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COEFFICIENT STABILITY OF OPERATOR–DIFFERENCE SCHEMES

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ABSTRACT

A priori estimates expressing continuous dependence of the solution of a first order evolutionary equation in Hilbert space on initial condition, right hand side and operator perturbations are obtained in time–integral norms. Analogous results hold for corresponding finite difference schemes.

1. INTRODUCTION

While solving differential equations, one is often in a situation that the coefficients are not given exactly, but approximately (e.g. as a result of a physical measurement). Therefore, the problem of stability for the variable coefficients differential equation is of great importance (coefficient stability). Analogous problem is present in corresponding numerical methods which approximate the differential equation. In spite of importance of this problem a priori estimates expressing continuous dependence of the solution on the right hand side and operator perturbation (so called strong stability) till recent times have been obtained only for stationary problems (see [2], [4], [5], [10], [12]). Strong stability of difference and operator–difference schemes were investigated in [3], [7], [8], [9] and [11]. In these papers corresponding a priori estimates were obtained in uniform in time norms. Using integral in time norm [7] we proved

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a priori estimates under very weak assumptions on the right hand side of the equation. We applied this approach investigating the accuracy of the solution of difference schemes for the problems with generalized solutions (see [3], [6], [9], [11]).

This paper deals with the construction of stability estimates for the first order evolutionary problems in Hilbert spaces, and corresponding finite difference schemes, in the case of perturbed operators, right hand sides and initial conditions. A priori error estimates in time–integral norms are obtained under some natural assumptions on perturbed operators.

2. STABILITY OF OPERATOR–DIFFERENTIAL SCHEMES

Let H be a Hilbert space with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let A be a constant, self-adjoint, positive definite linear operator in H, i.e.

$$A(t)=A(0), \qquad A=A^*\ge \delta E, \qquad \delta=\text{const}>0,$$

where E is the unit operator in H. We denote H_D, where $D=D^*>0$, the space with the following inner product and norm

$$(y,v)_D=(Dy,v), \qquad \|y\|_D=\sqrt{(Dy,y)}.$$

We consider the Cauchy problem

$$\frac{du}{dt}+Au=f(t), \quad 0<t<T; \qquad u(0)=u_0, \tag{2.1}$$

where u_0 is a given element of H, $f(t)$ is a given function and $u(t)$ is an unknown function with values in H.

Using the energy method and Fourier expansion we can prove the following assertion.

Lemma 2.1. *The solution of the problem* (2.1) *satisfies a priori estimates:*

$$\int_0^T \left(\|Au(t)\|^2+\left\|\frac{du(t)}{dt}\right\|^2\right)dt \le 2\,\|u_0\|_A^2+2\int_0^T \|f(t)\|^2\,dt, \tag{2.2}$$

$$\int_0^T \|u(t)\|_A^2\,dt+\int_0^T\int_0^T \frac{\|u(t)-u(t')\|^2}{|t-t'|^2}\,dt\,dt'$$

$$\le (1+3\pi)\,\|u_0\|^2+(1+4\pi)\int_0^T \|f(t)\|_{A^{-1}}^2\,dt, \tag{2.3}$$

$$\int_0^T \|u(t)\|^2\,dt \le \|u_0\|_{A^{-1}}^2+\int_0^T \|A^{-1}f(t)\|^2\,dt. \tag{2.4}$$

Setting $f(t) = dg(t)/dt$ in (2.1) one obtains the Cauchy problem in the form

$$\frac{du}{dt} + Au = \frac{dg}{dt}, \quad 0 < t < T; \qquad u(0) = u_0. \tag{2.5}$$

An analogous proposition holds for the problem (2.5).

Lemma 2.2. *The solution of the problem* (2.5) *satisfies a priori estimates:*

$$\begin{aligned}
&\int_0^T \|u(t)\|_A^2\,dt + \int_0^T\int_0^T \frac{\|u(t)-u(t')\|^2}{|t-t'|^2}\,dt\,dt' \\
&\le (\pi + 4\,\pi^2)\Big\{\|u_0\|^2 + \int_0^T\int_0^T \frac{\|g(t)-g(t')\|^2}{|t-t'|^2}\,dt\,dt' \\
&+ \int_0^T \Big(\frac{1}{t} + \frac{1}{T-t}\Big)\|g(t)\|^2\,dt\Big\},
\end{aligned} \tag{2.6}$$

$$\int_0^T \|u(t)\|^2\,dt \le \|u_0 - g(0)\|_{A^{-1}}^2 + \int_0^T \|g(t)\|^2\,dt. \tag{2.7}$$

