frac{8}{\gamma^2\beta^4}$, $\|Z\| = \frac{8}{8 - k\gamma^2\beta^4}\|U^{n-1}\| + 1$, we have $(g(Z), Z) > 0$. Then it follows from Lemma 1 that there exists $Z^* \in S_h$ such that $g(Z^*) = 0$. It is easily seen that $U^n = 2Z^* - U^{n-1}$ satisfies (2.8).

3.2. Uniqueness

Assume that the solution u of (1.1) is sufficiently regular and that the solution of (2.8) satisfies

$$\|U^n\|_\infty \leq c_0, \quad n = 0, 1, \ldots, N. \quad (3.5)$$

For uniqueness, suppose that $V^n \in S_h$ and $V^0 = u_{0h}$ satisfy

$$(\partial_t V^n, \chi) + (D^2 V^{n-\frac{1}{2}}, D^2 \chi) = (\phi(V^{n-\frac{1}{2}}), D^2 \chi), \quad \forall \chi \in S_h, \quad (3.6)$$

and

$$\|V^n\|_\infty \leq c_0, \quad n = 0, 1, \ldots, N.$$

Denoting $E^i = U^i - V^i$, with $E^0 = 0$, from (2.8) and (3.6), we have for $\chi \in S_h$

$$(\partial_t E^n, \chi) + (D^2 E^{n-\frac{1}{2}}, D^2 \chi) = (\phi(U^{n-\frac{1}{2}}) - \phi(V^{n-\frac{1}{2}}), D^2 \chi). \quad (3.7)$$

Now, supposing $E^{n-1} = 0$ and choosing $\chi = E^{n-\frac{1}{2}}$ in (3.7), we obtain

$$\frac{1}{2k}(\|E^n\|^2 - \|E^{n-1}\|^2) + |E^{n-\frac{1}{2}}|_2^2 \leq \frac{1}{4}\|\phi(U^{n-\frac{1}{2}}) - \phi(V^{n-\frac{1}{2}})\|^2 + |E^{n-\frac{1}{2}}|_2^2. \quad (3.8)$$

Using the continuous differentiability of $\phi(.)$, we get

$$\|\phi(U^{n-\frac{1}{2}}) - \phi(V^{n-\frac{1}{2}})\| \leq C\|E^{n-\frac{1}{2}}\|, \tag{3.9}$$

where C is a constant dependent on c_0. By (3.8) and (3.9), we find

$$\frac{1}{2k}(\|E^n\|^2 - \|E^{n-1}\|^2) \leq \frac{C^2}{4}\|E^{n-\frac{1}{2}}\|^2 \leq \frac{C^2}{8}(\|E^n\|^2 + \|E^{n-1}\|^2),$$

from which, for k sufficiently small, we get

$$\|E^n\| \leq \lambda\|E^{n-1}\|, \qquad \lambda = \left\{\frac{4 + kC^2}{4 - kC^2}\right\}^{\frac{1}{2}}.$$

We see that $E^n = 0$ and this completes the proof of the uniqueness.

4. Estimation of the Convergence Rate

In this section, we estimate the error of the solution of the fully discrete problem (2.8). We use the standard error decomposition with $u^n = u(t^n)$:

$$U^n - u^n = (U^n - P_h u^n) + (P_h u^n - u^n) = \theta^n + \rho^n. \tag{4.1}$$

Theorem 2. *Let us assume that regularity assumption* (3.5) *is satisfied, where* U^n *and* u *are the solutions of* (2.8) *and* (1.1), *respectively. Suppose that the solution* u *is sufficiently regular. If the initial data satisfy the estimate*

$$\|U^0 - u_0\| \leq Ch^r, \tag{4.2}$$

then we have for sufficiently small k *that*

$$\|U^n - u(t^n)\| \leq C(h^r + k^2), \tag{4.3}$$

where C *is a constant independent of* h *and* k.

