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SOME ESTIMATES OF SPECIAL CLASSES OF INTEGRALS

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ABSTRACT

We study the integrals $\int_a^b f(t)\exp(i|\ln rt|^{\sigma})\,dt$ and obtain asymptotic formula for these functions of non-regular growth. This is a peculiar kind of the theory asymptotic expansions. In particular, we get asymptotic formulae for different entire functions of non-regular growth. Asymptotic formulas for Levin-Pfluger entire functions of completely regular growth are well-known [1]. Our formulas allow to find limiting Azarin's [2] sets for some subharmonic functions. The kernel $\exp(i|\ln rt|^{\sigma})$ contains arbitrary parameter $\sigma > 0$. The integrals for $\sigma \in (0,1), \sigma = 1, \sigma > 1$ essentially differ. Our arguments can apply to more general kernels. We give a new variant of the classic lemma of Riemann and Lebesgue from the theory of the transformation of Fourier.

1. THE ANALOGY OF RIEMANN'S–LEBESGUE'S LEMMA

We will begin with the analogy of Riemann's–Lebesgue's lemma.

Lemma 1.1. *Let* $f(t) \in \mathbf{L}_1([a,b])$, $0 \le a < b \le \infty$, $\sigma > 1$. *Then*

$$\lim_{r\to\infty} \int_a^b f(t)\exp\left(i|\ln rt|^{\sigma}\right)\,dt = 0\,.$$

Proof. Assume that $a > 0$, $f \in \mathbf{C_1}([a,b])$. Integrating by parts, we get

$$\int_a^b f(t)\exp\left(i|\ln rt|^{\sigma}\right)\,dt = \left.\frac{f(t)t\exp\left(i(\ln rt)^{\sigma}\right)}{i\sigma(\ln rt)^{\sigma-1}}\right|_a^b -$$

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$$\frac{1}{i\sigma}\int_a^b \left[\frac{f'(t)t+f(t)}{(\ln rt)^{\sigma-1}} - \frac{(\sigma-1)f(t)t}{(\ln rt)^{\sigma}}\right] \exp\left(i(\ln rt)^{\sigma}\right)\, dt\,.$$

Limit of right part equals zero if $r \to \infty$.

Now let $f \in \mathbf{L_1}([a,b])$ and let function $f_1 \in \mathbf{C_1}([a,b])$ such that

$$\int_a^b |f(t) - f_1(t)|\, dt \le \epsilon\,,$$

where ϵ is any positive number. Then,

$$\left|\int_a^b f(t)\exp\left(i(\ln rt)^{\sigma}\right)\, dt\right| \le \left|\int_a^b f_1(t)\exp\left(i(\ln rt)^{\sigma}\right)\, dt\right| + \epsilon\,.$$

The desirable conclusion follows from above proved.

Let $a = 0$, $f \in \mathbf{L_1}([0,b])$, and let $\epsilon > 0$ be on arbitrary number. If $|\exp\left(i|\ln rt|^{\sigma}\right)| \le 1$, then there exists constants $\delta > 0$ such that

$$\left|\int_0^{\delta} f(t)\, dt\right| \le \epsilon\,.$$

Then

$$\left|\int_0^b f(t)\exp\left(i|\ln rt|^{\sigma}\right)\, dt\right| = \left|\int_0^{\delta} + \int_{\delta}^b\right| \le \epsilon + \left|\int_{\delta}^b f(t)\exp\left(i|\ln rt|^{\sigma}\right)\, dt\right|\,.$$

■

Remark 1.1. Lemma is true if kernels $\exp\left(i|\ln rt|^{\sigma}\right)$ are replaced by $\exp(i\varphi(rt)\ln rt)$, where φ is differentialable increasing function on the half axis $[0,\infty)$ such that $\lim_{t\to\infty}\varphi(t) = \infty$.

We do not evaluate the speed of convergence to zero of the integral. It can't be done $f \in \mathbf{L}_1([a,b])$.