Example 2.1. Let us consider the initial–boundary value problem for one–dimensional heat transfer equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(k(x)\,\frac{\partial u}{\partial x}\Big) + f(x,t), \qquad (x,t) \in Q_T = (0,1)\times(0,T),\\
&u(0,t) = u(1,t) = 0, \qquad u(x,0) = u_0(x).
\end{aligned} \tag{2.8}$$

The problem (2.8) can be represented in the form (2.1), setting $H = L_2(0,1)$ and $Av = -\frac{d}{dx}\Big(k(x)\,\frac{dv}{dx}\Big)$. If $k \in C^1[0,1]$ and $k(x) \ge k_0 > 0$ the operator A maps the set $D(A) = \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1) \subset L_2(0,1)$ on $L_2(0,1)$. The inverse operator A^{-1} is

$$\begin{aligned}
A^{-1}v(x) \quad = \quad & -\int_0^x \frac{1}{k(x')}\int_0^{x'} v(x'')\,dx''dx' \\
&+ \Big(\int_0^1 \frac{dx'}{k(x')}\Big)^{-1}\Big(\int_0^x \frac{dx'}{k(x')}\Big)\Big(\int_0^1 \frac{1}{k(x')}\int_0^{x'} v(x'')\,dx''dx'\Big).
\end{aligned} \tag{2.9}$$

The following inequalities hold (see [6])

$$\begin{aligned}
&c_1\,\|v\|_{W_2^1(0,1)}^2 \le \|v\|_A^2 = \int_0^1 k(x)\,|v'(x)|^2\,dx \le c_2\,\|v\|_{W_2^1(0,1)}^2\,,\\
&\quad v \in \overset{\circ}{W}{}_2^1(0,1),\\
&c_3\,\|v\|_{W_2^2(0,1)} \le \|Av\|_{L_2(0,1)} \le c_4\,\|v\|_{W_2^2(0,1)},\\
&\quad v \in \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1),
\end{aligned} \tag{2.10}$$

In such a way, inequalities (2.2), (2.3) and (2.4) show the stability of the problem (2.8) in spaces $W_2^{2,1}(Q_T)$, $W_2^{1,1/2}(Q_T)$ and $L_2(Q_T)$.

3. STRONG STABILITY OF OPERATOR–DIFFERENTIAL SCHEMES

Along with (2.1) and (2.5) let us consider analogous Cauchy problems with perturbed right hand side, initial condition and operator:

$$\frac{d\tilde{u}}{dt} + \tilde{A}\tilde{u} = \tilde{f}(t), \quad 0 < t < T; \qquad \tilde{u}(0) = \tilde{u}_0, \tag{3.1}$$

and

$$\frac{d\tilde{u}}{dt} + \tilde{A}\tilde{u} = \frac{d\tilde{g}(t)}{dt}, \quad 0 < t < T; \qquad \tilde{u}(0) = \tilde{u}_0. \tag{3.2}$$

We deal with the problem of estimating the perturbation of the solution

$$z(t) = \tilde{u}(t) - u(t)$$

with the values of perturbations of u_0, A and f (or g). Let the perturbed operators satisfy the analogous assumptions as the operator A:

$$\tilde{A}(t) = \tilde{A}(0), \qquad \tilde{A} = \tilde{A}^* \geq \tilde{\delta} E, \qquad \tilde{\delta} = \text{const} > 0.$$

As the measure of perturbation of operator we shall use the positive constant α in inequalities

$$\|(\tilde{A} - A)v\| \leq \alpha \|\tilde{A}v\|, \tag{3.3}$$

$$|((\tilde{A} - A)v, v)| \leq \alpha (Av, v), \tag{3.4}$$

$$\|A^{-1}\tilde{A}v - v\| \leq \alpha \|v\|. \tag{3.5}$$

From (2.1) and (3.1), (i.e. (2.5) and (3.2)), one obtains

$$\frac{dz}{dt} + Az = (\tilde{f}(t) - f(t)) - (\tilde{A} - A)\tilde{u}, \quad 0 < t < T, \quad z(0) = \tilde{u}_0 - u_0 \tag{3.6}$$

or

$$\frac{dz}{dt} + Az = \frac{d(\tilde{g} - g)}{dt} - (\tilde{A} - A)\tilde{u}, \quad 0 < t < T; \quad z(0) = \tilde{u}_0 - u_0. \tag{3.7}$$

Using lemmas 2.1 and 2.2 and conditions (2.3) – (2.5) we obtain the following result.