Proof. Since the estimate of ρ^n follows from (2.4), it is enough to estimate θ^n. Using the definition of the elliptic projection $P_h v$ in (2.3) with equations (1.1) and (2.8), we obtain the following equation, which is valid for all $\chi \in S_h$ such that $(\chi, 1) = 0$:

$$(\partial_t \theta^n, \chi) + (D^2\theta^{n-\frac{1}{2}}, D^2\chi) = (\phi(U^{n-\frac{1}{2}}) - \phi(u^{n-\frac{1}{2}}), D^2\chi)$$

$$- (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \chi) - (D^2(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}), D^2\chi). \tag{4.4}$$

By taking $\chi = \theta^{n-\frac{1}{2}}$, we obtain

$$(\partial_t \theta^n, \theta^{n-\frac{1}{2}}) + |\theta^{n-\frac{1}{2}}|_2^2 \leq I\,|\theta^{n-\frac{1}{2}}|_2 + J\,\|\theta^{n-\frac{1}{2}}\| + K\,|\theta^{n-\frac{1}{2}}|_2,$$

where

$$I = \|\phi(U^{n-\frac{1}{2}}) - \phi(u^{n-\frac{1}{2}})\|, \quad J = \|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\|,$$

$$K = \|D^2\Big(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}\Big)\|.$$

This yields

$$\frac{1}{2k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + |\theta^{n-\frac{1}{2}}|_2^2 \leq \frac{1}{2}I^2 + \frac{1}{2}|\theta^{n-\frac{1}{2}}|_2^2 + \frac{1}{2}J^2 + \frac{1}{2}\|\theta^{n-\frac{1}{2}}\|^2$$

$$+ \frac{1}{2}K^2 + \frac{1}{2}|\theta^{n-\frac{1}{2}}|_2^2.$$

The above inequality becomes

$$\frac{1}{k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \leq \|\theta^{n-\frac{1}{2}}\|^2 + I^2 + J^2 + K^2. \qquad (4.5)$$

We have the following estimate (see also Omrani [14, 15])

$$I = \|\phi(U^{n-\frac{1}{2}}) - \phi(u^{n-\frac{1}{2}})\| \leq C\|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\|,$$

here C is a constant dependent on c_0 and $\|u^{n-\frac{1}{2}}\|_\infty$. The continuous differentiability of $\phi(\cdot)$ have been used to derive this inequality.

Therefore we get the estimates

$$\|U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}}\| \leq \|U^{n-\frac{1}{2}} - P_h(\frac{u^n + u^{n-1}}{2})\| + \|P_h(\frac{u^n + u^{n-1}}{2}) - u^{n-\frac{1}{2}}\|$$

$$\leq \|\theta^{n-\frac{1}{2}}\| + I_1.$$

$$I_1 = \|P_h\Big(\frac{u^n + u^{n-1}}{2}\Big) - u^{n-\frac{1}{2}}\| \leq \frac{\|\rho^n\| + \|\rho^{n-1}\|}{2} + \|\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}\|$$

$$= \frac{\|\rho^n\| + \|\rho^{n-1}\|}{2} + \frac{1}{2}\|\int_{t_{n-1}}^{t_{n-\frac{1}{2}}}(s - t_{n-1})u_{tt}(s)ds - \int_{t_{n-\frac{1}{2}}}^{t_n}(s - t_n)u_{tt}(s)ds\|$$

$$\leq \frac{\|\rho^n\| + \|\rho^{n-1}\|}{2} + Ck^{\frac{3}{2}}\Big(\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|^2 ds\Big)^{\frac{1}{2}}.$$

Similarly we get the estimates

$$J = \|\partial_t P_h u^n - u_t^{n-\frac{1}{2}}\| \leq \|\partial_t P_h u^n - \partial_t u^n\| + \|\partial_t u^n - u_t^{n-\frac{1}{2}}\|$$

$$= \frac{1}{k}\|\rho^n - \rho^{n-1}\| + \frac{1}{2k}\|\int_{t_{n-1}}^{t_{n-\frac{1}{2}}}(s - t_{n-1})^2 u_{ttt}(s)ds + \int_{t_{n-\frac{1}{2}}}^{t_n}(s - t_n)^2 u_{ttt}(s)ds\|$$

$$\leq k^{-\frac{1}{2}}\Big(\int_{t_{n-1}}^{t_n}\|\rho_t(s)\|^2 ds\Big)^{\frac{1}{2}} + Ck^{\frac{3}{2}}\Big(\int_{t_{n-1}}^{t_n}\|u_{ttt}(s)\|^2 ds\Big)^{\frac{1}{2}},$$

$$K = \|D^2\Big(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}\Big)\| \leq Ck^{\frac{3}{2}}\Big(\int_{t_{n-1}}^{t_n}\|D^2 u_{tt}(s)\|^2 ds\Big)^{\frac{1}{2}}.$$