2. AZARIN LIMITING SETS

In this section we consider the following function:

$$\begin{aligned} u_1(z,\sigma) \quad &= \quad \frac{r\sin\theta}{\pi}\int_0^{\infty}\frac{\tau^{\rho}\exp(i\lambda|\ln\tau|^{\sigma})}{\tau^2 - 2\tau r\cos\theta + r^2}\, d\tau = \\ & \frac{r^{\rho}\sin\theta}{\pi}\int_0^{\infty}\frac{t^{\rho}\exp(i\lambda|\ln tr|^{\sigma})}{t^2 - 2t\cos\theta + 1}\, dt, \end{aligned} \tag{2.1}$$

$$u_2(z,\sigma) = \frac{1}{\pi}\int_0^\infty \frac{r(r-\tau\cos\theta)}{\tau^2-2\tau r\cos\theta+r^2}\exp(i\lambda|\ln\tau|^\sigma)\,d\tau = \frac{r^\rho}{\pi}\int_0^\infty \frac{1-t\cos\theta}{t^2-2t\cos\theta+1}\exp(i\lambda|\ln tr|^\sigma)\,dt,$$

$u_3(z,\sigma) = \operatorname{Re} u_1(z,\sigma)$, $u_4(z,\sigma) = \operatorname{Im} u_1(z,\sigma)$, $u_5(z,\sigma) = \operatorname{Re} u_2(z,\sigma)$, $u_6(z,\sigma) = \operatorname{Im} u_2(z,\sigma)$, $z = re^{i\theta}$, $\rho \in (0,1)$, $\sigma > 0, \lambda \geq 0$. If $\sigma = 1$, we do not write module.

Azarin limiting set $Fr\ u$ of subharmonic function $u(z)$ is its significant characteristics of the growth [2]. $Fr\,u$ is limiting set of the family of functions $u_t(z) = u(tz)/t^\rho$ (ρ be the order of u) by $t \to +\infty$ in the topology of the space of generalized Shwartz's functions. If $\rho \in (0,1)$, $\sigma \in (0,1)$, $\lambda > 0$ we have the following properties:

$$Fr\ u_3 = Fr\ u_4 = \left\{\alpha\frac{\sin\rho(\pi-\theta)}{\sin\rho\pi}r^\rho : \alpha \in [-1,1]\right\}, \tag{2.2}$$

$$Fr\ u_5 = Fr\ u_6 = \left\{\alpha\frac{\cos\rho(\pi-\theta)}{\sin\rho\pi}r^\rho : \alpha \in [-1,1]\right\}. \tag{2.3}$$

Let $h_k(\theta)$ be the Fragmen–Lindeljeff indicator of the function $u_k(z,\sigma)$. Then the following relations hold:

$$h_3(\theta) = h_4(\theta) = \frac{|\sin\rho(\pi-\theta)|}{\sin\rho\pi}, \quad h_5(\theta) = h_6(\theta) = \frac{|\cos\rho(\pi-\theta)|}{\sin\rho\pi}.$$

Theorem 2.1. *Let $\sigma = 1$, and let $\rho \in (0,1)$, $\lambda \geq 0$ be given numbers. Then the following relations hold:*

$$u_3(z,\sigma) = [A_\rho(\lambda,\theta)\cos\lambda\ln r - B_\rho(\lambda,\theta)\sin\lambda\ln r]r^\rho, \tag{2.4}$$

$$u_4(z,\sigma) = [B_\rho(\lambda,\theta)\cos\lambda\ln r + A_\rho(\lambda,\theta)\sin\lambda\ln r]r^\rho, \tag{2.5}$$

$$u_5(z,\sigma) = [C_\rho(\lambda,\theta)\cos\lambda\ln r - D_\rho(\lambda,\theta)\sin\lambda\ln r]r^\rho, \tag{2.6}$$

$$u_6(z,\sigma) = [D_\rho(\lambda,\theta)\cos\lambda\ln r + C_\rho(\lambda,\theta)\sin\lambda\ln r]r^\rho, \tag{2.7}$$

where

$$A_\rho(\lambda,\theta) = \operatorname{Re}\frac{\sin(\rho+i\lambda)(\pi-\theta)}{\sin(\rho+i\lambda)\pi}, \quad B_\rho(\lambda,\theta) = \operatorname{Im}\frac{\sin(\rho+i\lambda)(\pi-\theta)}{\sin(\rho+i\lambda)\pi}.$$

Analogous formulae for $C_\rho(\lambda,\theta)$ and $D_\rho(\lambda,\theta)$ come out if $\sin(\rho+i\lambda)(\pi-\theta)$ *is replaced by* $\cos(\rho+i\lambda)(\pi-\theta)$.

Corollary 2.1.

$$Fr\ u_5(z,1) = \{C_\rho(\lambda,\theta)\sin\varphi - D_\rho(\lambda,\theta)\cos\varphi : \varphi \in [0,2\pi]\}, \tag{2.8}$$

$$h_5(\theta) = \sqrt{C_\rho^2(\lambda,\theta) + D_\rho^2(\lambda,\theta)}, \tag{2.9}$$

and analogous formulae for $u_3(z,1)$, $u_4(z,1)$, $u_6(z,1)$ occur.