Theorem 3.1. *The perturbation of the solution of the problem* (2.1) *satisfies a priori estimates:*

$$\int_0^T \left(\|Az(t)\|^2 + \left\| \frac{dz(t)}{dt} \right\| \right) dt \le 2\,\|\tilde{u}_0 - u_0\|_A^2$$

$$+4 \int_0^T \|\tilde{f}(t) - f(t)\|^2 dt + 4\,\alpha^2 \left(\|\tilde{u}_0\|_{\tilde{A}}^2 + \int_0^T \|\tilde{f}(t)\|^2 dt \right),$$

if the condition (3.3) *holds;*

$$\int_0^T \|z(t)\|_A^2\, dt + \int_0^T \int_0^T \frac{\|z(t) - z(t')\|^2}{|t-t'|^2}\, dt\, dt' \le (2+15\pi) \Bigg[\|\tilde{u}_0 - u_0\|^2$$

$$+ \int_0^T \|\tilde{f}(t) - f(t)\|_{A^{-1}}^2 dt + \frac{\alpha^2}{1-\alpha} \left(\|\tilde{u}_0\|^2 + \int_0^T \|\tilde{f}(t)\|_{\tilde{A}^{-1}}^2 dt \right) \Bigg],$$

if the condition (3.4) *holds;*

$$\int_0^T \|z(t)\|^2\, dt \le \|\tilde{u}_0 - u_0\|_{A^{-1}}^2 + 2 \int_0^T \|A^{-1}(\tilde{f}(t) - f(t))\|^2\, dt$$

$$+2\,\alpha^2 \left(\|\tilde{u}_0\|_{\tilde{A}^{-1}}^2 + \int_0^T \|\tilde{A}^{-1} \tilde{f}(t)\|^2\, dt \right),$$

if the condition (3.5) *holds;*

The perturbation of the solution of the problem (2.5) *satisfies a priori estimates:*

$$\int_0^T \|z(t)\|_A^2\, dt + \int_0^T \int_0^T \frac{\|z(t) - z(t')\|^2}{|t-t'|^2}\, dt\, dt'$$

$$\le 34\,\pi^2 \Bigg\{ \|\tilde{u}_0 - u_0\|^2 + \int_0^T \int_0^T \frac{\|[\tilde{g}(t) - g(t)] - [\tilde{g}(t') - g(t')]\|^2}{|t-t'|^2}\, dt\, dt'$$

$$+ \int_0^T \left(\frac{1}{t} + \frac{1}{T-t} \right) \|\tilde{g}(t) - g(t)\|^2\, dt + \frac{\alpha^2}{1-\alpha} \Bigg[\|\tilde{u}_0\|^2$$

$$+ \int_0^T \int_0^T \frac{\|\tilde{g}(t) - \tilde{g}(t')\|^2}{|t-t'|^2}\, dt\, dt' + \int_0^T \left(\frac{1}{t} + \frac{1}{T-t} \right) \|\tilde{g}(t)\|^2\, dt \Bigg] \Bigg\},$$

if the condition (3.4) *holds;*

$$\int_0^T \|z(t)\|^2\, dt \le 2\,\|\tilde{u}_0 - u_0 - [\tilde{g}(0) - g(0)]\|_{A^{-1}}^2 + 2 \int_0^T \|\tilde{g}(t) - g(t)\|^2\, dt$$

$$+2\,\alpha^2 \left(\|\tilde{u}_0 - g(0)\|_{\tilde{A}^{-1}}^2 + \int_0^T \|\tilde{g}(t)\|^2\, dt \right),$$

if the condition (3.5) *holds.*

Example 3.1. Let as consider together with (2.8) a perturbed initial–boundary value problem

$$
\begin{aligned}
&\frac{\partial \tilde u}{\partial t} = \frac{\partial}{\partial x}\Big(\tilde k(x)\,\frac{\partial \tilde u}{\partial x}\Big) + \tilde f(x,t)\,, \qquad (x,t)\in Q_T=(0,1)\times(0,T)\\
&\tilde u(0,t)=\tilde u(1,t)=0\,, \qquad \tilde u(x,0)=\tilde u_0(x)\,.
\end{aligned}
\tag{3.8}
$$