Using (4.5) with the estimates of I, J and K, we obtain

$$\frac{1}{k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \leq C[\|\theta^{n-\frac{1}{2}}\|^2 + \|\rho^n\|^2 + \|\rho^{n-1}\|^2$$

$$+ \frac{1}{k}\int_{t_{n-1}}^{t_n}\|\rho_t(s)\|^2 ds + k^3 \int_{t_{n-1}}^{t_n}(\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|D^2 u_{tt}(s)\|^2)ds],$$

and hence

$$\|\theta^n\|^2 - \|\theta^{n-1}\|^2 \leq Ck\big(\|\theta^{n-\frac{1}{2}}\|^2 + R_n\big),$$

where the latter equality defines R_n. So we proved that

$$(1 - Ck)\|\theta^n\|^2 \leq (1 + Ck)\|\theta^{n-1}\|^2 + CkR_n,$$

and for small $k \leq k_0$ the stability estimate is valid:

$$\|\theta^n\|^2 \leq \Big(\frac{1 + Ck}{1 - Ck}\Big)\|\theta^{n-1}\|^2 + CkR_n.$$

After repeated application, this yields

$$\|\theta^n\|^2 \leq \Big(\frac{1 + Ck}{1 - Ck}\Big)^n\|\theta^0\|^2 + Ck\sum_{j=1}^n \Big(\frac{1 + Ck}{1 - Ck}\Big)^{n-j} R_j,$$

or

$$\|\theta^n\|^2 \leq C\|\theta^0\|^2 + Ck\sum_{j=1}^n R_j. \tag{4.6}$$

Noting that

$$\|\theta^0\|^2 \leq \|\rho^0\|^2 + \|U^0 - u_0\|^2, \tag{4.7}$$

and by (2.4), we obtain

$$\|\rho^n\| \leq Ch^r(\|u_0\|_r + \int_0^{t_n}\|u_t(s)\|_r ds). \tag{4.8}$$

Using (4.6), (4.7) and (4.8), we find

$$\|\theta^n\|^2 \leq C\Big(\|U^0 - u_0\|^2 + h^{2r}(\|u_0\|_r^2 + \int_0^T \|u_t(s)\|_r^2\, ds)$$

$$+ k^4 \int_0^T (\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|D^2 u_{tt}(s)\|^2)\, ds\Big).$$

It follows from (4.2) that estimate (4.3) holds. ∎

Next we will use the Nirenberg inequality [1].

Lemma 4. *For* $\dfrac{j}{m} \leq a \leq 1,\quad \dfrac{1}{p} = \dfrac{j}{n'} + a(\dfrac{1}{r} - \dfrac{m}{n'}) + (1 - a)\dfrac{1}{q},$ *there holds:*

$$\|D^j v\|_{L^p(\Omega)} \leq C\big(\|D^m v\|_{L^r(\Omega)}^a \|v\|_{L^q(\Omega)}^{1-a} + \|v\|_{L^q(\Omega)}\big),$$

where Ω is a bounded domain in $\mathbb{R}^{n'}$.

Theorem 3. *Let U^n be the solution of (2.8). Suppose that the solution u of (1.1) is sufficiently smooth and the initial data is defined as*

$$u_{0h} = P_h u_0. \tag{4.9}$$

Then, for k sufficiently small, the estimate

$$\max_{0 \leq n \leq N} \|U^n - u(t^n)\| \leq C(h^r + k^2)$$

holds, where C is a constant independent of h and k.

Proof.   Setting $\chi = \partial_t \theta^n$ in (4.4), we obtain

$$\|\partial_t \theta^n\|^2 + |\partial_t \theta^n|_2^2 \leq I \, |\partial_t \theta^n|_2 + J \, \|\partial_t \theta^n\| + K \, |\partial_t \theta^n|_2$$

$$\leq I^2 + \tfrac{1}{4}|\partial_t \theta^n|_2^2 + \tfrac{1}{4}J^2 + \|\partial_t \theta^n\|^2 + K^2 + \tfrac{1}{4}|\partial_t \theta^n|_2^2 \, .$$