Proof. We prove equality (2.4). We have

$$u_3(z,1) = \frac{r^\rho \sin\theta}{\pi} \Re r^{i\lambda} \int_0^\infty \frac{t^{\rho+i\lambda}}{t^2 - 2t\cos\theta + 1}\, dt\,.$$

We define the digit branch of function $t^{\rho+i\lambda}$ on cut plane by the semi-axis $[0,\infty)$ such that $\arg t = 0$ over the side of the cut, $\arg t = 2\pi$ under the side of the cut, and $0 < \arg t < 2\pi$ on the plane with the cut.

We define the contour of integration $L = L(\epsilon) \bigcup L(R) \bigcup L_1 \bigcup L_2$, where $L(\epsilon) = \{z : |z| \le \epsilon\}$, $L(R)$ is analogous circle with the radius R, L_1 is upper side of the cut of $[\epsilon, R]$, L_2 is the bottom of this cut with has contrary respect. Then we have

$$I = \int_L \frac{t^{\rho+i\lambda}}{t^2 - 2t\cos\theta + 1}\, dt = I(\epsilon) + I(R) + I_1 + I_2\,,$$

$$\lim_{\epsilon\to 0} I(\epsilon) = \lim_{R\to\infty} I(R) = 0\,, \tag{2.10}$$

and

$$I_1 = \int_\epsilon^R \frac{t^{\rho+i\lambda}}{t^2 - 2t\cos\theta + 1}\, dt\,,$$

$$I_2 = \int_R^\epsilon \frac{\exp(2\pi i(\rho+i\lambda)) t^{\rho+i\lambda}}{t^2 - 2t\cos\theta + 1}\, dt = -\exp(2\pi i(\rho+i\lambda)) I_1\,. \tag{2.11}$$

The integral function has simple poles at points $t_1 = \exp(i\theta)$ and $t_2 = \exp(i(2\pi - \theta))$ if $\theta \ne 0$. Applying the residue theorem, we have

$$I = 2\pi i \left(\operatorname*{Res}_{t=e^{i\theta}} \frac{t^{\rho+i\lambda}}{t^2 - 2t\cos\theta + 1} + \operatorname*{Res}_{t=e^{i(2\pi-\theta)}} \frac{t^{\rho+i\lambda}}{t^2 - 2t\cos\theta + 1} \right) =$$

$$\pi i \left(\frac{e^{i\theta(\rho+i\lambda)}}{e^{i\theta} - \cos\theta} + \frac{e^{i(2\pi-\theta)(\rho+i\lambda)}}{e^{i(2\pi-\theta)} - \cos\theta} \right) = \frac{\pi}{\sin\theta} \left(e^{i\theta(\rho+i\lambda)} - e^{i(2\pi-\theta)(\rho+i\lambda)} \right) =$$

$$\frac{\pi}{\sin\theta}[\cos\theta(\rho+i\lambda) + i\sin\theta(\rho+i\lambda) - \cos(2\pi-\theta)(\rho+i\lambda) - i\sin(2\pi-\theta)(\rho+i\lambda)]$$

$$= \frac{\pi}{\sin\theta}[2\sin(\pi-\theta)(\rho+i\lambda)\sin\pi(\rho+i\lambda) - 2i\sin(\pi-\theta)\cos\pi(\rho+i\lambda)] =$$

$$\frac{2\pi}{\sin\theta}\sin(\pi-\theta)(\rho+i\lambda)[\sin\pi(\rho+i\lambda) - i\cos\pi(\rho+i\lambda)]\,.$$

Equality (2.11) implies

$$I_1 + I_2 = (1 - \exp(2\pi(\rho+i\lambda)))I_1 = (1 - \cos 2\pi(\rho+i\lambda) -$$

$$i\sin 2\pi(\rho+i\lambda))I_1 = 2\sin\pi(\rho+i\lambda)[\sin\pi(\rho+i\lambda) - i\cos\pi(\rho+i\lambda)]I_1\,.$$

This and (2.10) give

$$\int_0^\infty \frac{t^{\rho+i\lambda}}{t^2-2t\cos\theta+1}\,dt = \frac{\pi}{\sin\theta}\,\frac{\sin(\pi-\theta)(\rho+i\lambda)}{\sin\pi(\rho+i\lambda)}\,.$$