We have $Av = -\frac{d}{dx}\Big(k(x)\,\frac{dv}{dx}\Big)$ and $\tilde A v = -\frac{d}{dx}\Big(\tilde k(x)\,\frac{dv}{dx}\Big)$. In such a way

$$
\big((\tilde A - A)v,\, v\big)_{L_2(0,1)} = \int_0^1 \big(\tilde k(x)-k(x)\big)\,|v'(x)|^2\,dx\,,
$$

where from follows that the condition (3.4) is satisfied for

$$
|\tilde k(x)-k(x)| \le \alpha\, k(x)\,.
$$

Using (2.10) we get

$$
\begin{aligned}
\|(\tilde A - A)\,v\|_{L_2(0,1)} &\le \|k-\tilde k\|_{C[0,1]}\,\|v''\|_{L_2(0,1)} + \|k'-\tilde k'\|_{C[0,1]}\,\|v'\|_{L_2(0,1)}\\
&\le \Big(\|k-\tilde k\|_{C[0,1]} + \|k'-\tilde k'\|_{C[0,1]}\Big)\,\|v\|_{W_2^2(0,1)}\\
&\le \tfrac{1}{c_3}\,\Big(\|k-\tilde k\|_{C[0,1]} + \|k'-\tilde k'\|_{C[0,1]}\Big)\,\|Av\|_{L_2(0,1)}\,.
\end{aligned}
$$

Consequently, if

$$
|\tilde k(x)-k(x)| \le \alpha_1 \qquad \text{and} \qquad |\tilde k'(x)-k'(x)| \le \alpha_2\,, \tag{3.9}
$$

then (3.3) holds with $\alpha = (\alpha_1+\alpha_2)/c_3$.

It follows from (2.9) that

$$
\begin{aligned}
A^{-1}\tilde A v(x) - v(x) = &-\int_0^x \bigg[\frac{\tilde k'(x')-k'(x')}{k(x')} + \frac{k'(x')\,(k(x')-\tilde k(x'))}{k^2(x')}\bigg]\\
&v(x')\,dx' + \bigg(\int_0^1 \frac{dx'}{k(x')}\bigg)^{-1}\bigg(\int_0^x \frac{dx'}{k(x')}\bigg)\int_0^1\bigg[\frac{\tilde k'(x')-k'(x')}{k(x')}\\
&+\frac{k'(x')\,(k(x')-\tilde k(x'))}{k^2(x')}\bigg]\,v(x')\,dx' + \frac{\tilde k(x)-k(x)}{k(x)}\,v(x)\,.
\end{aligned}
$$

Hence, when inequalities (3.9) are satisfied, then (3.5) holds with $\alpha = \alpha_1\,(k_0 + 2\,\|k'\|_{C[0,1]})/k_0^2 + 2\alpha_2/k_0$.

4. STABILITY OF OPERATOR–DIFFERENCE SCHEMES

Analogous results hold for operator–difference schemes. Let H_h be a finite dimensional Hilbert space, with inner product $(\cdot,\cdot)_h$ and norm $\|\cdot\|_h$. Let A_h be a constant, self-adjoint, positive definite linear operator in H_h, i.e.

$$A_h(t) = A_h(0), \qquad A_h = A_h^* \geq \delta_h E_h, \qquad \delta_h = \text{const} > 0,$$

where E_h is the unit operator in H_h. We denote H_{D_h}, where $D_h = D_h^* > 0$, the space with the following inner product and norm

$$(y,v)_{D_h} = (D_h y, v)_h, \qquad \|y\|_{D_h} = \sqrt{(D_h y, y)_h}.$$

Let ω_τ be an uniform mesh on $(0,T)$ with the step $\tau = T/m$, $\omega_\tau^- = \omega_\tau \cup \{0\}$ and $\bar{\omega}_\tau = \omega_\tau \cup \{0,T\}$. We will use the standard notation of the theory of difference schemes [1].