The above inequality gives the estimate

$$\frac{1}{k}(|\theta^n|_2^2 - |\theta^{n-1}|_2^2) \leq C(I^2 + J^2 + K^2). \tag{4.10}$$

Using the estimates of I, J and K with (4.3), we obtain

$$\frac{|\theta^n|_2^2 - |\theta^{n-1}|_2^2}{k} \leq C\Big(C(u)(h^r + k^2)^2 + \|\rho^n\|^2 + \|\rho^{n-1}\|^2 + \frac{1}{k}\int_{t_{n-1}}^{t_n}\|\rho_t(s)\|^2 \, ds$$

$$+ k^3 \int_{t_{n-1}}^{t_n} \left(\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|D^2 u_{tt}(s)\|^2 \right) ds \, \Big).$$

Hence the following stability inequality

$$|\theta^n|_2^2 \leq |\theta^{n-1}|_2^2 + C(u)k(h^r + k^2)^2 + CkR_n$$

is valid, where the inequality also defines R_n. Hence, by repeated application of the obtained estimate and (4.9) we get

$$|\theta^n|_2^2 \leq T \, C(u)(h^r + k^2)^2 + Ck\sum_{j=1}^{n} R_j \, .$$

Therefore,

$$|\theta^n|_2^2 \leq C\Big(C(u)(h^r + k^2)^2 + k\sum_{j=1}^{n} \|\rho^j\|^2 + \int_0^T \|\rho_t(s)\|^2 \, ds$$

$$+ k^4 \int_0^T \left(\|u_{ttt}(s)\|^2 + \|u_{tt}(s)\|^2 + \|D^2 u_{tt}(s)\|^2 \right) ds \, \Big).$$

Now, using (2.4), we conclude that, for some constant $C = C(u, T)$

$$|\theta^n|_2 \le C(h^r + k^2), \quad 0 \le n \le N, \qquad (4.11)$$

and, hence using (4.3), (4.11) and (3.2), we obtain

$$|\theta^n|_1 \le C(h^r + k^2), \quad 0 \le n \le N. \qquad (4.12)$$

Applying (4.3) and (4.12), it follows from Lemma 4 with $p = \infty$ that

$$\|\theta^n\|_\infty \le C(h^r + k^2), \quad 0 \le n \le N. \qquad (4.13)$$

Together the estimates (2.5) and (4.13) show the Theorem. ∎

5. A Linearized-Galerkin Method

The above method has the disadvantage that a nonlinear system has to be solved at each time step. For this reason we shall consider a linearized modification of the method in which the argument f is obtained by extrapolation from U^{n-1} and U^{n-2}, i.e. $\hat{U}^n = \frac{3}{2}U^{n-1} - \frac{1}{2}U^{n-2}$, for $n \ge 2$,

$$(\partial_t U^n, \chi) + (D^2 U^{n-\frac{1}{2}}, D^2\chi) = (\phi(\hat{U}^n), D^2\chi), \quad \forall \chi \in S_h. \qquad (5.1)$$

This method will require a separate prescription for calculating U^1 (see, e.g. Thomée [17], pp 218 − 222). We analyse a predictor corrector method for this purpose, which is formulated as follows:

$$\begin{cases} U^0 = u_{0h}, \\ \left(\dfrac{U^{1,0} - U^0}{k}, \chi\right) + \left(D^2\left(\dfrac{U^{1,0} + U^0}{2}\right), D^2\chi\right) = (\phi(U^0), D^2\chi), \\ (\partial_t U^1, \chi) + (D^2 U^{\frac{1}{2}}, D^2\chi) = (\phi(\frac{U^{1,0}+U^0}{2}), D^2\chi), \quad \forall \chi \in S_h. \end{cases} \qquad (5.2)$$

Remark 2. For $u(t)$ sufficiently smooth, we have

$$\hat{u}^n = \frac{3}{2}u^{n-1} - \frac{1}{2}u^{n-2} = u^{n-\frac{1}{2}} + O(k^2) \quad as \quad k \to 0. \qquad (5.3)$$

Now we will prove that the proposed approximation will give the second order accuracy.