This implies assertion (2.4) of the theorem. If $\theta = 0$ then (2.4) can be received by $\theta \to +0(2\pi - 0)$. ■

3. ASYMPTOTIC FORMULAE OF INTEGER FUNCTIONS OF IRREGULAR GROWTH

Let $\{a_k\}_{k=1}^\infty$ be a sequence of positive zeros of the integer function $f(z)$, and let $\rho \in (0,1)$ be the order of f. Define $\ln(1-z/a_k)$ by $\ln(1-z/a_k) > 0$, if $z \in (-\infty, 0)$, on the cut plane by $[0,\infty)$. Then we have

$$\ln f(z) = \sum_{k=1}^\infty \ln\left(1-\frac{z}{a_k}\right) = \int_0^\infty \ln\left(1-\frac{z}{t}\right)\,dn(t) = \int_0^\infty \frac{z}{z-t}\frac{n(t)}{t}\,dt\,,$$

where $n(t)$ is defined to be the number of zeros, counted with multiplicity, of f in the circle of radius t, excluding those at the origin. We have

$$\ln|f(z)| = \int_0^\infty \frac{r(r-t\cos\theta)}{t^2-2tr\cos\theta+r^2}\frac{n(t)}{t}\,dt\,.$$

Define $\varphi(t)$ by $\varphi(t) = t^\rho(a_0 + a_1\cos\lambda\ln t + b_1\sin\lambda\ln t)$.
If $a_0 \ge \sqrt{1+\lambda^2/\rho^2 i}\sqrt{a_1^2+b_1^2}$, then $\varphi(t)$ is the increasing function so as

$$\varphi'(t) = \rho t^{\rho-1}\left[a_0 + \cos\lambda\ln t\left(a_1+\frac{\lambda}{\rho}b_1\right) + \sin\lambda\ln t\left(b_1-\frac{\lambda}{\rho}a_1\right)\right] \ge 0\,.$$

Consider the function f with $n(t) = [\varphi(t)]$ (here $[\cdot]$ represents the integer part). Azarin limiting set $Fr\,f$ of the integer function f is Azarin limiting set of the subharmonic function $\ln|f(z)|$. Applying the theorem 2.1, we obtain

$$Fr\ f = \left\{\left(a_0\frac{\cos\rho(\pi-\theta)}{\sin\rho\pi} + (a_1C_\rho(\lambda,\theta)+b_1D_\rho(\lambda,\theta))\cos\varphi + \right.\right.$$

$$\left.\left.(-a_1D_\rho(\lambda,\theta)+b_1C_\rho(\lambda,\theta))\sin\varphi\right)r^\rho : \varphi\in[0,2\pi]\right\},$$

$$h_f(\theta) = a_0\frac{\cos\rho(\pi-\theta)}{\sin\rho\pi} + \sqrt{a_1^2+b_1^2}\sqrt{C_\rho^2(\lambda,\theta)+D_\rho^2(\lambda,\theta)}\,.$$

These relations hold if $a_0 < \sqrt{1+\lambda^2/\rho^2}\sqrt{a_1^2+b_1^2}$. However, function f will be meromorphic function in a general case. If

$$\varphi(t) = t^\rho \left(a_0 + \sum_{k=1}^{n} (a_k \cos\lambda_k \ln t + b_k \sin\lambda_k \ln t) \right)$$

then using the theorem 2.1, we obtain asymptotic formulae for $\ln|f(z)|$. If $\varphi(t)$ is the increasing function then f is the integer function. In this way, we can obtain asymptotic formulae for a general class of integer functions of irregular growth. In the book of B.Ya.Levin [1], asymptotic formulae for a class of integer functions of regular growth are represented.

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SPECALIŲJŲ INTEGRALŲ KLASIŲ ĮVERČIAI

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Darbe nagrinėjami integralai $\int_a^b f(t)\exp(i|\ln r|^\sigma)dt$ ir tiriamos šių nereguliaraus augimo greičių funkcijų asimptotinės formulės. Gautos naujos asimptotinės formulės, leidžiančios rasti Azarino aibes kai kurioms subharmoninėms funkcijoms. Branduolys $\exp(i|\ln rt|^\sigma)$ priklauso nuo vieno parametro $\sigma > 0$. Trys atvejai, kai $0 < \sigma < 1$, $\sigma = 1$ ir $\sigma > 0$, yra esminiai skirtingi. Darbo metodika gali būti naudojama ir bendresniems branduolių atvejams. Įrodytas naujas Rimano ir Lebego lemos varijantas, kuris naudojamas Furje transformacijos teorijoje.