We consider the two–level weighted finite difference scheme

$$v_t + A_h v^{(\sigma)} = \varphi(t), \quad t \in \omega_\tau^-; \qquad v(0) = v_0, \tag{4.1}$$

where $v^{(\sigma)} = \sigma\hat{v} + (1-\sigma)v = \sigma v(t+\tau) + (1-\sigma)v(t)$, $\sigma \geq (1+\varepsilon)/2 - 1/(\tau\,\|A_h\|)$ is the weight parameter, v_0 is a given element of H_h, $\varphi(t)$ is a given function and $v(t)$ is unknown function with values in H_h. We also consider the finite difference scheme

$$v_t + A_h v^{(\sigma)} = \psi_t, \quad t \in \omega_\tau^-; \qquad v(0) = v_0, \tag{4.2}$$

where $\psi(t)$ is a given mesh function with values in H_h.

The following analogous of lemmas 2.1 and 2.2 hold.

Lemma 4.1. *The solution of the finite difference scheme* (4.1) *satisfies a priori estimates:*

$$\tau \sum_{t\in\bar{\omega}_\tau}{}' \|A_h v(t)\|_h^2 + \tau \sum_{t\in\omega_\tau^-} \|v_t(t)\|_h^2 \leq C\left(\|v_0\|_{A_h}^2 + \tau\,\|A_h v_0\|_h^2 + \tau \sum_{t\in\omega_\tau^-} \|\varphi(t)\|_h^2\right),$$

$$\begin{aligned}
&\tau \sum_{t\in\bar{\omega}_\tau}{}' \|v(t)\|_{A_h}^2 + \tau^2 \sum_{t\in\bar{\omega}_\tau}{}' \sum_{t'\in\bar{\omega}_\tau,\ t'\neq t}{}' \frac{\|v(t)-v(t')\|_h^2}{|t-t'|^2} \\
&\quad \leq C\left(\|v_0\|_h^2 + \tau\,\|v_0\|_{A_h}^2 + \tau \sum_{t\in\omega_\tau^-} \|\varphi(t)\|_{A_h^{-1}}^2\right),
\end{aligned}$$

$$\tau \sum_{t\in\bar{\omega}_\tau}{}' \|v(t)\|^2 \leq C\left(\|v_0\|_{A_h^{-1}}^2 + \tau\,\|v_0\|_h^2 + \tau \textstyle\sum_{t\in\omega_\tau^-} \|A_h^{-1}\varphi(t)\|_h^2\right).$$

Here we denoted $\sum\limits_{t\in\bar\omega_\tau}{}' w(t) = \frac{w(0)}{2} + \sum_{t\in\omega_\tau} w(t) + \frac{w(T)}{2}$.

Lemma 4.2. *The solution of the finite difference scheme* (4.2) *satisfies a priori estimates:*

$$\begin{gathered}
\tau \sum_{t\in\bar\omega_\tau}{}' \|v(t)\|^2_{A_h} + \tau^2 \sum_{t\in\bar\omega_\tau}{}' \sum_{t'\in\bar\omega_\tau,\ t'\neq t}{}' \frac{\|v(t)-v(t')\|^2_h}{|t-t'|^2} \le C\bigg\{ \|v_0\|^2_h + \tau\,\|v_0\|^2_{A_h} \\
+\tau^2 \sum_{t\in\bar\omega_\tau}{}' \sum_{t'\in\bar\omega_\tau,\ t'\neq t}{}' \frac{\|\psi(t)-\psi(t')\|^2_h}{|t-t'|^2} + \tau \sum_{t\in\omega_\tau} \Big(\frac{1}{t} + \frac{1}{T-t}\Big) \|\psi(t)\|^2_h \bigg\}, \\
\tau \sum_{t\in\bar\omega_\tau}{}' \|v(t)\|^2_h \le C\bigg(\|v_0-\psi(0)\|^2_{A_h^{-1}} + \tau\,\|v_0\|^2_h + \tau \sum_{t\in\bar\omega_\tau}{}' \|\psi(t)\|^2_h \bigg).
\end{gathered}$$

Example 4.1. Let ω_h be an uniform mesh in $(0,1)$ with the step $h = 1/n$, $\omega_h^- = \omega_h \cup \{0\}$, $\bar\omega_h = \omega_h \cup \{0,1\}$ and $\bar\omega_{h\tau} = \bar\omega_h \times \bar\omega_\tau$. We approximate the initial–boundary value problem (2.8) with the standard weighted difference scheme

$$\begin{gathered}
v_t = \Big(a\, v_x^{(\sigma)}\Big)_{\bar x} + f\,, \quad x \in \omega_h\,,\ t \in \omega_\tau^-\,; \\
v(0,\,t) = v(1,\,t)\,, \quad t \in \bar\omega_\tau\,; \qquad v(x,\,0) = u_0(x)\,, \quad x \in \omega_h\,.
\end{gathered} \tag{4.3}$$

Here a is some stencil functional of k, e.g. $a(x) = k(x+h/2)$.