Theorem 4. *Let U^n be the solution of (5.1), with U^0 and U^1 defined by (5.2). Suppose that the solution u of (1.1) is sufficiently regular, and $\|U^n\|_\infty, \|u^n\|_\infty$ are bounded. If the condition (4.9) is valid, then, for k sufficiently small, we have the following estimate*

$$\max_{0 \le n \le N} \|U^n - u(t^n)\| \le C(h^r + k^2) \qquad (5.4)$$

with $C = C(u, T)$.

Proof.   Using (2.4), it is sufficient to estimate θ^n. From (2.3), (1.1) and (5.1), we obtain for $n \geq 2$ the equation for θ^n

$$(\partial_t \theta^n, \chi) + (D^2 \theta^{n-\frac{1}{2}}, D^2 \chi) = (\phi(\hat{U}^n) - \phi(u^{n-\frac{1}{2}}), D^2 \chi)$$

$$- (\partial_t P_h u^n - u_t^{n-\frac{1}{2}}, \chi) - \left(D^2\left(\frac{u^n + u^{n-1}}{2} - u^{n-\frac{1}{2}}\right), D^2 \chi\right). \quad (5.5)$$

Setting $\chi = \theta^{n-\frac{1}{2}}$, we find

$$\frac{1}{2k}\left(\|\theta^n\|^2 - \|\theta^{n-1}\|^2\right) + |\theta^{n-\frac{1}{2}}|_2^2 \leq \frac{1}{2}I'^2 + \frac{1}{2}|\theta^{n-\frac{1}{2}}|_2^2 + \frac{1}{2}J^2$$

$$+ \frac{1}{2}\|\theta^{n-\frac{1}{2}}\|^2 + \frac{1}{2}K^2 + \frac{1}{2}|\theta^{n-\frac{1}{2}}|_2^2,$$

where $I' = \|\phi(\hat{U}^n) - \phi(u^{n-\frac{1}{2}})\|$. Thus

$$\frac{1}{k}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) \leq \|\theta^{n-\frac{1}{2}}\|^2 + I'^2 + J^2 + K^2. \quad (5.6)$$

Using the differentiability of $\phi(\cdot)$, we find for some constant C dependent on $\|\hat{U}^n\|_\infty$ and $\|u^{n-\frac{1}{2}}\|_\infty$

$$I' \leq C\|\hat{U}^n - u^{n-\frac{1}{2}}\| \leq C\left(\|\hat{\theta}^n\| + \|\hat{\rho}^n\| + \|\hat{u}^n - u^{n-\frac{1}{2}}\|\right).$$

By (2.4) and (5.3), we have

$$I' \leq C(\|\theta^{n-1}\| + \|\theta^{n-2}\|) + C(u)\left(h^r + k^2\right).$$

Using (2.4), (5.6) and the estimates of I', J, K, we obtain

$$\|\theta^n\|^2 \leq (1 + Ck)\|\theta^{n-1}\|^2 + Ck\|\theta^{n-2}\|^2 + C(u)k(h^r + k^2)^2.$$

This yields

$$\|\theta^n\|^2 + Ck\|\theta^{n-1}\|^2 \leq (1 + 2Ck)\left(\|\theta^{n-1}\|^2 + Ck\|\theta^{n-2}\|^2\right) + C(u)\,k(h^r + k^2)^2.$$

Therefore, for $nk \leq T$ and $n \geq 2$

$$\|\theta^n\|^2 \leq C\left(\|\theta^1\|^2 + k\|\theta^0\|^2 + (h^r + k^2)^2\right). \quad (5.7)$$

Next we shall estimate $\|\theta^1\|$. From equations (5.2) we get the estimate

$$\frac{1}{k}(\|\theta^{1,0}\|^2 - \|\theta^0\|^2) \leq C\left(\|U^0 - u^{\frac{1}{2}}\|^2 + (h^r + k^2)^2\right),$$

here $\theta^{1,0} = U^{1,0} - P_h u^1$, $\theta^{0,0} = \theta^0$. It follows from (4.8) that

$$\|U^0 - u^{\frac{1}{2}}\| \leq \|\theta^0\| + \|\rho^0\| + \|u^0 - u^{\frac{1}{2}}\| \leq \|\theta^0\| + C(h^r + k),$$

which yields

$$\frac{1}{k}(\|\theta^{1,0}\|^2 - \|\theta^0\|^2) \le C\left(\|\theta^0\|^2 + h^{2r} + k^2\right).$$