Let H_h be the set of functions defined on the mesh $\bar\omega_h$, which vanish in the nodes $x=0$ and $x=1$, $(v,\,w)_h = h \sum\limits_{x\in\omega_h} v(x)\,w(x)$ and $A_h w = -(a\,w_x)_{\bar x}$. The characteristics of the operator A_h are well known [1, p. 120], in particular $A_h = A_h^* \ge 8\,k_0\,E_h$. In such a way, the difference scheme (4.3) can be represented in the form of operator–difference scheme (4.1). The inverse operator A_h^{-1} is

$$\begin{gathered}
A_h^{-1} w(x) = -\sum_{x'=0}^{x-h} \frac{h}{a(x')} \sum_{x''=h}^{x'} h\,w(x'') \\
+\Bigg(\sum_{x'=0}^{1-h} \frac{h}{a(x')}\Bigg)^{-1} \Bigg(\sum_{x'=0}^{x-h} \frac{h}{a(x')}\Bigg) \Bigg(\sum_{x'=0}^{1-h} \frac{h}{a(x')} \sum_{x''=h}^{x'} h\,w(x'')\Bigg)\,.
\end{gathered} \tag{4.4}$$

The following inequalities hold

$$k_0\,|[w_x\|^2_h \le \|w\|^2_{A_h} = |[\sqrt{a}\,w_x\|^2_h \le k_1\,|[w_x\|^2_h\,, \tag{4.5}$$

where $|[w\|^2_h = h \sum\limits_{x\in\omega_h^-} w^2(x)$, and

$$k_2\,\|w_{x\bar x}\|^2_h \le \|A_h w\|^2_h \le k_3\,\|w_{x\bar x}\|^2_h\,. \tag{4.6}$$

5. STRONG STABILITY OF OPERATOR–DIFFERENCE SCHEMES

Along with (4.1) and (4.2) let us consider analogous finite difference schemes with perturbed right hand sides, initial conditions and operators:

$$\tilde{v}_t + \tilde{A}_h \tilde{v}^{(\sigma)} = \tilde{\varphi}(t), \quad t \in \omega_\tau^-; \qquad \tilde{v}(0) = \tilde{v}_0 \tag{5.1}$$

and

$$\tilde{v}_t + \tilde{A}_h \tilde{v}^{(\sigma)} = \tilde{\psi}_t, \quad t \in \omega_\tau^-; \qquad \tilde{v}(0) = \tilde{v}_0. \tag{5.2}$$

We deal with the problem of estimating the perturbation of the solution

$$z(t) = \tilde{v}(t) - v(t)$$

with perturbations of v_0, A_h and φ (or ψ).

Let the perturbed operator satisfy the analogous assumptions as the operator A_h:

$$\tilde{A}_h(t) = \tilde{A}_h(0), \qquad \tilde{A}_h = \tilde{A}_h^* \geq \tilde{\delta}_h E_h, \qquad \tilde{\delta}_h = \text{const} > 0.$$

As the measure of perturbation of operator we shall use the positive constant α in inequalities

$$\|(\tilde{A}_h - A_h) w\|_h \leq \alpha \, \|\tilde{A}_h w\|_h \,, \tag{5.3}$$

$$|((\tilde{A}_h - A_h) w, \, w)_h| \leq \alpha \, (A_h w, \, w)_h, \tag{5.4}$$

$$\|A_h^{-1} \tilde{A}_h w - w\|_h \leq \alpha \, \|w\|_h. \tag{5.5}$$

From (4.1) and (5.1), consequently (4.2) and (5.2), we obtain the following difference schemes for the perturbation of the solution

$$z_t + A_h z^{(\sigma)} = (\tilde{\varphi}(t) - \varphi(t)) - (\tilde{A}_h - A_h) \, \tilde{v}^{(\sigma)}, \quad t \in \omega_\tau^-; \quad z(0) = \tilde{v}_0 - v_0 \tag{5.6}$$

and

$$z_t + A_h z^{(\sigma)} = (\tilde{\psi} - \psi)_t - (\tilde{A}_h - A_h) \, \tilde{v}^{(\sigma)}, \quad t \in \omega_\tau^-; \quad z(0) = \tilde{v}_0 - v_0. \tag{5.7}$$

The following analog of the theorem 3.1 holds.

Theorem 5.1. *The perturbation of the solution of difference scheme* (4.1) *satisfies a priori estimates:*

$$\tau \sum_{t \in \bar{\omega}_\tau}{}' \|A_h z(t)\|_h^2 + \tau \sum_{t \in \omega_\tau^-} \|z_t(t)\|_h^2 \leq C \Big\{ \|\tilde{v}_0 - v_0\|_{A_h}^2 + \tau \, \|A_h (\tilde{v}_0 - v_0)\|_h^2$$
$$+ \tau \sum_{t \in \omega_\tau^-} \|\tilde{\varphi}(t) - \varphi(t)\|_h^2 + \alpha^2 \Big(\|\tilde{v}_0\|_{\tilde{A}_h}^2 + \tau \sum_{t \in \omega_\tau^-} \|\tilde{\varphi}(t)\|_h^2 \Big) \Big\},$$

if the condition (5.3) *holds;*

$$\tau \sum_{t\in\bar\omega_\tau}{}' \|z(t)\|_{A_h}^2 + \tau^2 \sum_{t\in\bar\omega_\tau}{}' \sum_{t'\in\bar\omega_\tau,\ t'\neq t}{}' \frac{\|z(t)-z(t')\|_h^2}{|t-t'|^2} \le C\left\{\|\tilde v_0 - v_0\|_h^2\right.$$
$$\left. +\tau\,\|\tilde v_0 - v_0\|_{A_h}^2 + \tau \sum_{t\in\omega_\tau^-} \|\tilde\varphi(t)-\varphi(t)\|_{A_h^{-1}}^2 + \alpha^2\Big(\|\tilde v_0\|_h^2 + \tau\sum_{t\in\omega_\tau^-}\|\tilde\varphi(t)\|_{\tilde A_h^{-1}}^2\Big)\right\},$$

if the condition (5.4) *holds;*

$$\tau \sum_{t\in\bar\omega_\tau}{}' \|z(t)\|_h^2 \le C\left\{\|\tilde v_0 - v_0\|_{A_h^{-1}}^2 + \tau\,\|\tilde v_0 - v_0\|_h^2\right.$$
$$\left. +\tau \sum_{t\in\omega_\tau^-} \|A_h^{-1}(\tilde\varphi(t)-\varphi(t))\|_h^2 + \alpha^2\Big(\|\tilde v_0\|_{\tilde A_h^{-1}}^2 + \tau\sum_{t\in\omega_\tau^-}\|\tilde A_h^{-1}\tilde\varphi(t)\|_h^2\Big)\right\},$$

if the condition (5.5) *holds.*

The perturbation of the solution of difference scheme (4.2) *satisfies a priori estimates:*

$$\tau \sum_{t\in\bar\omega_\tau}{}' \|z(t)\|_{A_h}^2 + \tau^2 \sum_{t\in\bar\omega_\tau}{}' \sum_{t'\in\bar\omega_\tau,\ t'\neq t}{}' \frac{\|z(t)-z(t')\|_h^2}{|t-t'|^2} \le C\left\{\|\tilde v_0 - v_0\|_h^2\right.$$
$$+\tau\,\|\tilde v_0 - v_0\|_{A_h}^2 + \tau^2 \sum_{t\in\bar\omega_\tau}{}' \sum_{t'\in\bar\omega_\tau,\ t'\neq t}{}' \frac{\|[\tilde\psi(t)-\psi(t)]-[\tilde\psi(t')-\psi(t')]\|_h^2}{|t-t'|^2}$$
$$+\tau \sum_{t\in\omega_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\|\tilde\psi(t)-\psi(t)\|_h^2 + \alpha^2\Big[\|\tilde v_0\|_h^2$$
$$\left. +\tau^2 \sum_{t\in\bar\omega_\tau}{}' \sum_{t'\in\bar\omega_\tau,\ t'\neq t}{}' \frac{\|\tilde\psi(t)-\tilde\psi(t')\|_h^2}{|t-t'|^2} + \tau\sum_{t\in\omega_\tau}\Big(\frac{1}{t}+\frac{1}{T-t}\Big)\|\tilde\psi(t)\|_h^2\Big]\right\},$$