Thus,

$$\|\theta^{1,0}\|^2 \le (1 + Ck)\|\theta^0\|^2 + Ck(h^{2r} + k^2) \le C\left(\|\theta^0\|^2 + h^{2r} + k^3\right). \quad (5.8)$$

In the same way as above we obtain from (5.2)

$$\frac{1}{k}(\|\theta^1\|^2 - \|\theta^0\|^2) \le C\left(\|\frac{U^{1,0} + U^0}{2} - u^{\frac{1}{2}}\|^2 + (h^r + k^2)^2\right). \quad (5.9)$$

Therefore, using (5.8), we estimate the error

$$\|\frac{U^{1,0} + U^0}{2} - u^{\frac{1}{2}}\| \le \|\frac{\theta^{1,0} + \theta^0}{2}\| + \|P_h u^{\frac{1}{2}} - u^{\frac{1}{2}}\|$$

$$\le \frac{1}{2}(\|\theta^{1,0}\| + \|\theta^0\|) + C(u)\left(h^r + k^2\right)$$

$$\le C\|\theta^0\| + C(u)\left(h^r + k^{\frac{3}{2}}\right).$$

With these estimates, (5.9) becomes

$$\|\theta^1\|^2 \le (1 + Ck)\|\theta^0\|^2 + Ck(h^{2r} + k^3) \le C\left(\|\theta^0\|^2 + (h^r + k^2)^2\right). \quad (5.10)$$

It follows from (4.9), (5.6) and (5.10), that the estimate

$$\|\theta^n\| \le C(h^r + k^2), \quad 0 \le n \le N \qquad (5.11)$$

is valid for some constant $C = C(u, T)$. Choosing $\chi = \partial_t \theta^n$ in (5.5), we obtain for $n \ge 2$

$$\|\partial_t \theta^n\|^2 + |\partial_t \theta^n|_2^2 \le I' \, |\partial_t \theta^n|_2 + J \, \|\partial_t \theta^n\| + K \, |\partial_t \theta^n|_2$$

$$\le I'^2 + \frac{1}{4}|\partial_t \theta^n|_2^2 + \frac{1}{4}J^2 + \|\partial_t \theta^n\|^2 + K^2 + \frac{1}{4}|\partial_t \theta^n|_2^2.$$

Therefore, by the estimates of I, J, K and (2.4), we obtain

$$\frac{1}{k}(|\theta^n|_2^2 - |\theta^{n-1}|_2^2) \le C(\|\theta^{n-1}\|^2 + \|\theta^{n-2}\|^2) + C(u)(h^r + k^2)^2.$$

Using (5.11), we get

$$\frac{1}{k}(|\theta^n|_2^2 - |\theta^{n-1}|_2^2) \le C(u)(h^r + k^2)^2.$$

Consequently, for $n \ge 2$ and $nk \le T$, we have

$$|\theta^n|_2^2 \le C(u, T)\left(|\theta^1|_2^2 + (h^r + k^2)^2\right).$$

Similarly to the analysis given above, we obtain from (5.2) instead of (4.10) the following estimate

$$|\theta^1|_2^2 \le C(h^r + k^2)^2.$$

Therefore

$$|\theta^n|_2 \le C(u, T)(h^r + k^2), \quad 0 \le n \le N. \tag{5.12}$$

Then using (3.2), (5.11) and (5.12), we obtain

$$|\theta^n|_1 \le C(u, T)(h^r + k^2), \quad 0 \le n \le N. \tag{5.13}$$

From (5.11), (5.13) and Lemma 4 with $p = \infty$, we get

$$\|\theta^n\|_\infty \le C(h^r + k^2), \quad 0 \le n \le N. \tag{5.14}$$

Now, the result follows from (5.14) and (2.5). ∎

Acknowledgments

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Pilnai diskrečioji Galerkino aproksimacija Cahn-Hilliard lygčiai

K. Omrani

Straipsnyje analizuojama standartinė Galerkino aproksimacija nestacionariajai Canh-Hilliard lygčiai, panaudojant glodžius splainus. Aptarta pilnai diskrečios Cranko-Nikolsono baigtinių skirtumų schemos sprendinio egzistencija, vienatis ir konvergavimas. Pabaigoje pateikta tiesinė Galerkino diskrečioji schema, kuri yra antros eilės tikslumo pagal laiką.