if the condition (5.4) *holds;*

$$\tau \sum_{t\in\bar\omega_\tau}{}' \|z(t)\|_h^2 \le C\left\{\|\tilde v_0 - v_0 - [\tilde\psi(0)-\psi(0)]\|_{A_h^{-1}}^2 + \tau\,\|\tilde v_0 - v_0\|_h^2\right.$$
$$\left. +\tau \sum_{t\in\bar\omega_\tau}{}' \|\tilde\psi(t)-\psi(t)\|_h^2 + \alpha^2\left(\|\tilde v_0 - \tilde\psi(0)\|_{A_h^{-1}}^2 + \tau\sum_{t\in\bar\omega_\tau}{}'\|\tilde\psi(t)\|_h^2\right)\right\},$$

if the condition (5.5) *holds.*

Example 5.1. Let us consider, together with (4.3), a perturbed finite difference scheme

$$\begin{gathered} \tilde v_t = \left(\tilde a\,\tilde v_x^{(\sigma)}\right)_{\bar x} + \tilde f\,, \quad x\in\omega_h\,,\ \ t\in\omega_\tau^-\,; \\ \tilde v(0,\,t) = \tilde v(1,\,t)\,, \quad t\in\bar\omega_\tau\,; \qquad \tilde v(x,\,0) = \tilde u_0(x)\,, \quad x\in\omega_h\,. \end{gathered} \tag{5.8}$$

Let us set $A_h w = -\,(a\,w_x)_{\bar{x}}$ and $\tilde{A}_h w = -\,(\tilde{a}\,w_x)_{\bar{x}}$. From (4.5) follows

$$((\tilde{A}_h - A_h)\,w,\,w)_h = h \sum_{x\in\omega_h^-} (\tilde{a}(x) - a(x))\,w_x^2(x)\,,$$

wherefrom we can see, that the condition (5.4) is satisfied when

$$|\tilde{a}(x) - a(x)| \le \alpha\,a(x)\,.$$

Using (4.6) we obtain

$$\begin{aligned}\|(\tilde{A}_h - A_h)\,w\|_h &\le \|\tilde{a} - a\|_{C,h}\,\|w_{x\bar{x}}\|_h + \|\tilde{a}_x - a_x\|_{C,h}\,|[w_x\|_h \\ &\le \frac{1}{k_2}\left(\|\tilde{a} - a\|_{C,h} + \frac{1}{8}\,\|\tilde{a}_x - a_x\|_{C,h}\right)\|A_h w\|_h\,,\end{aligned}$$

where $\|\cdot\|_{C,h}$ is the mesh C–norm. In such a way, if

$$|\tilde{a}(x) - a(x)| \le \alpha_1 \quad \text{i} \quad |\tilde{a}_x(x) - a_x(x)| \le \alpha_2\,, \tag{5.9}$$

then (5.3) holds with $\alpha = \alpha_1/k_2 + \alpha_2/(8\,k_2)$.

The condition (5.9) is also sufficient for the inequality (5.5). Using (4.4) we obtain (5.5) with $\alpha = \alpha_1\,(k_0 + 2\,\|a_x\|_{C,h})/k_0^2 + 2\,\alpha_2/k_0$.

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OPERATORINIŲ-DIFERENCIALINIŲ SCHEMŲ KOEFICIENTINIS STABILUMAS

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Darbe tiriamas pirmosios eilės nestacionarių diferencialinių lygčių, apibrėžtų Hilberto erdvėse, stabilumas pradinės sąlygos, laisvųjų narių ir operatoriaus koeficientų atžvilgiu. Stabilumo įverčiai įrodomi integralinėse laiko atžvilgiu normose. Analogiški rezultatai gauti ir baigtinių skirtumų schemoms. Įrodyti stipraus stabilumo įverčiai, kai modifikuojami lygties operatorius ir pradinė sąlyga. Teoriniai rezultatai pritaikyti vienmačiam šilumos laidumo uždaviniui ir jį aproksimuojančiai baigtinių skirtumų schemai.